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Web-based Supplementary Material for "Exact Inference on the Random-Effects Model for Meta-Analyses with Few Studies"

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K		DL	HE	SJ	ML	REML	EB	HS	Bayes	Exact CI								
										c_0								
										0	0.5	1	2	4	8	16	32	1024
3	coverage	0.81	0.78	0.86	0.65	0.79	0.81	0.66	1	0.98	0.98	**0.98**	0.98	0.98	0.98	0.98	0.98	0.98
	length	9.08	9.21	10.14	6.49	9.1	9.36	5.62	82.94	23.9	23.52	**23.33**	23.18	23.15	23.42	24.31	25.63	29.02
4	coverage	0.83	0.79	0.89	0.74	0.83	0.83	0.74	1	0.96	0.96	**0.96**	0.96	0.96	0.96	0.96	0.97	0.97
	length	7.83	7.88	8.82	6.32	7.92	8.09	5.67	21.53	15.49	15.15	**15.08**	15.08	15.28	15.8	16.51	17.22	18.47
6	coverage	0.86	0.84	0.91	0.81	0.86	0.86	0.81	0.97	0.95	**0.95**	0.95	0.95	0.95	0.96	0.96	0.97	0.97
	length	6.82	6.78	7.55	6.01	6.82	6.96	5.6	10.83	10.04	**9.82**	9.83	9.97	10.29	10.74	11.18	11.54	12.01
8	coverage	0.89	0.86	0.93	0.86	0.88	0.89	0.85	0.97	0.95	**0.95**	0.96	0.96	0.96	0.97	0.97	0.97	0.97
	length	5.83	5.68	6.4	5.32	5.82	5.87	5.1	8.1	8.1	**7.97**	8.02	8.17	8.47	8.83	9.16	9.37	9.66
10	coverage	0.9	0.87	0.93	0.87	0.91	0.9	0.86	0.97	0.96	**0.96**	0.96	0.97	0.97	0.97	0.98	0.98	0.98
	length	5.41	5.3	5.9	5.05	5.4	5.45	4.88	6.81	6.99	**6.87**	6.92	7.09	7.35	7.65	7.91	8.07	8.28
15	coverage	0.94	0.92	0.96	0.92	0.93	0.93	0.92	0.96	**0.96**	0.96	0.96	0.96	0.98	0.98	0.99	0.99	0.99
	length	4.44	4.4	4.85	4.26	4.45	4.48	4.17	5.14	**5.25**	5.19	5.26	5.42	5.65	5.88	6.06	6.17	6.3
20	coverage	0.93	0.92	0.96	0.93	0.94	0.94	0.92	0.94	**0.96**	0.96	0.96	0.96	0.97	0.97	0.97	0.97	0.98
	length	3.89	3.83	4.21	3.77	3.89	3.9	3.71	4.33	**4.46**	4.42	4.5	4.64	4.84	5.03	5.17	5.26	5.36

Table 1: Simulation comparing coverage rates and lengths for 95% asymptotic and exact CIs of μ, the grand mean. The proposed estimator, using a range of c_0 values, is compared with the DerSimonian-Laird, Hedges-Olkin, Sidak-Jonkman, ML, REML, empirical bayes, and Hunter-Schmidt estimators, as well as a bayesian estimator using a typical non-informative prior. Highlighted in bold are cells corresponding to the suggested tuning parameter for the proposed estimator.

