PNAS
www.pnas.org

Supplementary Information for

Tree Clusters in Savannas result from islands of soil moisture

Ignacio Rodriguez-Iturbe, Zijuan Chen, A. Carla Staver, Simon A. Levin

Zijuan Chen

Email: zijuan@stat.tamu.edu

This PDF file includes:

Supplementary text
Figs. S1 to S6
References for SI reference citations

www.pnas.org/cgi/doi/10.1073/pnas.1819389116

Supporting Materials and methods

The Cox and Isham rainfall model (1)

The rainfall process is used as input in the space-time soil moisture balance equation. Rainfall occurrences are modelled by a sequence of circular rain cells that occur in a Poisson process of rate λ in space and time. Each cell has a random radius, W, a random duration, D and random average intensity, X, resulting in a total depth equal to XD. The random variables described above are mutually independent and the triples (W, D, X) are independent and identically distributed over the cells. In this model, the cells overlap temporally as well as spatially so that, at any particular spatial location $\boldsymbol{u}$ and time t, the rainfall process is the superposition of all cells overlapping the space-time $(\boldsymbol{u}, t)$. If W and D are all exponentially distributed with parameters ρ and η respectively ($\rho = {\mu_W}^{-1}$, $\eta = {\mu_D}^{-1}$), then the mean and covariance function of the total intensity $Y(\boldsymbol{u}, t)$ are (1, 2)

$$E\big(Y(\boldsymbol{u}, t)\big) = \pi\lambda E(W^2)\mu_D\mu_X = \frac{2\pi\lambda\mu_X}{\eta\rho^2},$$

$$\Gamma_Y(l, h) \cong \frac{2\pi\lambda E(X^2)}{\eta\rho^2}\left(1 + \frac{\rho l}{4}\right)e^{-\eta h - \frac{\rho l}{2}},$$

where $\Gamma_Y(l, h)$ denotes the covariance between the rain intensity at two locations at a distance l and with a temporal lag $h \geq 0$.

The jitter process and the soil moisture field (3)

The jitter process $Z(\boldsymbol{u}, t)$ is a non-negative stationary process correlated in space and time with mean one. The mean and covariance function of $Z(\boldsymbol{u}, t)$ are denoted by μ_Z and $\Gamma_Z(l, h)$. Since the jitter process should only increase the variability of soil moisture without changing its expected value, we impose $E(Z(\boldsymbol{u}, t)) = 1$. The covariance function of the jitter process is modelled as

$$\Gamma_Z(l, h) = \sigma_Z^2 e^{-\alpha l - \beta h}.$$

The parameters σ_Z^2, α and β are estimated from the data. With the jitter process, the modeled soil moisture field $\tilde{S}(\boldsymbol{u}, t)$ has an analytically known correlation structure (3) that is able to represent the spatial and temporal variability of soil moisture as observed in data. This is illustrated in Fig. S1.

Simulation of the soil moisture field

The probabilistic structure of the soil moisture field, $\tilde{S}(\boldsymbol{u}, t)$, is studied via simulation with the parameter values given in Table 1 and 500 runs for each set of parameters. For each run the rainfall process $Y(\boldsymbol{u}, t)$ is generated in a 500 km × 500 km square field. We then divide the 1 km × 1 km square at the center of the rainfall field into 100 × 100 square pixels ($\boldsymbol{u}_i, i = 1, 2, \dots, 10000$) of 10 m ×10 m each. At the center of each pixel, we calculate the soil moisture solely driven by rainfall, $S(\boldsymbol{u}, t)$, for 30 consecutive days, ($S(\boldsymbol{u}_i, j)$, $j = 1, 2, \dots, 30$) through the balance equation with $k = 0$. The jitter process is also simulated at the center of each pixel for 30 days and the soil moisture, $\tilde{S}(\boldsymbol{u}, t)$ is obtained by multiplying $S(\boldsymbol{u}, t)$ by $Z(\boldsymbol{u}, t)$. Focusing on the impact of soil moisture on vegetation the average of $\tilde{S}(\boldsymbol{u},t)$ over a number of days, $\tilde{S}_T(\boldsymbol{u},t)$, is specially relevant. This average over every pixel was studied at different levels of temporal aggregation with very similar results. This paper presents the results of $\tilde{S}(\boldsymbol{u},t)$, for $T = 30$ days.

