Appendix

Generalized-α time integration

After spatial discretization of (46)–(47) we get the following degenerate hyperbolic system

$$\begin{aligned} \rho_0 \boldsymbol{M}_h \ddot{\mathbf{u}}(t) + \boldsymbol{C}_h \dot{\mathbf{u}}(t) + \mathbf{R}_{\text{upper}}(\mathbf{u}(t), \mathbf{p}(t)) &= \mathbf{0}, \\ \mathbf{R}_{\text{lower}}(\mathbf{u}(t), \mathbf{p}(t)) &= \mathbf{0}, \\ \mathbf{u}(0) &= \mathbf{u}_0, \\ \dot{\mathbf{u}}(0) &= \mathbf{u}_0, \end{aligned}$$

where $\boldsymbol{M}_h$ denotes the mass matrix; $\boldsymbol{C}_h$ denotes an optional damping matrix; $\ddot{\mathbf{u}}(t)$ denote the unknown nodal accelerations; $\dot{\mathbf{u}}(t)$ denote the unknown nodal velocities; $\mathbf{u}(t)$ denote the unknown nodal displacements; and $\mathbf{p}(t)$ denote the unknown nodal pressure values. We will use the modified generalized-α method proposed in [50]. To this end we introduce the auxiliary velocity $\mathbf{v} = \dot{\mathbf{u}}$. Then, applying the standard generalized-α integrator from [28] we obtain

$$\boldsymbol{M}_h \dot{\mathbf{u}}_{n+\alpha_{\text{m}}} - \boldsymbol{M}_h \mathbf{v}_{n+\alpha_{\text{f}}} = \mathbf{0}, \tag{48}$$

$$\rho_0 \boldsymbol{M}_h \dot{\mathbf{v}}_{n+\alpha_{\text{m}}} + \boldsymbol{C}_h \mathbf{v}_{n+\alpha_{\text{f}}} + \mathbf{R}^{n+\alpha_{\text{f}}}_{\text{upper}} = \mathbf{0}, \tag{49}$$

$$\mathbf{R}^{n+\alpha_{\text{f}}}_{\text{lower}} = \mathbf{0}, \tag{50}$$

where

$$\begin{aligned} \mathbf{R}^{n+\alpha_{\text{f}}}_{\text{upper}} &:= \alpha_{\text{f}} \mathbf{R}_{\text{upper}}(\mathbf{u}_{n+1}, \mathbf{p}_{n+1}), \\ &\quad + (1-\alpha_{\text{f}}) \mathbf{R}_{\text{upper}}(\mathbf{u}_n, \mathbf{p}_n), \\ \mathbf{R}^{n+\alpha_{\text{f}}}_{\text{lower}} &:= \alpha_{\text{f}} \mathbf{R}_{\text{lower}}(\mathbf{u}_{n+1}, \mathbf{p}_{n+1}), \\ &\quad + (1-\alpha_{\text{f}}) \mathbf{R}_{\text{lower}}(\mathbf{d}_n, \mathbf{p}_n), \end{aligned}$$

and

$$\dot{\mathbf{u}}_{n+\alpha_{\text{m}}} := \alpha_{\text{m}} \dot{\mathbf{u}}_{n+1} + (1-\alpha_{\text{m}}) \dot{\mathbf{u}}_n, \tag{51}$$

$$\dot{\mathbf{v}}_{n+\alpha_{\text{m}}} := \alpha_{\text{m}} \dot{\mathbf{v}}_{n+1} + (1-\alpha_{\text{m}}) \dot{\mathbf{v}}_n, \tag{52}$$

$$\mathbf{v}_{n+\alpha_{\text{f}}} := \alpha_{\text{f}}\ \mathbf{v}_{n+1} + (1-\alpha_{\text{f}})\ \mathbf{v}_n. \tag{53}$$