K		DL	HE	SJ	ML	REML	EB	HS	Bayes	Exact CI								
										c_0								
										0	0.5	1	2	4	8	16	32	1024
2	coverage	0.89	0.89	0.93	0.83	0.89	0.89	0.85		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	7.52	7.52	8.29	4.27	7.52	7.52	4.25		57.03	56.98	56.93	56.83	56.69	56.56	56.53	56.78	70.36
3	coverage	0.89	0.87	0.94	0.83	0.87	0.88	0.83	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	1.00
	length	5.80	6.26	7.30	4.33	5.72	6.08	4.25	64.53	17.90	17.81	17.78	17.76	17.81	17.99	18.41	19.12	21.86
4	coverage	0.92	0.91	0.96	0.87	0.91	0.92	0.87	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	5.07	5.46	6.53	4.08	4.97	5.32	4.09	15.58	10.75	10.64	10.60	10.59	10.69	10.97	11.40	11.89	12.92
6	coverage	0.86	0.89	0.97	0.85	0.88	0.87	0.86	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	4.28	4.42	5.50	3.76	4.25	4.42	3.77	7.55	6.48	6.44	6.44	6.48	6.61	6.81	7.06	7.28	7.58
8	coverage	0.93	0.91	0.97	0.92	0.93	0.92	0.91	0.99	0.98	0.98	0.97	0.96	0.96	0.96	0.97	0.97	0.97
	length	3.62	3.80	4.76	3.31	3.59	3.74	3.32	5.46	4.81	4.77	4.77	4.82	4.89	5.04	5.21	5.33	5.49
10	coverage	0.93	0.90	0.98	0.91	0.93	0.93	0.91	0.99	0.98	0.98	0.98	0.98	0.98	0.99	0.99	0.99	0.99
	length	3.39	3.60	4.44	3.07	3.34	3.53	3.16	4.58	4.29	4.26	4.28	4.34	4.47	4.65	4.80	4.92	5.07
15	coverage	0.95	0.93	0.98	0.89	0.93	0.95	0.91	0.97	0.97	0.96	0.96	0.96	0.97	0.97	0.97	0.97	0.97
	length	2.78	3.00	3.70	2.57	2.70	2.90	2.65	3.32	3.37	3.36	3.37	3.42	3.53	3.66	3.76	3.83	3.91
20	coverage	0.94	0.94	0.99	0.93	0.93	0.95	0.93	0.97	0.94	0.94	0.94	0.95	0.95	0.95	0.95	0.97	0.97
	length	2.40	2.48	3.15	2.27	2.35	2.44	2.31	2.79	2.69	2.70	2.73	2.80	2.91	3.01	3.09	3.14	3.21

Table 2: Comparing coverage rates and lengths for 95% asymptotic and exact CIs of μ, the grand mean, under a misspecified model. The proposed estimator, using a range of c_0 values, is compared with the DerSimonian-Laird, Hedges-Olkin, Sidak-Jonkman, ML, REML, empirical bayes, and Hunter-Schmidt estimators, as well as a bayesian estimator using a typical non-informative prior. The true between-study distribution is a centered chi-squared distribution, which is misspecified by a Gaussian model to test the robustness of the method. The empty cells in the "Bayes" column occur when the R function fails to yield a valid result.

K		DL	HE	SJ	ML	REML	EB	HS	Bayes	Exact CI								
										c_0								
										0	0.5	1	2	4	8	16	32	1024
2	coverage	0.88	0.88	0.92	0.81	0.88	0.88	0.83		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	7.62	7.62	8.42	4.33	7.62	7.62	4.26		60.60	60.55	60.50	60.42	60.27	60.14	60.15	60.34	-Inf
3	coverage	0.91	0.89	0.96	0.85	0.90	0.90	0.85	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	6.23	6.41	7.58	4.44	6.23	6.44	4.38	62.44	15.84	15.74	15.70	15.69	15.75	15.95	16.43	17.19	19.91
4	coverage	0.91	0.90	0.96	0.87	0.91	0.91	0.88	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	5.22	5.75	6.77	4.13	5.15	5.60	4.17	16.06	9.73	9.66	9.65	9.68	9.77	10.00	10.37	10.75	11.64
6	coverage	0.93	0.92	0.99	0.90	0.93	0.93	0.91	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
	length	4.31	4.62	5.65	3.79	4.30	4.54	3.78	7.49	6.47	6.43	6.43	6.46	6.57	6.79	7.05	7.28	7.65
8	coverage	0.93	0.93	0.99	0.91	0.92	0.93	0.91	0.97	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98
	length	3.67	3.89	4.84	3.34	3.65	3.80	3.36	5.53	5.12	5.09	5.11	5.17	5.30	5.51	5.72	5.88	6.10
10	coverage	0.92	0.90	0.98	0.90	0.91	0.92	0.90	0.99	0.96	0.96	0.96	0.95	0.95	0.95	0.96	0.96	0.97
	length	3.30	3.51	4.40	3.04	3.25	3.43	3.07	4.47	4.29	4.28	4.30	4.36	4.49	4.66	4.81	4.92	5.07
15	coverage	0.91	0.91	0.97	0.90	0.91	0.91	0.90	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.98
	length	2.80	2.93	3.68	2.63	2.76	2.87	2.67	3.34	3.26	3.25	3.26	3.32	3.43	3.55	3.66	3.73	3.81
20	coverage	0.94	0.92	0.97	0.92	0.94	0.93	0.93	0.95	0.97	0.97	0.97	0.97	0.97	0.97	0.98	0.98	0.98
	length	2.44	2.53	3.20	2.33	2.42	2.49	2.36	2.72	2.84	2.83	2.86	2.92	3.03	3.14	3.23	3.29	3.35

Table 3: Comparing coverage rates and lengths for 95% asymptotic and exact CIs of μ, the grand mean, under a misspecified model. The proposed estimator, using a range of c_0 values, is compared with the DerSimonian-Laird, Hedges-Olkin, Sidak-Jonkman, ML, REML, empirical bayes, and Hunter-Schmidt estimators, as well as a bayesian estimator using a typical non-informative prior. The true between-study distribution is a Cauchy distribution, which is misspecified by a Gaussian model to test the robustness of the method. The empty cells in the "Bayes" column occur when the R function fails to yield a valid result.