With $k = 0$, we are able to calculate the soil moisture process S by (2)

$$S(\boldsymbol{u},t) = b\int_0^\infty e^{-av}Y(\boldsymbol{u},t-v)dv = b\int_{-\infty}^t e^{-a(t-v)}Y(\boldsymbol{u},v)dv.$$

By replacing ∞ by some large number T, $S(\boldsymbol{u},t)$ can be approximated by

$$S(\boldsymbol{u},t) \approx b\int_{-T}^t e^{-a(t-v)}Y(\boldsymbol{u},v)dv.$$

Here we use $T = 1$ year, and the rainfall occurrences are simulated in the time period $[-T,0]$. Then $S(\boldsymbol{u},t)$ is simulated at $\boldsymbol{u} = \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{10000}$ and $t = 1,2,\dots,30$ (30 consecutive days).

The jitter process, Z, is simulated as a Chi-squared field on the 1 km × 1 km area. For $\boldsymbol{u} = \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{10000}$ and $t = 1,2,\dots,30$, we firstly simulate a Gaussian Random Field $X(\boldsymbol{u},t)$ with mean and covariance function

$$E\big(X(\boldsymbol{u},t)\big) = 0,$$
$$\Gamma_X(l,h) = e^{-\alpha l/2-\beta h/2}.$$

Then $Z(\boldsymbol{u},t)$ is calculated as

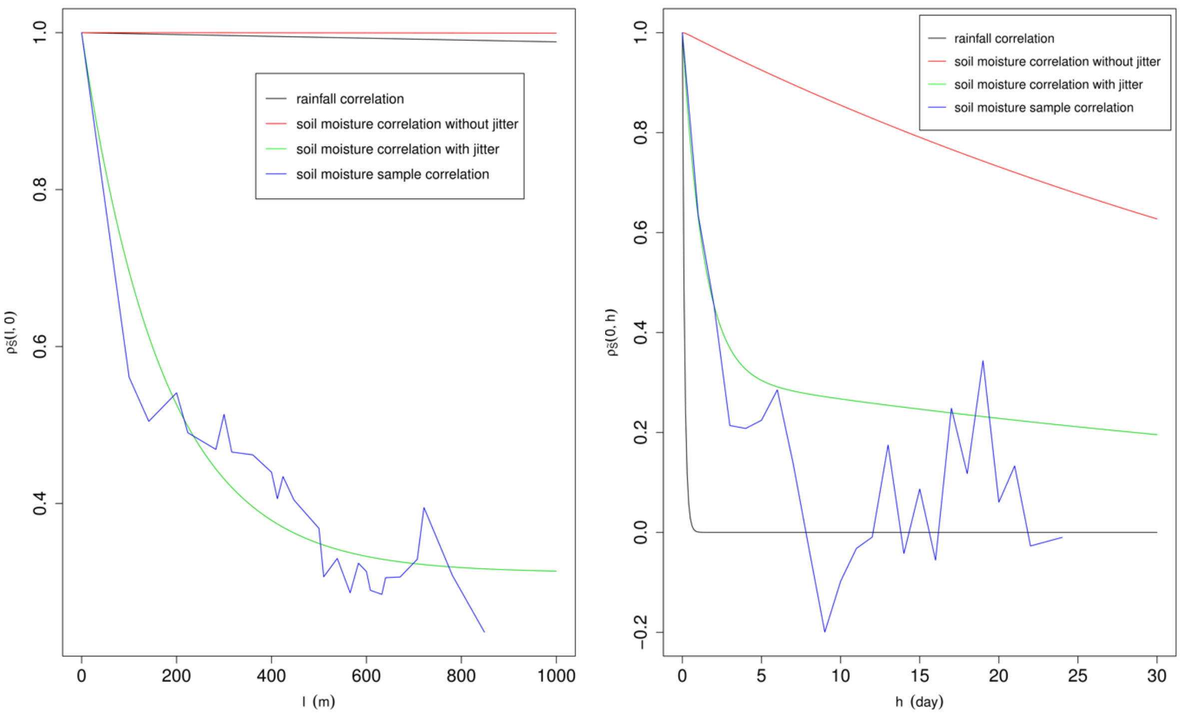
$$Z(\boldsymbol{u},t) = 1 + \frac{\sigma}{\sqrt{2}}(X^2(\boldsymbol{u},t) - 1).$$