Moreover, we employ Newmark's approximations, [61],

$$\dot{\mathbf{u}}_{n+1} = \frac{1}{\gamma \Delta t} (\mathbf{u}_{n+1} - \mathbf{u}_n) + \frac{\gamma - 1}{\gamma} \dot{\mathbf{u}}_n, \tag{54}$$

$$\mathbf{v}_{n+1} = \frac{1}{\gamma \Delta t} (\mathbf{v}_{n+1} - \mathbf{v}_n) + \frac{\gamma - 1}{\gamma} \dot{\mathbf{v}}_n \tag{55}$$

Using (48) we observe

$$\dot{\mathbf{u}}_{n+\alpha_{\text{m}}} = \mathbf{v}_{n+\alpha_{\text{f}}}$$

and combining this with (49)–(55) we conclude

$$\begin{aligned} \mathbf{v}_{n+1} &= \frac{\alpha_{\text{m}}}{\alpha_{\text{f}} \gamma \Delta t} (\mathbf{u}_{n+1} - \mathbf{u}_n) + \frac{\gamma - \alpha_{\text{m}}}{\gamma \alpha_{\text{f}}} \dot{\mathbf{u}}_n + \frac{\alpha_{\text{f}} - 1}{\alpha_{\text{f}}} \mathbf{v}_n, \\ \dot{\mathbf{v}}_{n+1} &= \frac{\alpha_{\text{m}}}{\alpha_{\text{f}} \gamma^2 \Delta t^2} (\mathbf{u}_{n+1} - \mathbf{u}_n) - \frac{1}{\alpha_{\text{f}} \gamma \Delta t} \mathbf{v}_n + \frac{\gamma - 1}{\gamma} \dot{\mathbf{v}}_n \\ &\quad + \frac{\gamma - \alpha_{\text{m}}}{\alpha_{\text{f}} \gamma^2 \Delta t} \dot{\mathbf{u}}_n. \end{aligned}$$

Thus, a dependence of $\mathbf{v}_{n+1}$ and $\dot{\mathbf{v}}_{n+1}$ on $\mathbf{u}_{n+1}$ can be established. Having this the unknown values $\mathbf{u}_{n+1}, \mathbf{p}_{n+1}$ can be computed with the Newton–Raphson method. Based on [50] we set the parameters depending only on $\rho_\infty \in [0, 1)$ by

$$\begin{aligned} \alpha_{\text{f}} &:= \frac{1}{1+\rho_\infty}, \\ \alpha_{\text{m}} &:= \frac{3-\rho_\infty}{2(1+\rho_\infty)}, \\ \gamma &:= \frac{1}{2} + \alpha_{\text{m}} - \alpha_{\text{f}}. \end{aligned}$$

In all our simulations we used a value of $\rho_\infty = 0.5$.

Remark on the implementation of the pressure-projection stabilized equal order pair

Considering the bilinear form $s_h(p_h, q_h)$ defined in (38) we can rewrite this with a simple calculation into

$$s_h(p_h, q_h) := \sum_{l=1}^{n_{\text{el}}} \left(\int_{K_l} p_h q_h \,\mathrm{d}\mathbf{x} - \frac{1}{|\tau_l|} \int_{K_l} p_h \,\mathrm{d}\mathbf{x} \int_{K_l} q_h \,\mathrm{d}\mathbf{x} \right).$$

Denoting by $\{\phi_i\}_{i=1}^n$ the chosen ansatz functions the element contribution for an arbitrary element K to the matrix $\boldsymbol{C}_h$ is given by

$$\int_K \phi_i \phi_j \,\mathrm{d}\mathbf{x} - \frac{1}{|K|} \int_K \phi_i \,\mathrm{d}\mathbf{x} \int_K \phi_j \,\mathrm{d}\mathbf{x}.$$

This corresponds to an element mass matrix minus a rank-one correction.

