Online Supplement – Appendix: Proofs

Proof for Lemma 1: We calculate the determinant of the Hessian of the R's profit $\Pi_R(p_1,p_2;w_1,w_2) = (p_1-w_1)(\alpha_1-\beta_1 p_1+p_2)+(p_2-w_2)(\alpha_2+p_1-\beta_2 p_2)$ as:

$$\begin{aligned} |H\Pi_R(p_1,p_2;w_1,w_2)| &= \begin{vmatrix} \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1^2} & \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1\partial p_2} \\ \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_2\partial p_1} & \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1^2} \end{vmatrix}, \\ &= \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1^2}\frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_2^2} - \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1\partial p_2}\frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_2\partial p_1}. \end{aligned}$$

We calculate the derivatives and then the determinant as follows:

$$\begin{aligned} \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1^2} &= \text{-}2\beta_1, \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1\partial p_2} &= 2, \\ \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_2^2} &= \text{-}2\beta_2, \frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_2\partial p_1} &= 2, \\ \det(H\Pi_R(p_1,p_2;w_1,w_2)) &= 4\beta_1\beta_2-4 = 4(\beta_1\beta_2-1). \end{aligned}$$

Since, $\frac{\partial^2\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1^2} < 0$, when $\beta_1\beta_2 - 1 \geq 0$ the critical point obtained by solving $\frac{\partial\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_1} = 0$ and $\frac{\partial\Pi_R(p_1,p_2;w_1,w_2)}{\partial p_2} = 0$ will be the local maximum of the profit, i.e., $\Pi_R(p_1,p_2;w_1,w_2)$ is concave in (p_1,p_2). A sufficient condition to preserve concavity is to assume $\beta_i \geq 0 \implies \beta_1\beta_2 \geq 1$, for $i \in \{1,2\}$. □

Proof for Lemma 2: From (2), we have: $p_i^*(w_i) = \frac{\alpha_i\beta_{-i}+\alpha_{-i}}{2(\beta_i\beta_{-i}-1)} + \frac{w_i}{2}$. The first order derivatives, w.r.t. w_i, α_i, α_{-i}, β_i, and β_{-i} are:

$$\begin{aligned} \frac{\partial p_i^*(w_i)}{w_i} &= \frac{1}{2} > 0, \\ \frac{\partial p_i^*(w_i)}{\alpha_i} &= \frac{\beta_{-i}}{2(\beta_i\beta_{-i}-1)} > 0, \frac{\partial p_i^*(w_i)}{\alpha_{-i}} = \frac{1}{2(\beta_i\beta_{-i}-1)} > 0 \text{ and } \frac{\partial p_i^*(w_i)}{\alpha_i} = \beta_{-i}\frac{\partial p_i^*(w_i)}{\alpha_{-i}} \geq \frac{\partial p_i^*(w_i)}{\alpha_{-i}}, \\ \frac{\partial p_i^*(w_i)}{\beta_i} &= -\frac{\alpha_i\beta_{-i}^2}{2(\beta_i\beta_{-i}-1)^2} < 0, \\ \frac{\partial p_i^*(w_i)}{\beta_{-i}} &= \frac{\alpha_i(\beta_i\beta_{-i}-1)-\beta_i(\alpha_i\beta_{-i}+\alpha_{-i})}{2(\beta_i\beta_{-i}-1)^2} = -\frac{\alpha_i+\beta_i\alpha_{-i}}{2(\beta_i\beta_{-i}-1)^2} < 0. \end{aligned}$$

From the above first order conditions to be positive or negative, we see that $p_i^*(w_i)$ is increasing in the wholesale price w_i, market sizes (α_1,α_2), and decreasing in product's own price elasticity β_i and other product's price elasticity β_{-i}. □.

Proof for Lemma 3: Supplier A's profit is given by: $\Pi_A(w_1,w_2) = (w_1-c_1)D_1(p_1^*(w_1),p_2^*(w_2)) = (w_1-c_1)(\alpha_1-\beta_1 p_1^*(w_1)+p_2^*(w_2))$. In order to examine the concavity of $\Pi_A(w_1,w_2)$ in w_1 and w_2, we derive the second order derivatives with respect to w_i as:

$$\frac{\partial^2\Pi_A(w_1,w_2)}{\partial w_1^2} = -\frac{\beta_1}{2} < 0, \frac{\partial\Pi_A(w_1,w_2)}{\partial w_2} = \frac{(w_1-c_1)}{2} \geq 0, \text{ and } \frac{\partial^2\Pi_A(w_1,w_2)}{\partial w_2^2} = 0.$$

Since, the second order derivative of supplier A's profit w.r.t. w_1 is negative, the profit is concave in w_1. Furthermore, we see that the first order derivative of the profit w.r.t. w_2 is independent of w_2 and non-negative; also, the second order derivative is zero. This implies that the profit is linearly increasing in w_2. We can derive similar expressions for supplier B. $\square$