Simple calculations can show that this transformation will result in a non-central Chi-squared distribution of Z with $E(Z(\boldsymbol{u},t)) = 1$ and $\Gamma_Z(l,h) = \sigma_Z^2 e^{-\alpha l-\beta h}$.

The power-law clustering of soil moisture islands and the spatial correlation of the soil moisture field

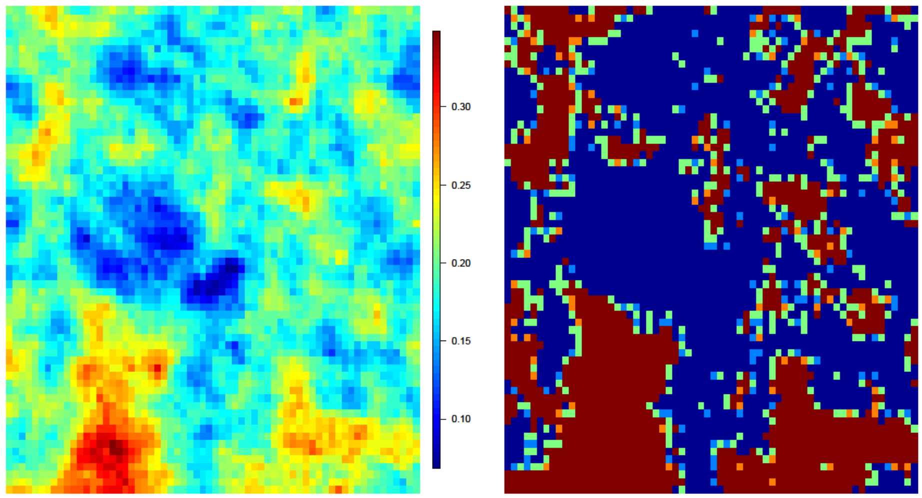
The power-law distribution of the size of soil moisture islands is closely related to the spatial correlation structure of the soil moisture field. Without the jitter process, the spatial correlation function of the soil moisture decays very slowly (above 0.95 within 1 km) and the soil moisture islands would not have a power law distribution. More specifically, without incorporating the spatial correlation structure of jitter process, the soil moisture islands would not have a power law distribution. This is illustrated in Fig. S6.