Static condensation

For completeness we provide a summary for the static condensation used for the MINI element. Consider a finite element $K \in \mathcal{T}_h$ with a local ordering of the unknowns $\mathbf{u}$

$$\mathbf{u} = \big(u_x^1, u_y^1, u_z^1, \ldots, u_x^{\text{ndofs}_N}, u_y^{\text{ndofs}_\text{N}}, u_z^{\text{ndofs}_\text{N}}, \\ u_{x,\text{B}}^1, u_{y,\text{B}}^1, u_{z,\text{B}}^1, \ldots, u_{x,\text{B}}^{\text{ndofs}_\text{B}}, u_{y,\text{B}}^{\text{ndofs}_\text{B}}, u_{z,\text{B}}^{\text{ndofs}_\text{B}}\big)$$

and $\mathbf{p}$ as

$$\mathbf{p} = \left(p^1, p^2, \ldots, p^{\text{ndofs}_\text{N}}\right).$$

Here, ndofs_N corresponds to the nodal degrees of freedom per element and ndofs_B to the bubble degrees of freedom (one for tetrahedral elements and two for hexahedral elements). Then the element contribution to the global saddle-point system can be written as

$$\begin{pmatrix} \boldsymbol{K}_{\text{NN}} & \boldsymbol{K}_{\text{NB}} & \boldsymbol{B}_\text{N}^\top \\ \boldsymbol{K}_{\text{BN}} & \boldsymbol{K}_{\text{BB}} & \boldsymbol{B}_\text{B}^\top \\ \boldsymbol{B}_\text{N} & \boldsymbol{B}_\text{B} & \boldsymbol{C}_\text{N} \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u}_\text{N} \\ \Delta \mathbf{u}_\text{B} \\ \Delta \mathbf{p} \end{pmatrix} = \begin{pmatrix} -\mathbf{R}_\text{N}^{\text{upper}} \\ -\mathbf{R}_\text{B}^{\text{upper}} \\ -\mathbf{R}_\text{N}^{\text{lower}} \end{pmatrix}.$$

The bubble part of the stiffness matrix, $\boldsymbol{K}_{\mathrm{BB}}$ is local to the element and can be directly inverted. This gives the condensed system

$$\begin{pmatrix} \boldsymbol{K}_{\mathrm{eff}} & \boldsymbol{B}_{\mathrm{eff}}^T \\ \boldsymbol{B}_{\mathrm{eff}} & \boldsymbol{C}_{\mathrm{eff}} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{u}_{\mathrm{N}} \\ \Delta\mathbf{p} \end{pmatrix} = \begin{pmatrix} -\mathbf{R}_{\mathrm{eff}}^{\mathrm{upper}} \\ -\mathbf{R}_{\mathrm{eff}}^{\mathrm{lower}} \end{pmatrix},$$

where the effective matrices and vectors are given as

$$\begin{aligned} \boldsymbol{K}_{\mathrm{eff}} &:= \boldsymbol{K}_{\mathrm{NN}} - \boldsymbol{K}_{\mathrm{NB}}\boldsymbol{K}_{\mathrm{BB}}^{-1}\boldsymbol{K}_{\mathrm{BN}} \\ \boldsymbol{B}_{\mathrm{eff}} &:= \boldsymbol{B}_{\mathrm{N}} - \boldsymbol{B}_{\mathrm{B}}\boldsymbol{K}_{\mathrm{BB}}^{-1}\boldsymbol{K}_{\mathrm{BN}}, \\ \boldsymbol{C}_{\mathrm{eff}} &:= \boldsymbol{C}_{\mathrm{N}} - \boldsymbol{B}_{\mathrm{B}}\boldsymbol{K}_{\mathrm{BB}}^{-1}\boldsymbol{B}_{\mathrm{B}}^{\top}, \\ \mathbf{R}_{\mathrm{eff}}^{\mathrm{upper}} &:= \mathbf{R}_{\mathrm{N}}^{\mathrm{upper}} - \boldsymbol{K}_{\mathrm{NB}}\boldsymbol{K}_{\mathrm{BB}}^{-1}\mathbf{R}_{\mathrm{B}}^{\mathrm{upper}}, \\ \mathbf{R}_{\mathrm{eff}}^{\mathrm{lower}} &:= \mathbf{R}_{\mathrm{N}}^{\mathrm{lower}} - \boldsymbol{B}_{\mathrm{B}}\boldsymbol{K}_{\mathrm{BB}}^{-1}\mathbf{R}_{\mathrm{B}}^{\mathrm{upper}}. \end{aligned}$$