Proof for Proposition 1:

i) Consider the first order derivative of w_1^e (from (3)) w.r.t. β_1 and β_2 as:

$$\begin{aligned}
\frac{\partial w_1^e}{\partial \beta_1} &= \frac{2\beta_2 c_1(4\beta_1\beta_2 - 1) - 4\beta_2[\alpha_2 + \beta_2(2(\alpha_1 + \beta_1 c_1) + c_2)]}{(4\beta_1\beta_2 - 1)^2} = -\frac{2\beta_2 c_1 + 4\beta_2[\alpha_2 + \beta_2(2\alpha_1 + c_2)]}{(4\beta_1\beta_2 - 1)^2} < 0, \\
&= -\frac{[2\beta_2(2\alpha_2 + c_1) + 4\beta_2^2(2\alpha_1 + c_2)]}{(4\beta_1\beta_2 - 1)^2}, \\
\frac{\partial w_1^e}{\partial \beta_2} &= \frac{[2(\alpha_1 + \beta_1 c_1) + c_2](4\beta_1\beta_2 - 1) - 4\beta_1[\alpha_2 + \beta_2(2(\alpha_1 + \beta_1 c_1) + c_2)]}{(4\beta_1\beta_2 - 1)^2} < 0, \\
&= -\frac{[(2\alpha_1 + c_2) + 2\beta_1(2\alpha_2 + c_1)]}{(4\beta_1\beta_2 - 1)^2} > -\frac{[2\beta_2(2\alpha_2 + c_1) + 4\beta_2^2(2\alpha_1 + c_2)]}{(4\beta_1\beta_2 - 1)^2} = \frac{\partial w_1^e}{\partial \beta_1} \text{ if } \beta_2 > \beta_1.
\end{aligned}$$

Since, w_1^e and w_2^e are symmetric, we present the first order derivatives for only w_1^e w.r.t. β_1 and β_2. Checking the sensitivity of p_1^e w.r.t. β_1 and β_2:

$$\begin{aligned}
p_1^e &= \frac{\alpha_1\beta_2 + \alpha_2}{2(\beta_1\beta_2 - 1)} + \frac{w_1^e}{2}, \\
\frac{\partial p_1^e}{\partial \beta_1} &= \frac{0 - \beta_2[\alpha_1\beta_2 + \alpha_2]}{2(\beta_1\beta_2 - 1)^2} + \frac{1}{2}\frac{\partial w_1^e}{\partial \beta_1} < 0, \text{ as } \frac{\partial w_1^e}{\partial \beta_1} < 0, \\
\frac{\partial p_1^e}{\partial \beta_2} &= \frac{\alpha_1[\beta_1\beta_2 - 1] - \beta_1[\alpha_1\beta_2 + \alpha_2]}{2(\beta_1\beta_2 - 1)^2} + \frac{1}{2}\frac{\partial w_1^e}{\partial \beta_2} = -\frac{\alpha_1 + \beta_1\alpha_2}{2(\beta_1\beta_2 - 1)^2} + \frac{1}{2}\frac{\partial w_1^e}{\partial \beta_2} < 0, \text{ as } \frac{\partial w_1^e}{\partial \beta_2} < 0.
\end{aligned}$$

ii) We look at the difference of $w_i^e - w_{-i}^e$:

$$\begin{aligned}
w_i^e - w_{-i}^e &= \frac{[\alpha_{-i} + \beta_{-i}(2(\alpha_i + \beta_i c_i) + c_{-i})] - [\alpha_i + \beta_i(2(\alpha_{-i} + \beta_{-i}c_{-i}) + c_i)]}{4\beta_1\beta_2 - 1}, \\
\implies (w_i^e - w_{-i}^e)(4\beta_1\beta_2 - 1) &= (\alpha_i - \beta_i c_i)(2\beta_{-i} - 1) - (\alpha_{-i} - \beta_{-i}c_{-i})(2\beta_i - 1) \\
\implies \frac{(w_i^e - w_{-i}^e)(4\beta_1\beta_2 - 1)}{(2\beta_i - 1)(2\beta_{-i} - 1)} &= \frac{\alpha_i - \beta_i c_i}{2\beta_i - 1} - \frac{\alpha_{-i} - \beta_{-i}c_{-i}}{2\beta_{-i} - 1}, \\
\implies w_i^e \geq w_{-i}^e \quad &\text{if} \quad \frac{\alpha_i - \beta_i c_i}{2\beta_i - 1} \geq \frac{\alpha_{-i} - \beta_{-i}c_{-i}}{2\beta_{-i} - 1}.
\end{aligned}$$

Therefore, if the market parameters for products 1 and 2 satisfy the above condition, then $w_i^e \geq w_{-i}^e$.