K		DL	HE	SJ	ML	REML	EB	HS	Bayes	Exact CI								
										c_0								
										0	0.5	1	2	4	8	16	32	1024
2	coverage	0.92	0.92	0.96	0.89	0.92	0.92	0.90		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	7.30	7.30	8.20	4.27	7.30	7.30	4.20		56.93	56.92	56.90	56.87	56.83	56.77	56.79	57.06	69.08
3	coverage	0.91	0.90	0.98	0.86	0.90	0.91	0.89	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	5.84	6.18	7.27	4.28	5.82	6.13	4.24	58.44	15.94	15.87	15.84	15.82	15.83	15.98	16.43	17.24	19.91
4	coverage	0.94	0.91	0.97	0.90	0.94	0.94	0.91	1.00	0.99	0.99	0.99	1.00	1.00	1.00	0.99	1.00	1.00
	length	4.94	5.39	6.49	4.00	4.92	5.22	4.02	14.83	9.22	9.16	9.15	9.16	9.25	9.46	9.81	10.20	11.06
6	coverage	0.94	0.92	0.98	0.93	0.93	0.94	0.92	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
	length	4.04	4.18	5.30	3.57	3.99	4.14	3.58	7.13	6.14	6.12	6.11	6.14	6.26	6.46	6.69	6.90	7.22
8	coverage	0.95	0.94	0.98	0.93	0.94	0.95	0.94	0.99	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.99
	length	3.50	3.78	4.76	3.18	3.46	3.65	3.22	5.22	4.93	4.92	4.92	4.96	5.06	5.24	5.42	5.57	5.77
10	coverage	0.93	0.92	0.98	0.92	0.93	0.93	0.92	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98
	length	3.14	3.34	4.24	2.93	3.13	3.25	2.96	4.21	4.08	4.05	4.06	4.10	4.20	4.34	4.48	4.58	4.72
15	coverage	0.93	0.92	0.98	0.92	0.93	0.93	0.92	0.97	0.98	0.97	0.97	0.97	0.97	0.97	0.98	0.98	0.98
	length	2.60	2.80	3.54	2.45	2.56	2.67	2.49	3.14	3.06	3.05	3.07	3.11	3.21	3.33	3.43	3.50	3.58
20	coverage	0.95	0.93	0.99	0.95	0.95	0.95	0.94	0.97	0.97	0.97	0.97	0.98	0.98	0.98	0.98	0.98	0.98
	length	2.26	2.37	3.07	2.17	2.25	2.30	2.19	2.62	2.60	2.59	2.60	2.65	2.74	2.84	2.92	2.97	3.03

Table 4: Comparing coverage rates and lengths for 95% asymptotic and exact CIs of μ, the grand mean, under a misspecified model. The proposed estimator, using a range of c_0 values, is compared with the DerSimonian-Laird, Hedges-Olkin, Sidak-Jonkman, ML, REML, empirical bayes, and Hunter-Schmidt estimators, as well as a bayesian estimator using a typical non-informative prior. The true between-study distribution is a centered exponential distribution, which is misspecified by a Gaussian model to test the robustness of the method. The empty cells in the "Bayes" column occur when the R function fails to yield a valid result.