SUPPLEMENTARY FIGURES



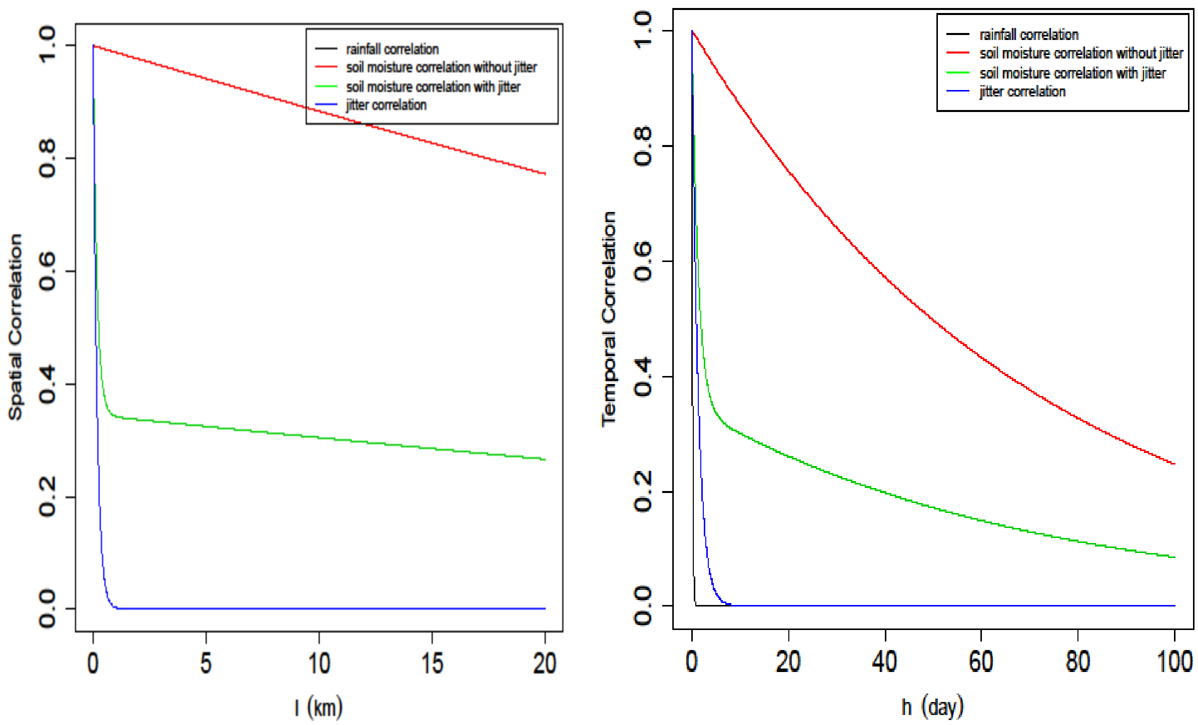
a **b**

Figure S1: Correlation functions of the rainfall process and the soil moisture field (3). **a,** The spatial correlation functions of the rainfall field and the soil moisture field. The sample correlation function is estimated from soil moisture data. Other correlation functions are from the models with estimated parameters. **b,** The temporal correlation functions of the rainfall field and the soil moisture field. The sample correlation function is estimated from soil moisture data. Other correlation functions are from the models with estimated parameters. Here $a = a_1 = 0.014\ \text{day}^{-1}$ and $b = b_1 = 0.002\ \text{mm}^{-1}$ are used, and the rainfall parameters and jitter parameters are estimated from the data, as shown in the 'average' column of Table 1, resulting in a mean growing rainfall equal to 497 mm/year and mean soil moisture equal to 0.195. These two plots show that the soil moisture field with jitter process, $\tilde{S}(u,t)$ captures the spatial and temporal variability of soil moisture while the soil moisture field without jitter, $S(u,t)$, fails to do so being dominated by the correlation structure of the rainfall field (explicit expression of the spatial-temporal covariance function is shown somewhere else (3)).