Next, we consider the difference of $p_i^e - p_{-i}^e$:

$$\begin{aligned}
p_i^e - p_{-i}^e &= \frac{\alpha_i\beta_{-i} + \alpha_{-i} - \alpha_{-i}\beta_i - \alpha_i}{2(\beta_1\beta_2 - 1)} + \frac{w_i^e - w_{-i}^e}{2}, \\
&= \frac{\alpha_i(\beta_{-i} - 1) - \alpha_{-i}(\beta_i - 1)}{2(\beta_1\beta_2 - 1)} + \frac{w_i^e - w_{-i}^e}{2}, \\
\implies p_i^e \geq p_{-i}^e \quad &\text{if} \quad \frac{\alpha_i}{\beta_i - 1} \geq \frac{\alpha_{-i}}{\beta_{-i} - 1} \text{ and } w_i^e \geq w_{-i}^e.
\end{aligned}$$

We know that if $\frac{\alpha_i - \beta_i c_i}{2\beta_i - 1} \geq \frac{\alpha_{-i} - \beta_{-i}c_{-i}}{2\beta_{-i} - 1}$, then $w_i^e \geq w_{-i}^e$. Combining with the above results, we have: $p_i^e \geq p_{-i}^e$ if $\frac{\alpha_i}{\beta_i - 1} \geq \frac{\alpha_{-i}}{\beta_{-i} - 1}$ and $\frac{\alpha_i - \beta_i c_i}{2\beta_i - 1} \geq \frac{\alpha_{-i} - \beta_{-i}c_{-i}}{2\beta_{-i} - 1}$. $\square$

Proof for Theorem 1: i) We first analyze the Stackelberg game where A leads, from (3) and (6), we have

$$\begin{aligned} w^e_{1,s_A} - w^e_{1,n} &= \frac{\alpha_2 + 2\alpha_1\beta_2 + \beta_2 c_2}{2(2\beta_1\beta_2 - 1)} + \frac{c_1}{2} - \frac{\alpha_2 + 2\alpha_1\beta_2 + \beta_2 c_2 + 2\beta_1\beta_2 c_1}{4\beta_1\beta_2 - 1}, \\ &= \frac{\alpha_2 + 2\alpha_1\beta_2 + \beta_2 c_2}{2(2\beta_1\beta_2 - 1)(4\beta_1\beta_2 - 1)} + \frac{c_1}{2} - \frac{1}{2}\frac{4\beta_1\beta_2 c_1 - c_1 + c_1}{4\beta_1\beta_2 - 1}, \\ &= \frac{1}{4\beta_1\beta_2 - 1}\left(\frac{\alpha_2 + 2\alpha_1\beta_2 + \beta_2 c_2}{2(2\beta_1\beta_2 - 1)} - \frac{c_1}{2}\right) = \frac{1}{4\beta_1\beta_2 - 1}\left(w^e_{1,s_A} - c_1\right) > 0. \end{aligned}$$

As $w^e_{2,s_A} = \dfrac{\alpha_2 + w^e_{1,s_A}}{2\beta_2} + \dfrac{c_2}{2}$ and $w^e_{2,n} = \dfrac{\alpha_2 + w^e_{1,n}}{2\beta_2} + \dfrac{c_2}{2}$, it is easy to obtain that $w^e_{2,s_A} > w^e_{2,n}$ by using $w^e_{1,s_A} > w^e_{1,n}$. Similarly, we can prove that $p^e_{i,s_A} > p^e_{i,n}$ using (2).

We now use the "similarity" between the Stackelberg games to present the expressions for the case where B leads the SC.

$$w^e_{2,s_B} - w^e_{2,n} = \frac{1}{4\beta_1\beta_2 - 1}\left(w^e_{2,s_B} - c_2\right) > 0, \text{ and } w^e_{1,s_B} - w^e_{1,n} = \frac{1}{2\beta_1}\left(w^e_{2,s_B} - w^e_{2,n}\right) > 0.$$

ii) Again, we first analyze the Stackelberg game where A leads. We know that $D^e_{1,s} = \alpha_1 - \beta_1 p^e_{1,s} + p^e_{2,s}$ and $D^e_{1,n} = \alpha_1 - \beta_1 p^e_{1,n} + p^e_{2,n}$:

$$\begin{aligned} D^e_{1,s_A} - D^e_{1,n} &= p^e_{2,s_A} - p^e_{2,n} - \beta_1(p^e_{1,s_A} - p^e_{1,n}) = \frac{w^e_{2,s_A} - w^e_{2,n}}{2} - \frac{\beta_1}{2}(w^e_{1,s_A} - w^e_{1,n}), \\ &= \frac{1}{2\beta_2}\left(\frac{w^e_{1,s_A} - w^e_{1,n}}{2}\right) - \frac{\beta_1}{2}(w^e_{1,s_A} - w^e_{1,n}) = -\frac{2\beta_1\beta_2 - 1}{2\beta_2}\left(\frac{w^e_{1,s_A} - w^e_{1,n}}{2}\right) < 0 \text{ as } w^e_{1,s_A} > w^e_{1,n}. \end{aligned}$$