The effective matrices and vectors can then be assembled in a standard way into the global system. The bubble update contributions can be calculated once $\Delta\mathbf{u}_N$ and $\Delta\mathbf{p}_N$ are know as

$$\Delta\mathbf{u}_B = -\boldsymbol{K}_{\mathrm{BB}}^{-1}\left(\mathbf{R}_{\mathrm{B}}^{\mathrm{upper}} + \boldsymbol{K}_{\mathrm{BN}}\Delta\mathbf{u}_{\mathrm{N}} + \boldsymbol{B}_{\mathrm{B}}^{\top}\Delta\mathbf{p}_{\mathrm{N}}\right).$$

Tensor calculus

We use the following results from tensor calculus, for more details we refer to, e.g., [45, 84].

$$\begin{aligned} \frac{\partial\overline{\boldsymbol{C}}}{\partial\boldsymbol{C}} &= J^{-\frac{2}{3}}\mathbb{P} = J^{-\frac{2}{3}}\left(\mathbb{I} - \frac{1}{3}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}\right), \\ \frac{\partial\boldsymbol{C}^{-1}}{\partial\boldsymbol{C}} &= -\boldsymbol{C}^{-1}\odot\boldsymbol{C}^{-1}, \\ (\boldsymbol{A}\odot\boldsymbol{A})_{ijkl} &:= \frac{1}{2}\left(A_{ik}A_{jl} + A_{il}A_{jk}\right). \end{aligned}$$

For symmetric $\boldsymbol{A}$ it holds

$$\mathbb{P} : \boldsymbol{A} = \mathrm{Dev}(\boldsymbol{A}) = \boldsymbol{A} - \frac{1}{3}(\boldsymbol{A} : \boldsymbol{C})\boldsymbol{C}^{-1}.$$

The isochoric part of the second Piola–Kirchhoff stress tensor as well as the isochoric part of the fourth order elasticity tensor are given as

$$\boldsymbol{S}_{\mathrm{isc}} := 2\frac{\partial\overline{\Psi}(\overline{\boldsymbol{C}})}{\partial\boldsymbol{C}} = J^{-\frac{2}{3}}\mathrm{Dev}(\overline{\boldsymbol{S}}), \tag{56}$$

$$\overline{\boldsymbol{S}} := 2\frac{\partial\overline{\Psi}(\overline{\boldsymbol{C}})}{\partial\overline{\boldsymbol{C}}},$$

$$\begin{aligned} \mathbb{C}_{\mathrm{isc}} &:= 4\frac{\overline{\Psi}(\overline{\boldsymbol{C}})}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \\ &= J^{-\frac{4}{3}}\mathbb{P}\overline{\mathbb{C}}\mathbb{P}^{\top} + J^{-\frac{2}{3}}\frac{2}{3}\mathrm{tr}(\boldsymbol{C}\overline{\boldsymbol{S}})\widetilde{\mathbb{P}} \\ &\quad - \frac{4}{3}\boldsymbol{S}_{\mathrm{isc}} \overset{\mathrm{S}}{\otimes} \boldsymbol{C}^{-1}, \end{aligned} \tag{57}$$

$$\begin{aligned} \overline{\mathbb{C}} &:= 4\frac{\partial\overline{\Psi}(\overline{\boldsymbol{C}})}{\partial\overline{\boldsymbol{C}}\partial\overline{\boldsymbol{C}}}, \\ \widetilde{\mathbb{P}} &:= \boldsymbol{C}^{-1}\odot\boldsymbol{C}^{-1} - \frac{1}{3}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}^{-1}, \\ \boldsymbol{A}\overset{\mathrm{S}}{\otimes}\boldsymbol{B} &:= \frac{1}{2}\left(\boldsymbol{A}\otimes\boldsymbol{B} + \boldsymbol{B}\otimes\boldsymbol{A}\right). \end{aligned}$$