K		DL	HE	SJ	ML	REML	EB	HS	Bayes	Exact CI								
										c_0								
										0	0.5	1	2	4	8	16	32	1024
2	coverage	0.68	0.68	0.78	0.46	0.68	0.68	0.50		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	length	10.02	10.02	10.55	5.22	10.02	10.02	4.68		71.73	71.67	71.62	71.53	71.39	71.28	71.30	71.59	89.39
3	coverage	0.73	0.70	0.83	0.60	0.73	0.73	0.62	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.98
	length	7.83	7.63	8.86	5.54	7.88	7.96	5.04	75.36	20.36	20.17	20.10	20.04	20.09	20.35	21.08	22.25	25.38
4	coverage	0.78	0.77	0.87	0.68	0.77	0.79	0.69	0.99	0.95	0.95	0.95	0.95	0.95	0.96	0.95	0.96	0.96
	length	6.91	7.06	8.10	5.51	6.95	7.17	5.11	20.22	13.73	13.52	13.47	13.52	13.69	14.14	14.84	15.50	16.67
6	coverage	0.83	0.79	0.90	0.77	0.83	0.84	0.76	0.97	0.95	0.95	0.95	0.95	0.95	0.96	0.96	0.96	0.97
	length	5.80	5.65	6.64	5.05	5.82	5.87	4.84	9.81	9.29	9.16	9.17	9.26	9.54	9.93	10.33	10.63	11.03
8	coverage	0.86	0.83	0.93	0.83	0.86	0.87	0.82	0.95	0.94	0.94	0.94	0.95	0.95	0.96	0.96	0.96	0.97
	length	5.25	5.13	5.95	4.83	5.29	5.32	4.61	7.17	7.06	6.99	7.03	7.17	7.43	7.76	8.06	8.27	8.54
10	coverage	0.88	0.85	0.93	0.85	0.89	0.89	0.85	0.97	0.93	0.94	0.94	0.94	0.94	0.96	0.97	0.97	0.97
	length	4.80	4.65	5.39	4.47	4.80	4.83	4.35	6.13	6.02	5.98	6.04	6.17	6.40	6.66	6.89	7.04	7.22
15	coverage	0.91	0.90	0.95	0.90	0.92	0.92	0.90	0.95	0.96	0.96	0.96	0.96	0.97	0.97	0.98	0.98	0.98
	length	3.99	3.90	4.45	3.83	4.01	4.01	3.75	4.59	4.71	4.69	4.75	4.89	5.08	5.28	5.43	5.53	5.64
20	coverage	0.94	0.91	0.97	0.93	0.93	0.93	0.92	0.96	0.95	0.94	0.95	0.96	0.97	0.98	0.98	0.98	0.98
	length	3.48	3.40	3.87	3.37	3.48	3.49	3.33	3.81	3.99	3.96	4.02	4.15	4.33	4.50	4.63	4.70	4.79

Table 5: Comparing coverage rates and lengths for 95% asymptotic and exact CIs of μ, the grand mean, under a misspecified model. The proposed estimator, using a range of c_0 values, is compared with the DerSimonian-Laird, Hedges-Olkin, Sidak-Jonkman, ML, REML, empirical bayes, and Hunter-Schmidt estimators, as well as a bayesian estimator using a typical non-informative prior. The true between-study distribution is a uniform distribution, which is misspecified by a Gaussian model to test the robustness of the method. The empty cells in the "Bayes" column occur when the R function fails to yield a valid result.

Study	K	DL	HE	SJ	ML	REML	EB	HS	Bayes	Permutation	Proposal
lumbar spine	27	0.83–1.68	0.85–1.3	0.75–1.83	0.82–1.72	0.81–1.74	0.80–1.75	0.85–1.66	0.76–1.79	0.79–1.65	0.79–1.73
total hip	6	0.50–1.85	0.56–1.74	0.51–1.83	0.54–1.77	0.50–1.85	0.52–1.82	0.56–1.73	0.21–2.32	0.00–2.30	0.23–2.14
forearm	5	0.21–3.38	-0.22–4.15	-0.14–4.02	0.47–2.58	0.26–3.25	-0.04–3.86	0.31–3.12	-0.72–3.70	*	-0.49–4.16
total body	3	0.27–1.78	0.03–2.11	0.07–2.06	0.28–1.76	0.09–2.03	0.06–2.08	0.38–1.55	-8.27–10.54	*	-0.78–2.83

Table 6: Random effects meta-analyses of the effect of calcium supplements on percentage change in bone mineral density (Tai et al. (2015), Figs. 1, 3, and 7). The meta-analyses were carried out using the DerSimonian-Laird, Hedges, Hunter-Schmidt, Sidik-Jonkman, ML, REML, and empirical Bayes estimators; a bayesian estimator with a non-informative prior; the permutation test of Wang and Tian (2017) , applicable to meta-analyses with 6 or more studies; and the proposed exact method. A nominal 95% CI is reported. On the two smaller meta-analyses ($K = 3, 5$) the proposed exact method fails to reject the null of no change, whereas most of the other estimators reject the null (except for Bayes, Sidik-Jonkman, empirical Bayes and Hunter-Schmidt methods). The Hedges, Sidik-Jonkman, and Bayes estimators fail to reject the null in the meta analysis for forearm with $K = 5$. The CI based on the Bayes method is very wide in the meta analysis for total body with $K = 3$, a potential indication of its failure.

References

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Wang, Y. and Tian, L. (2017). An efficient numerical algorithm for exact inference in meta analysis. *Journal of Statistical Computation and Simulation* page under review.