Similar to the above, we consider:

$$\begin{aligned} D^e_{2,s_A} - D^e_{2,n} &= p^e_{1,s_A} - p^e_{1,n} - \beta_2(p^e_{2,s_A} - p^e_{2,n}) = \frac{w^e_{1,s_A} - w^e_{1,n}}{2} - \frac{\beta_2}{2}(w^e_{2,s_A} - w^e_{2,n}), \\ &= \frac{w^e_{1,s_A} - w^e_{1,n}}{2} - \frac{\beta_2}{2}\left(\frac{1}{\beta_2}\frac{w^e_{1,s_A} - w^e_{1,n}}{2}\right) = \frac{1}{2}\left(\frac{w^e_{1,s_A} - w^e_{1,n}}{2}\right) > 0 \text{ as } w^e_{1,s_A} > w^e_{1,n}. \end{aligned}$$

When B leads the SC, we have:

$$D^e_{2,s_B} - D^e_{2,n} = -\frac{2\beta_1\beta_2 - 1}{4\beta_1}\left(w^e_{2,s_B} - w^e_{2,n}\right) < 0, \text{ and } D^e_{1,s_B} - D^e_{1,n} = \frac{1}{4}\left(w^e_{2,s_B} - w^e_{2,n}\right) > 0.$$

Therefore, we have $D^e_{1,s_B} > D^e_{2,n} > D^e_{2,s_A}$ and $D^e_{2,s_B} < D^e_{2,n} < D^e_{2,s_A}$.

iii) Considering $\Pi^*_{R,n}(w_1, w_2) = \max_{(p_1,p_2)}(p_1 - w_1)(\alpha_1 - \beta_1 p_1 + p_2) + (p_2 - w_2)(\alpha_2 - \beta_2 p_2 + p_1)$, and (2), we have:

$$\begin{aligned} \Pi^*_{R,n}(w_1, w_2) &= \left(\frac{\alpha_1\beta_2 + \alpha_2}{2(\beta_1\beta_2 - 1)} - \frac{w_1}{2}\right)\left(\alpha_1 - \beta_1\left[\frac{\alpha_1\beta_2 + \alpha_2}{2(\beta_1\beta_2 - 1)} + \frac{w_1}{2}\right] + \frac{\alpha_2\beta_1 + \alpha_1}{2(\beta_1\beta_2 - 1)} + \frac{w_2}{2}\right) \\ &+ \left(\frac{\alpha_2\beta_1 + \alpha_1}{2(\beta_1\beta_2 - 1)} - \frac{w_2}{2}\right)\left(\alpha_2 - \beta_2\left[\frac{\alpha_2\beta_1 + \alpha_1}{2(\beta_1\beta_2 - 1)} + \frac{w_2}{2}\right] + \frac{\alpha_1\beta_2 + \alpha_2}{2(\beta_1\beta_2 - 1)} + \frac{w_1}{2}\right) \end{aligned}$$

We check the first order derivatives of $\Pi^*_{R,n}(w_1, w_2)$ w.r.t. to w_1: $\dfrac{\partial \Pi^*_{R,n}(w_1, w_2)}{\partial w_1} = -(\alpha_1 - \beta_1 w_1 + w_2) < 0$, as $\alpha_1 - \beta_1 w_1 + w_2 > 0$ because the market demand is non-negative for product 1 even when the retailer sets the market prices at the wholesale prices $D_1(p_1 = w_1, p_2 = w_2) > 0$.

iv) We present the proof when A leads, the proof for the case when B is the Stackelberg leader is identical. From i) and ii) above, we know that $w^e_{2,s_A} > w^e_{2,n}$ and $D^e_{2,s_A} > D^e_{2,n}$, respectively. Therefore, $(w^e_{2,s_A} - c_2)D^e_{2,s_A} > (w^e_{2,n} - c_2)D^e_{2,n}$ or $\Pi^{B,e}_{s_A} > \Pi^{B,e}_{n}$.

v) Using (3)–(4) and (6)–(7), we obtain the expressions for $\Pi_{s_A}^{A,e}$ and $\Pi_n^{A,e}$ as follows:

$$\Pi_{s_A}^{A,e} = \frac{\beta_1\left[\alpha_2 + c_1 + \beta_2(2\alpha_1 - 2c_1\beta_1 + c_2)\right]^2}{2(4\beta_1\beta_2 - 1)^2},$$
$$\Pi_n^{A,e} = \frac{\left[\alpha_2 + c_1 + \beta_2(2\alpha_1 - 2c_1\beta_1 + c_2)\right]^2}{16\beta_2(2\beta_1\beta_2 - 1)}.$$

