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Supplementary Information for

Indirect Reciprocity with Simple Records

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Fig. S1
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Supporting Information Text

Related Work

Work on equilibrium cooperation in repeated games began with studies of reciprocal altruism with general stage games where a fixed set of players interacts repeatedly with a commonly known start date and a common notion of calendar time (1–3), and has been expanded to allow for various sorts of noise and imperfect observability (4–8). In contrast, most evolutionary analyses of repeated games have focused on the prisoner's dilemma (9–23), though a few evolutionary analyses have considered more complex stage games (24, 25). Similarly, most laboratory and field studies of the effects of repeated interaction have also focused on the prisoner's dilemma (9, 26–28), though some papers consider variants with an additional third action (29, 30).

Reciprocal altruism is an important force in long-term relationships among a relatively small number players, such as business partnerships or collusive agreements among firms, but there are many social settings where people manage to cooperate even though direct reciprocation is impossible. These interactions are better modelled as games with repeated random matching (31). When the population is small compared to the discount factor, cooperation in the prisoner's dilemma can be enforced by contagion equilibria even when players have no information at all about each other's past actions (32–34). These equilibria do not exist when the population is large compared to the discount factor, so they are ruled out by our assumption of a continuum population.

Previous research on indirect reciprocity in large populations has studied the enforcement of cooperation as an equilibrium using first-order information. Takahashi (35) shows that cooperation can be supported as a strict equilibrium when the PD exhibits strategic complementarity; however, his model does not allow noise or the inflow of new players, and assumes players can use a commonly known calendar to coordinate their play. Heller and Mohlin (36) show that, under strategic complementarity, the presence of a small share of players who always defect allows cooperation to be sustained as a stable (though not necessarily strict) equilibrium when players are infinitely lived and infinitely patient and are restricted to using stationary strategies. The broader importance of strategic complementarity has long been recognized in economics (37, 38) and game theory (39, 40).

Many papers study the evolutionary selection of cooperation using image scoring (41–52). With image scoring, each player has first-order information about their partner, but conditions their action only on their partner's record and not on their own record. These strategies are never a strict equilibrium, and are typically unstable in environments with noise (47, 53). With more complex "higher order" record systems such as standing, cooperation can typically be enforced in a wide range of games (32, 44, 54–62). Most research has focused on the case where each player has only two states: for instance, Ohtsuko and Iwasa (44, 63) consider all possible record systems of this type, and show that only 8 of them allow an ESS with high levels of coooperation. Our first-order records can take on any integer values, so they do not fall into this class, even though behavior is determined by a binary classification of the records. Another innovation in our model is to consider steady-state equilibria in a model with a constant inflow of new players, even without any evolutionary dynamics. This approach has previously been used to model industry dynamics in economics (64, 65), but is novel in the context of models of cooperation and repeated games.

The key novel aspects of our framework may thus be summarized as follows:

1. Information ("records") depends only on a player's own past actions, but players condition their behavior on their own record as well as their current partner's record.
2. The presence of strategic complementarity implies that such two-sided conditioning can generate strict incentives for cooperation.
3. Records are integers, and can therefore remain "good" even if they are repeatedly hit by noise (as is inevitable when players are long-lived).
4. The presence of a constant inflow of new players implies that the population share with "good" records can remain positive even in steady state.

Model Description

Here we formally present the model and the steady-state and equilibrium concepts.

Time is discrete and doubly infinite: $t \in \{\ldots,-2,-1,0,1,2,\ldots\}$. There is a unit mass of individuals, each with survival probability $\gamma \in (0,1)$, and an inflow of $1-\gamma$ newborns each period to keep the population size constant.

Every period, individuals randomly match in pairs to play the PD (**Fig. 1**). Each individual carries a *record* $k \in \mathbb{N} := \{0,1,2,...\}$. Newborns have record 0. When two players meet, they observe each other's records and nothing else. A *strategy* is a mapping $\mathbf{s} : \mathbb{N}\times\mathbb{N} \to \{C,D\}$. All players use the same strategy. When the players use strategy $\mathbf{s}$, the distribution over next-period records of a player with record k who meets a player with record k' is given by

$$\phi_{k,k'}(\mathbf{s}) = \begin{cases} r_k(C) \text{ w/ prob. } 1-\varepsilon,\ r_k(D) \text{ w/ prob. } \varepsilon & \text{if } \mathbf{s}(k,k')=C \\ r_k(D) \text{ w/ prob. } 1 & \text{if } \mathbf{s}(k,k')=D endequation* \end{cases},$$

where $r_k(C)$ is the next-period record when a player with current record k is recorded as playing C and $r_k(D)$ is the next-period record when a player with current record k is recorded as playing D. For the Counting D's record system, $r_k(C)=k$ and $r_k(D)=k+1$ for all $k\in\mathbb{N}$. More generally, for each $k\in\mathbb{N}$, $r_k(C)$ and $r_k(D)$ can be arbitrary integers.

The *state* of the system $\mu \in \Delta(\mathbb{N})$ describes the share of the population with each record, where $\mu_k \in [0,1]$ denotes the share with record k. The evolution of the state over time under strategy $\mathbf{s}$ is described by the update map $f_{\mathbf{s}} : \Delta(\mathbb{N}) \to \Delta(\mathbb{N})$, given by

$$f_{\mathbf{s}}(\mu)[0] := 1 - \gamma + \gamma \sum_{k'} \sum_{k''} \mu_{k'} \mu_{k''} \phi_{k',k''}(\mathbf{s})[0],$$

$$f_{\mathbf{s}}(\mu)[k] := \gamma \sum_{k'} \sum_{k''} \mu_{k'} \mu_{k''} \phi_{k',k''}(\mathbf{s})[k] \text{ for } k \neq 0.$$

A *steady state* under strategy $\mathbf{s}$ is a state μ such that $f_{\mathbf{s}}(\mu) = \mu$.

