Supplemental Material for "Topological phases with long-range interactions"

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In this supplemental material, we give a detailed analytical field theory calculation on how to obtain the $1/r^{\alpha+4}$ long-distance tail in the ground-state spin-spin correlation $\mathcal{C}_{ij} = \langle S_i^z S_j^z\rangle_0$ [where $\langle\cdots\rangle_m$ denotes the expectation value in the state $|m\rangle$ defined in Fig. (1a) of the main text] of H_α [defined in the Eq. (1) of the main text], as well as the $1/r^{\alpha+2}$ long-distance tail in the edge excitation amplitude $\langle S_i^z\rangle_1$ of H_α. As we stated in the main text, these power-law tails come from the non-analytic $O(|q|^{\alpha-1})$ term in the dispersion of the field $\boldsymbol{l}(q)$. Directly calculating the corrections due to these terms turns out to be complicated. Here we adopt a different field theoretic approach that is simpler in obtaining the long-distance behavior of $\mathcal{C}_{ij}$ and $\langle S_i^z\rangle_1$.

This alternative field theory is based on decomposing each spin-1 into two spin-1/2's and mapping them to two bosonic fields $\psi_{1,2}$ [1]. This field theory was originally designed for a nearest-neighbor spin-1 XXZ chain with ferromagnetic XY interaction. We work with a long-range interacting version of such a model:

$$\tilde{H}_\alpha = -\sum_{i>j}\frac{(-1)^{i-j-1}}{(i-j)^\alpha}\left(S_i^x S_j^x + S_i^y S_j^y\right) + \sum_{i>j}\frac{S_i^z S_j^z}{(i-j)^\alpha}, \tag{S1}$$

which is equivalent to H_α upon flipping every other spin in the x-y plane.

Following Ref. [1], we map each spin-1 to two fields $\psi_{1,2}(x)$ and their conjugate fields $X_{1,2}(x)$ as:

$$S^+(x) = \frac{1}{\pi}e^{-iX_1(x)/\sqrt{2}}\left\{\cos\left[\frac{X_2(x)}{\sqrt{2}}\right] + e^{i\pi x}e^{-i\sqrt{2}\psi_1(x)}\cos\left[\sqrt{2}\psi_2(x) + \frac{X_2(x)}{\sqrt{2}}\right]\right\}, \tag{S2}$$

$$S^z(x) = -\frac{\sqrt{2}}{\pi}\frac{\partial\psi_1}{\partial x} + \frac{2}{\pi}e^{i\pi x}\cos\left[\sqrt{2}\psi_2(x)\right]\cos\left[\sqrt{2}\psi_1(x)\right]. \tag{S3}$$

In the $\alpha\to\infty$ limit, and keeping only the most relevant terms, the above Hamiltonian gets mapped to two commuting parts $H_1 + H_2$ [1]:

$$H_1 = \frac{1}{2\pi}\int dx\left[\left(\frac{\partial X_1}{\partial x}\right)^2 + K_1^2\left(\frac{\partial\psi_1}{\partial x}\right)^2\right] + \frac{1}{\pi^2}\int dx\mu_1\cos\left[\sqrt{8}\psi_1(x)\right], \tag{S4}$$

$$H_2 = \frac{1}{2\pi}\int dx\left[\left(\frac{\partial X_2}{\partial x}\right)^2 + K_2^2\left(\frac{\partial\psi_2}{\partial x}\right)^2\right] + \frac{1}{\pi^2}\int dx\mu_2\cos\left[\sqrt{8}\psi_2(x)\right] + \frac{1}{\pi^2}\int dx\mu_3\cos\left[\sqrt{2}X_2(x)\right], \tag{S5}$$

where $\mu_1=\mu_2=\mu_3=-1$, $K_1=\sqrt{1+\frac{6}{\pi}}$, and $K_2=\sqrt{1+\frac{2}{\pi}}$.

The renormalization group (RG) flow equations for $\mu_{1,2,3}$ under the scaling change $x\to xe^l$ are given by

$$\frac{d\mu_1(l)}{dl} = (2-2/K_1)\,\mu_1(l), \qquad \frac{d\mu_2(l)}{dl} = (2-2/K_2)\,\mu_2(l), \qquad \frac{d\mu_3(l)}{dl} = (2-K_2/2)\,\mu_3(l). \tag{S6}$$

Therefore, the term $\cos(\sqrt{8}\psi_1)$ in Eq. (S4) is a relevant operator which gives rise to a gap, and one similarly finds the field ψ_2 to be gapped. This picture is consistent with the gapped excitations predicted by the nonlinear sigma model used in the main text.