Therefore, we have

$$\frac{\Pi_n^{A,e}}{\Pi_{s_A}^{A,e}} = \frac{4\beta_1\beta_2(4\beta_1\beta_2 - 2)}{(4\beta_1\beta_2 - 1)^2} = 1 - \frac{1}{(4\beta_1\beta_2 - 1)^2} < 1.$$

Also, $\beta_1\beta_2 \geq 1 \implies (4\beta_1\beta_2 - 1)^2 \geq 9$ and $\frac{\Pi_n^{A,e}}{\Pi_{s_A}^{A,e}} = 1 - \frac{1}{(4\beta_1\beta_2 - 1)^2} \geq \frac{8}{9}$.

□

Proof for Proposition 2: From the expressions for the emergency order's wholesale price and the equilibrium retail prices, it is easy to see that all the prices are linearly decreasing in δ. However, the level of disruption is decreasing in δ. Therefore, the equilibrium prices are increasing in the level of disruption. □

Proof for Lemma 4: From (12) and (15), we can explore $w_{E,s_A}^e - w_{E,n}^e$:

$$w_{E,s_A}^e - w_{E,n}^e = \frac{1}{\beta_1\beta_2 - 1}\left[\delta(q_{1,n}^e - q_{1,s_A}^e) + \beta_1(q_{2,n}^e - q_{2,s_A}^e)\right],$$

Using $(q_{1,n}^e - q_{1,s_A}^e) = \frac{2\beta_1\beta_2 - 1}{4\beta_2}\left(w_{1,s_A}^e - w_{1,n}^e\right)$ and $(q_{2,n}^e - q_{2,s_A}^e) = -\frac{1}{4}\left(w_{1,s_A}^e - w_{1,n}^e\right)$, we have

$$= \frac{(w_{1,s_A}^e - w_{1,n}^e)}{4(\beta_1\beta_2 - 1)}\left[\delta\left(\frac{2\beta_1\beta_2 - 1}{\beta_2}\right) - \beta_1\right] = \frac{(w_{1,s_A}^e - w_{1,n}^e)(2\beta_1\beta_2 - 1)}{4\beta_2(\beta_1\beta_2 - 1)}\left[\delta - \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right].$$

Therefore, if $\delta > \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}$ then $w_{E,s_A}^e > w_{E,n}^e$.

When B leads, we investigate $w_{E,s_B}^e - w_{E,n}^e$ similar to the above case when A leads, we will have:

$$w_{E,s_B}^e - w_{E,n}^e = \frac{1}{\beta_1\beta_2 - 1}\left[\delta(q_{1,n}^e - q_{1,s_B}^e) + \beta_1(q_{2,n}^e - q_{2,s_B}^e)\right],$$

Using $(q_{1,n}^e - q_{1,s_B}^e) = -\frac{1}{4}\left(w_{2,s_B}^e - w_{2,n}^e\right)$ and $(q_{2,n}^e - q_{2,s_B}^e) = \frac{2\beta_1\beta_2 - 1}{4\beta_1}\left(w_{2,s_B}^e - w_{2,n}^e\right)$, we have

$$= \frac{(w_{2,s_B}^e - w_{2,n}^e)}{4(\beta_1\beta_2 - 1)}\left[2\beta_1\beta_2 - 1 - \delta\right].$$

Therefore, if $\delta < 2\beta_1\beta_2 - 1$, then $w_{E,s_B}^e > w_{E,n}^e$. Note that $\beta_i > 1$, therefore $2\beta_1\beta_2 - 1 - \delta > 0$. Thus, the inequality is always satisfied. □

Proof for Lemma 5: It is immediate from (15) and (18) that $w_E^e > w_{E,s_A}^e$. Consider $w_E^e - w_{E,s_B}^e = \frac{1}{\beta_1\beta_2}\left[\delta(q_{1,s_B}^e - q_{1,s_A}^e) + \beta_1 q_{2,s_B}^e\right] > 0$, as $q_{1,s_B}^e > q_{1,s_A}^e$. From Lemma 4, we know that $w_{E,s_B}^e > w_{E,n}^e$, therefore $w_E^e > w_{E,s_B}^e > w_{E,n}^e$. □