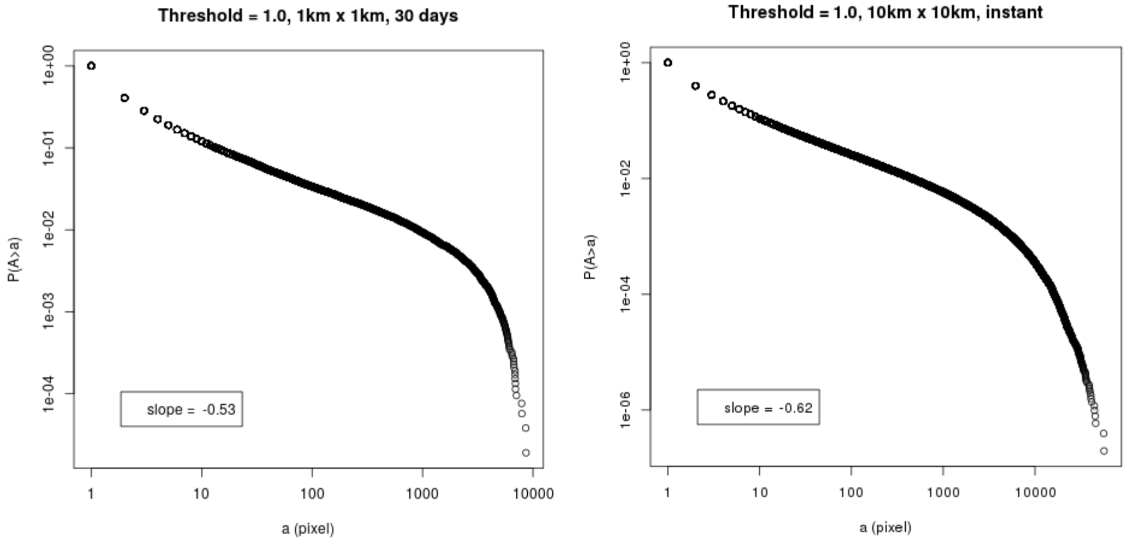
a **b**

Figure S2: Examples of the soil moisture field and soil moisture islands. **a,** The soil moisture field at a moment in time. **b,** Soil moisture islands (in red) above a threshold 0.19.



a **b**

Figure S3: Correlation functions of the rainfall process (Y), the soil moisture field driven by rainfall (S), the soil moisture field with jitter ($\tilde{S}$), and the jitter process (Z) with parameters estimated in (3). **a,** The spatial correlation for $0 \leq l \leq 20$ km. Note that the spatial correlation function of (S) and (Y) are the same (1). **b,** The temporal correlation for $0 \leq h \leq 100$ days.



a **b**

Figure S4: Distributions of jitter islands plotted in log-log scale. The threshold for both plots is 1.0, and a pixel for both plots is 10 m by 10 m. **a,** The jitter process is simulated on a 1 km by 1 km field (10^4 pixels in total). The power law exists until the number of pixels is close to the maximum number of pixels. **b,** The jitter process is simulated on a 10 km by 10 km field (10^6 pixels in total). The curve becomes bent before 10^4 pixels, which cannot result from an edge effect since 10^4 pixels only cover 1% of the total area.