For a finite $\alpha>0$, let us first consider the effect of the long-range $S_i^x S_j^x + S_i^y S_j^y$ interactions. Using Eq. (S2), we find that the most relevant terms in the RG sense are given by

$$\int dxdy\left\{-\frac{1}{\pi}\frac{1}{|x-y|^\alpha}\cos\left[\frac{X_1(x)-X_1(y)}{\sqrt{2}} + \pi(x-y)\right]\cos\frac{X_2(x)}{\sqrt{2}}\cos\frac{X_2(y)}{\sqrt{2}} + \text{less relevant terms}\right\}. \tag{S7}$$

The oscillatory character of these terms makes them irrelevant beyond the wavelength of the oscillation, i.e. $|x-y|\gtrsim 1$ [2]. We can thus disregard their effects at long distances.

Next we consider the long-range $S_i^z S_j^z$ interactions. Using Eq. (S3), we obtain

$$\int dxdy \frac{S^z(x)S^z(y)}{|x-y|^\alpha} = \int dxdy \left[\frac{2}{\pi^2}\frac{\partial\psi_1}{\partial x}\frac{1}{|x-y|^\alpha}\frac{\partial\psi_1}{\partial y} + \text{oscillating terms}\right]. \tag{S8}$$

The oscillating terms have the phase $e^{i\pi(x-y)}$ and thus are similarly suppressed; the first term in Eq. (S8) will change how ground-state correlations decay. This occurs because it contributes to the field $\psi_1(q)$ a dispersion proportional to $|q|^{\alpha+1}$, which is non-analytic and will generate a power-law decay of the correlations of $\psi_1(x)$ in the ground state. Formally, by expanding the $\mu_1 \cos\left[\sqrt{8}\psi_1(x)\right]$ term in Eq. (S4) to second order in $\psi_1(x)$, we can write the full dispersion for $\psi_1(q)$ as $\epsilon_1(q) = \mu_1 + q^2 + |q|^{\alpha+1}$ (ignoring constant coefficients). Consequently, the correlation function in the ground state $|0\rangle$ at large separation $|x-y|$ is given by

$$\langle\psi_1(x)\psi_1(y)\rangle_0 \propto \int \frac{e^{iq(x-y)}}{\epsilon_1(q)} \propto \frac{1}{|x-y|^{\alpha+2}}. \tag{S9}$$

Using Eq. (S3), we end up with $\langle S^z(x)S^z(y)\rangle_0 \propto 1/|x-y|^{\alpha+4}$ as stated in the main text.

Next, we calculate $\langle S^z(x)\rangle$ in the edge excited state $|1\rangle$ by using the edge-bulk coupling Hamiltonian $H_c = \sum_{i=2}^{L-1} \boldsymbol{S}_i \cdot \left[\frac{\boldsymbol{\tau}_L}{(i-1)^\alpha} + \frac{\boldsymbol{\tau}_R}{(L-i)^\alpha}\right]$ defined in the main text. Rewriting H_c in the continuum limit, we have

$$H_c = \frac{1}{2}\int dx \left[\frac{1}{(x-1)^\alpha} + \frac{1}{(L-x)^\alpha}\right] S^z(x). \tag{S10}$$

Under the mapping Eq. (S3), and ignoring oscillating terms, we have

$$H_c \propto \int dx \left[\frac{1}{(x-1)^{\alpha+1}} - \frac{1}{(L-x)^{\alpha+1}}\right] \psi_1(x). \tag{S11}$$

Applying first order perturbation theory to H_c, we find that $\langle\psi_1(q)\rangle_1$ is given by:

$$(q^2 + |q|^{\alpha+1} + \mu_1)\langle\psi_1(q)\rangle_1 \propto (e^{-iq} - e^{iqL})|q|^\alpha. \tag{S12}$$

As a result, we obtain $\langle\psi_1(q)\rangle_1 \propto (e^{-iq} - e^{iqL})|q|^\alpha$ + higher order terms. For spins far way from both ends, we obtain

$$\langle S^z(x)\rangle_1 \propto \frac{1}{(x-1)^{\alpha+2}} + \frac{1}{(L-x)^{\alpha+2}} + \text{higher-order terms.} \tag{S13}$$

[1] H. J. Schulz, Phys. Rev. B **34**, 6372 (1986).
[2] T. Giamarchi, *Quantum physics in one dimension* (Oxford University Press, 2004).