Proof for Lemma 6: i)

$$\begin{aligned}
p_{1,s_A}^{\delta,e} - p_{1,n}^{\delta,e} &= \frac{\delta(q_{1,n}^e - q_{1,s_A}^e) + \beta_1(q_{2,n}^e - q_{2,s_A}^e)}{2\beta_1(\beta_1\beta_2 - 1)} + \frac{\delta(q_{1,n}^e - q_{1,s_A}^e)}{\beta_1}, \\
\implies 2\beta_1(2\beta_1\beta_2 - 1)(p_{1,s_A}^{\delta,e} - p_{1,n}^{\delta,e}) &= (2\beta_1\beta_2 - 1)\delta(q_{1,n}^e - q_{1,s_A}^{)} + \beta_1(q_{2,n}^e - q_{2,s_A}^e), \\
\text{Using } (q_{1,n}^e - q_{1,s_A}^e) &= \frac{2\beta_1\beta_2 - 1}{4\beta_2}\left(w_{1,s_A}^e - w_{1,n}^e\right)
\end{aligned}$$

and $(q_{2,n}^e - q_{2,s_A}^e) = -\dfrac{w_{1,s_A}^e - w_{1,n}^e}{4}$ and $(q_{2,n}^e - q_{2,s_A}^e)$,

$$\implies 8\beta_1\beta_2(2\beta_1\beta_2 - 1)(p_{1,s_A}^{\delta,e} - p_{1,n}^{\delta,e}) = \left(w_{1,s_A}^e - w_{1,n}^e\right)\left[\delta(2\beta_1\beta_2 - 1)^2 - 4\beta_1\beta_2\right].$$

Therefore, if $\delta > \dfrac{4\beta_1\beta_2}{(2\beta_1\beta_2 - 1)^2}$, then $p_{1,s_A}^{\delta,e} > p_{1,n}^{\delta,e}$, else $p_{1,s_A}^{\delta,e} < p_{1,n}^{\delta,e}$. □

ii) Firstly, we consider $p_{1,s_B}^{\delta,e} - p_{1,n}^{\delta,e}$:

$$\begin{aligned}
p_{1,s_B}^{\delta,e} - p_{1,n}^{\delta,e} &= \frac{\delta(q_{1,n}^e - q_{1,s_B}^e) + \beta_1(q_{2,n}^e - q_{2,s_B}^e)}{2\beta_1(\beta_1\beta_2 - 1)} + \frac{\delta(q_{1,n}^e - q_{1,s_B}^e)}{\beta_1}, \\
\implies 2\beta_1(2\beta_1\beta_2 - 1)(p_{1,s_B}^{\delta,e} - p_{1,n}^{\delta,e}) &= (2\beta_1\beta_2 - 1)\delta(q_{1,n}^e - q_{1,s_B}^e) + \beta_1(q_{2,n}^e - q_{2,s_B}^e), \\
\text{Using } (q_{1,n}^e - q_{1,s_B}^e) &= -\frac{1}{4}\left(w_{2,s_B}^e - w_{2,n}^e\right) \\
\text{and } (q_{2,n}^e - q_{2,s_B}^e) &= \frac{2\beta_1\beta_2 - 1}{4\beta_1}\left(w_{2,s_B}^e - w_{2,n}^e\right), \text{ we have:} \\
\implies 2\beta_1(2\beta_1\beta_2 - 1)(p_{1,s_B}^{\delta,e} - p_{1,n}^{\delta,e}) &= \left(w_{2,s_B}^e - w_{2,n}^e\right)\left[-\frac{(2\beta_1\beta_2 - 1)\delta}{4} + 2\beta_1\beta_2 - 1\right], \\
2\beta_1(p_{1,s_B}^{\delta,e} - p_{1,n}^{\delta,e}) &= \left(w_{2,s_B}^e - w_{2,n}^e\right)\left[1 - \frac{\delta}{4}\right] > 0.
\end{aligned}$$

Now, we consider $p_1^{\delta,e} - p_{1,s_B}^{\delta,e}$

$$p_1^{\delta,e} - p_{1,s_B}^{\delta,e} = \frac{\delta(q_{1,s_B}^e - q_{1,s_A}^e) + \beta_1(q_{2,s_B}^e)}{2\beta_1(\beta_1\beta_2 - 1)} + \frac{\delta q_{1,s_B}^e}{\beta_1} > 0, \text{ as } q_{1,s_B}^e > q_{1,s_A}^e.$$

Combining the above two inequalities, we have $p_1^{\delta,e} > p_{1,s_B}^{\delta,e} > p_{1,n}^{\delta,e}$. □

iii) From (11), we see that $p_2^*(w_E) = \dfrac{\alpha_1 + \alpha_2\beta_1}{2(\beta_1\beta_2 - 1)} + \dfrac{w_E}{2}$. Therefore, from Lemma 4 and Lemma 5, we immediately have the above relationship. □