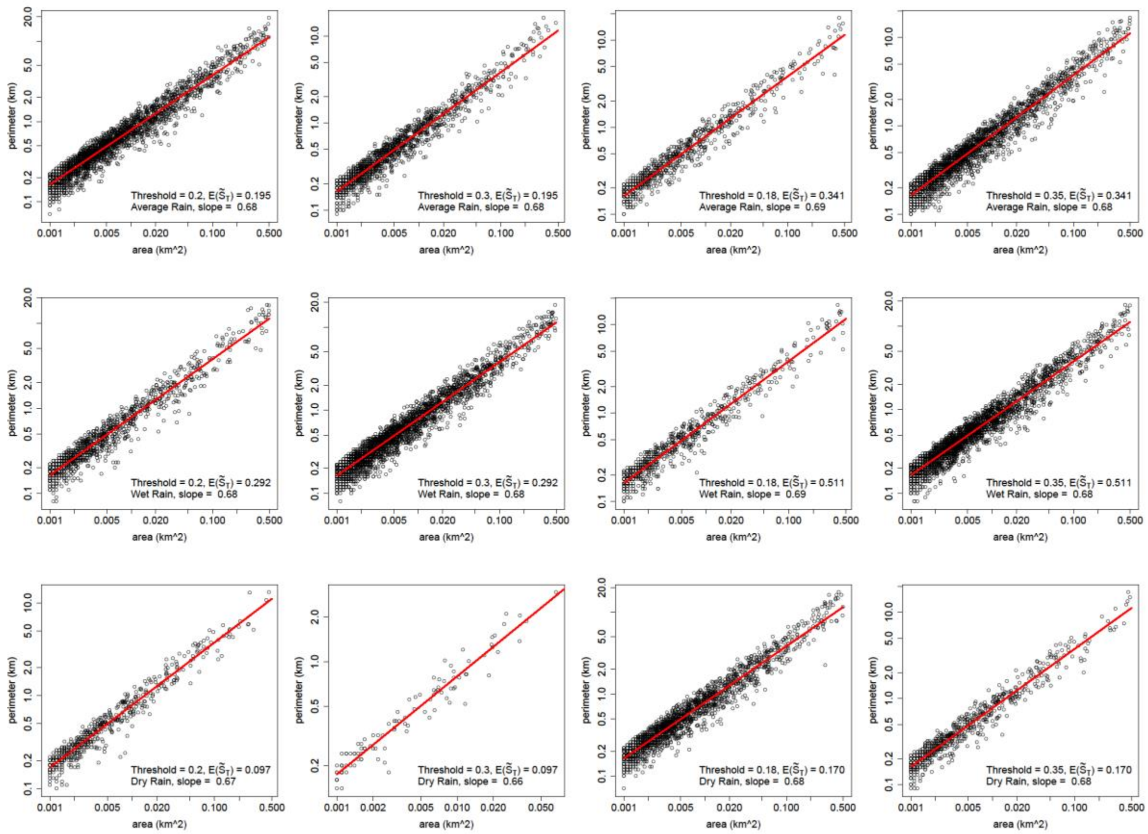
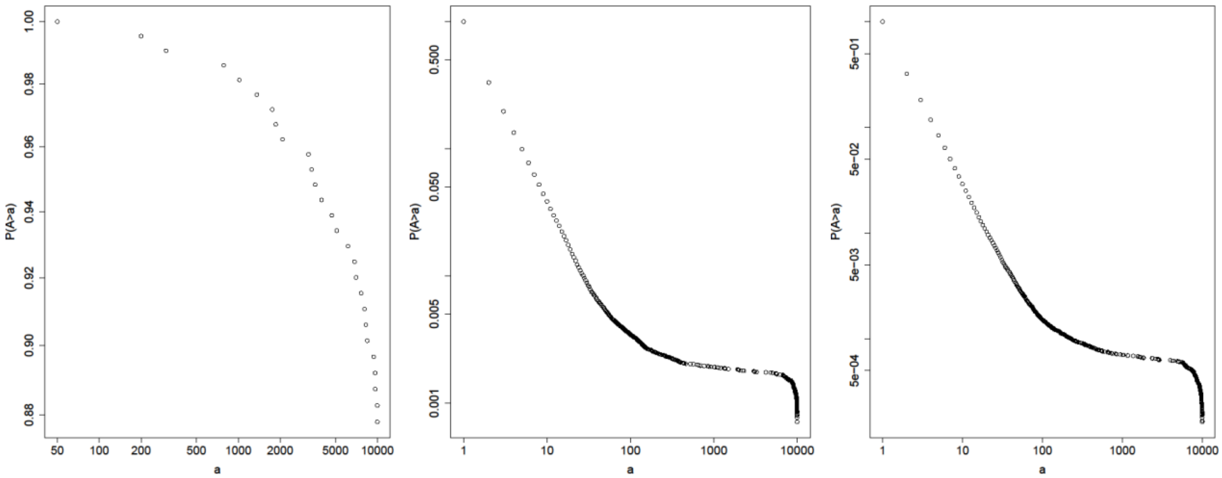


Figure S5: Area –vs– perimeter of soil moisture islands. Perimeter versus area are plotted at log-log scale with two different values of a and b ((a_1, b_1) and (a_2, b_2)) and three different mean growing season rainfall (average, wet and dry). These plots show that the slopes are very close (from 0.66 to 0.69) under different cases.



a **b** **c**

Figure S6: Distributions of soil moisture islands with and without jitter processes. **a,** The soil moisture islands on the soil moisture field without jitter. **b,** The soil moisture islands on the soil moisture field with independent jitter in both space and time. **c,** The soil moisture islands on the soil moisture field with jitter independent in space but correlated in time. Here $a = a_1 = 0.014$ day^{-1} and $b = b_1 = 0.002$ mm^{-1} are used, and the rainfall parameters and jitter parameters are from the 'average' column in Table 1. The soil moisture islands are calculated with threshold equal to 0.2.

REFERENCES

1. Rodriguez-Iturbe I, Cox DR, Isham V (1987) Some models for rainfall based on stochastic point processes. *Proceedings of the Royal Society A* 410:269-288.
2. Isham V, Cox DR, Rodriguez-Iturbe I, Porporato A, Manfreda S (2005) Representation of space-time variability of soil moisture. *Proceedings of the Royal Society A* 461:4035–4055.
3. Chen Z, Mohanty BP, Rodriguez-Iturbe I (2017) Space-time modeling of soil moisture. *Advances in Water Resources* 109:343-354.