Proof for Proposition 3: $D_{2,s_A}^{\delta,e} - D_{2,n}^{\delta,e} = \beta_2(p_{2,n}^{\delta,e} - p_{2,s_A}^{\delta,e}) + (p_{1,s_A}^{\delta,e} - p_{1,n}^{\delta,e})$. Consider:

$$\begin{aligned}
D_{2,s_A}^{\delta,e} - D_{2,n}^{\delta,e} &= \beta_2(p_{2,n}^{\delta,e} - p_{2,s_A}^{\delta,e}) + (p_{1,s_A}^{\delta,e} - p_{1,n}^{\delta,e}), \\
&= \frac{\beta_2}{2}(w_{E,n}^{\delta,e} - w_{E,s_A}^{\delta,e}) + \frac{1}{2\beta_1}\left((w_{E,s_A}^{\delta,e} - w_{E,n}^{\delta,e}) - 2\delta(q_{1,s_A}^{\delta,e} - q_{1,n}^{\delta,e})\right), \\
&= \frac{\beta_1\beta_2 - 1}{2\beta_1}(w_{E,n}^{\delta,e} - w_{E,s_A}^{\delta,e}) + \frac{\delta}{\beta_1}\left(q_{1,n}^{\delta,e} - q_{1,s_A}^{\delta,e}\right),
\end{aligned}$$

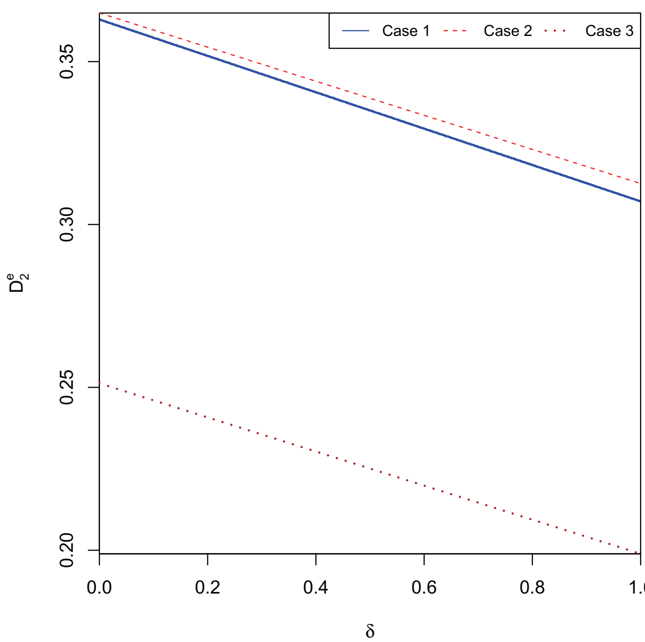
Using $w_{E,n}^{\delta,e} - w_{E,s_A}^{\delta,e} = -\dfrac{(w_{1,s_A}^e - w_{1,n}^e)(2\beta_1\beta_2 - 1)}{4\beta_2(\beta_1\beta_2 - 1)}\left[\delta - \dfrac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right]$, and $q_{1,n}^{\delta,e} - q_{1,s_A}^{\delta,e} = \dfrac{2\beta_1\beta_2 - 1}{4\beta_2}\left(w_{1,s_A}^e - w_{1,n}^e\right)$, we have:

$$\begin{aligned}
D^{\delta,e}_{2,s_A} - D^{\delta,e}_{2,n} &= -\frac{\beta_1\beta_2 - 1}{2\beta_1}\frac{(w^e_{1,s_A} - w^e_{1,n})(2\beta_1\beta_2 - 1)}{4\beta_2(\beta_1\beta_2 - 1)}\left[\delta - \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right] + \frac{\delta}{\beta_1}\frac{2\beta_1\beta_2 - 1}{4\beta_2}\left(w^e_{1,s_A} - w^e_{1,n}\right), \\
&= \frac{\left(w^e_{1,s_A} - w^e_{1,n}\right)(2\beta_1\beta_2 - 1)}{8\beta_1\beta_2}\left\{-\left[\delta - \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right] + 2\delta\right\}, \\
&= \frac{\left(w^e_{1,s_A} - w^e_{1,n}\right)(2\beta_1\beta_2 - 1)}{8\beta_1\beta_2}\left\{\delta + \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right\} > 0.
\end{aligned}$$

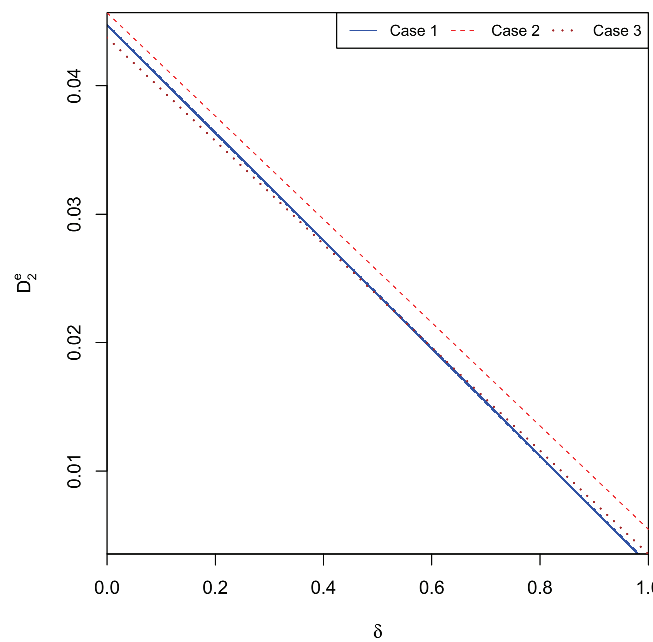
Similar to the above derivation, we have: $D^{\delta,e}_{2} - D^{\delta,e}_{2,n} = \frac{\beta_1\beta_2 - 1}{2\beta_1}(w^{\delta,e}_{E,n} - w^{\delta,e}_{E}) + \frac{\delta}{\beta_1}\left(q^{\delta,e}_{1,n} - q^{\delta,e}_{1,s_A}\right)$, $w^{\delta,e}_{E,n} - w^{\delta,e}_{E} = \frac{1}{\beta_1\beta_2 - 1}\left(\delta(q^e_{1,n} - q^e_{1,s_A}) - \beta_1 q^e_{2,n}\right)$, and $q^{\delta,e}_{1,n} - q^{\delta,e}_{1,s_A} = \frac{2\beta_1\beta_2 - 1}{4\beta_2}\left(w^e_{1,s_A} - w^e_{1,n}\right)$. Thus, we have

$$\begin{aligned}
D^{\delta,e}_{2} - D^{\delta,e}_{2,n} &= \frac{1}{2\beta_1}\left(\delta(q^e_{1,s_A} - q^e_{1,n}) - \beta_1 q^e_{2,n}\right) + \frac{\delta}{\beta_1}\left(q^{\delta,e}_{1,n} - q^{\delta,e}_{1,s_A}\right), \\
&= \frac{1}{2\beta_1}\left(\delta(q^e_{1,n} - q^e_{1,s_A}) - \beta_1 q^e_{2,n}\right). \qquad \text{(A1)}
\end{aligned}$$

RHS of (A1) increases in δ as $q^e_{1,n} > q^e_{1,s_A}$. At $\delta = 0$, RHS of (A1) is negative. Also, if $q^e_{2,n} \geq q^e_{1,n}$ RHS of (A1) is again negative and for the special case when the products are symmetric, i.e., $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, RHS of (A1) is again negative. For a relatively low order quantity $q^e_{2,n}$ (due to the small market size α_2 and/or high values of elasticity for product β_2), we can have a positive RHS of (A1). This is illustrated in Figure 9.



(a) $\alpha_2 = 1.0, \beta_2 = 2$



(b) $\alpha_2 = 0.5, \beta_2 = 3$

Figure 9: Variation of product 2 demand with δ for different values of market potential α_2 and product 2 price elasticity β_2.

Also, $D^{\delta,e}_{2,s_A} - D^{\delta,e}_{2} = \frac{\beta_1\beta_2 - 1}{2\beta_1}(w^{\delta,e}_{E} - w^{\delta,e}_{E,s_A}) > 0$ as $w^{\delta,e}_{E} > w^{\delta,e}_{E,s_A}$ from Lemma 4.

Next, consider $D_{2,s_A}^{\delta,e} - D_{2,s_B}^{\delta,e} = \frac{\beta_1\beta_2 - 1}{2\beta_1}(w_{E,s_B}^{\delta,e} - w_{E,s_A}^{\delta,e}) + \frac{\delta}{\beta_1}\left(q_{1,s_B}^{e} - q_{1,s_A}^{e}\right) > 0$, as $w_{E,s_B}^{\delta,e} > w_{E,s_A}^{\delta,e}$ from Lemma 4, and $q_{1,s_B}^{e} > q_{1,s_A}^{e}$ from Theorem 1.ii), respectively. Similarly, we have $D_{2,n}^{\delta,e} - D_{2,s_B}^{\delta,e} > 0$. Therefore, we have $D_{2,s_A}^{\delta,e} > D_{2,n}^{\delta,e} > D_{2,s_B}^{\delta,e}$.

From Lemma 5 and Corollary 1, the RHS in the above equality is positive if $\delta \in \left[0, \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right)$. Therefore, $D_{2,s_A}^{\delta,e} > D_{2,n}^{\delta,e}$ if $\delta \in \left[0, \frac{\beta_1\beta_2}{2\beta_1\beta_2 - 1}\right)$. Similarly, from Lemma 5 and Corollary 1, we obtain $D_{2}^{\delta,e} < D_{2,s_B}^{\delta,e} < D_{2,n}^{\delta,e}$. $\square$.