Given a strategy $\mathbf{s}$ and state μ, the expected flow payoff of a player with record k is $\pi_k(\mathbf{s}, \mu) = \sum_{k'} \mu_{k'} u(\mathbf{s}(k,k'), \mathbf{s}(k',k))$, where u is the (normalized) PD payoff function given by

$$u(a_1, a_2) = \begin{cases} 1 & \text{if } (a_1,a_2) = (C,C) \\ -l & \text{if } (a_1,a_2) = (C,D) \\ 1+g & \text{if } (a_1,a_2) = (D,C) \\ 0 & \text{if } (a_1,a_2) = (D,D) \end{cases}.$$

Denote the probability that a player with current record k has record k' t periods in the future by $\phi_k(\mathbf{s},\mu)^t(k')$. The continuation payoff of a player with record k is then $V_k(\mathbf{s}, \mu) = (1-\gamma) \sum_{t=0}^{\infty} \gamma^t \sum_{k'} \phi_k(\mathbf{s},\mu)^t(k') \pi_{k'}(\mathbf{s},\mu)$. A player's objective is to maximize their expected lifetime payoff.

A pair $(\mathbf{s}, \mu)$ is an *equilibrium* if μ is a steady-state under $\mathbf{s}$ and, for each own record k and opponent's record k', $\mathbf{s}(k,k') \in \{C, D\}$ maximizes $(1-\gamma)u(a, \mathbf{s}(k',k)) + \gamma \sum_{k''} (\rho(k,a)[k'']) V_{k''}(\mathbf{s},\mu)$ over $a \in \{C, D\}$, where $\rho(k,a)[k'']$ denotes the probability that a player with record k who takes action a acquires next-period record k''. An equilibrium is *strict* if the maximizer is unique for all pairs (k, k').

This equilibrium definition encompasses two forms of strategic robustness. First, we allow agents to maximize over all possible strategies, as opposed to only strategies from some pre-selected set. Second, we focus on strict equilibria, which remain equilibria under "small" perturbations of the model.

Limit Cooperation under $GrimK$ Strategies

Under $GrimK$ strategies, a matched pair of players cooperate if and only if both records are below a pre-specified cutoff K: that is, $s(k,k') = C$ if $\max\{k,k'\} < K$ and $s(k,k') = D$ if $\max\{k,k'\} \geq K$.

We call an individual a *cooperator* if their record is below K and a *defector* otherwise. Note that each individual may be a cooperator for some periods of their life and a defector for other periods.

Given an equilibrium strategy $GrimK$, let $\mu^C = \sum_{k=0}^{K-1} \mu_k$ denote the corresponding steady-state share of cooperators. Note that, in a steady state with cooperator share μ^C, mutual cooperation is played in share $(\mu^C)^2$ of all matches. Let $\overline{\mu}^C(\gamma, \varepsilon)$ be the maximal share of cooperators in any $GrimK$ equilibrium (allowing for every possible K) when the survival probability is γ and the noise level is ε.

The following theorem characterizes the performance of equilibria in $GrimK$ strategies in the double limit of interest (33, 35, 44, 63, 66) where the survival probability approaches 1—so that players expect to live a long time and the "shadow of the future" looms large—and the noise level approaches 0—so that players who play C are unlikely to be recorded as playing D.

Theorem 1.

$$\lim_{(\gamma,\varepsilon)\to(1,0)} \overline{\mu}^C_{gr}(\gamma, \varepsilon) = \begin{cases} \frac{l}{1+l} & \text{if } g < \frac{l}{1+l} \\ 0 & \text{if } g > \frac{l}{1+l} \end{cases}.$$

To prove the theorem, let $\beta : (0,1) \times (0,1) \times (0,1) \to (0,1)$ be the function given by

$$\beta(\gamma, \varepsilon, \mu^C) = \frac{\gamma(1 - (1-\varepsilon)\mu^C)}{1 - \gamma(1-\varepsilon)\mu^C}. \quad [1]$$

When players use $GrimK$ strategies and the share of cooperators is μ^C, $\beta(\gamma, \varepsilon, \mu^C)$ is the probability that a player with cooperator record k survives to reach record $k+1$. (This probability is the same for all $k < K$.)

Lemma 2. *There is a $GrimK$ equilibrium with cooperator share μ^C if and only if the following conditions hold:*

1. *Feasibility:*

$$\mu^C = 1 - \beta(\gamma, \varepsilon, \mu^C)^K. \quad [2]$$

2. *Incentives:*

$$\frac{(1-\varepsilon)(1-\mu^C)}{1-(1-\varepsilon)\mu^C} \mu^C > g, \quad [3]$$

$$\mu^C < \frac{1}{\gamma(1-\varepsilon)} \frac{l}{1+l}. \quad [4]$$

Note that $\mu^C = 0$ solves [2] when $K = 0$. For any $K > 0$, $0 < 1 - \beta(\gamma, \varepsilon, \mu^C)^K$ and $1 > 1 - \beta(\gamma, \varepsilon, 1)^K$, so by the intermediate value theorem, [2] has some solution $\mu \in (0, 1)$. Thus, there is at least one steady state for every *GrimK* strategy. For some strategies, there are multiple steady states, but never more than $K + 1$, because [2] can be rewritten as a polynomial equation in μ^C with degree $K + 1$.

The upper bounds on the equilibrium share of cooperators in Figure 2 are the suprema of the $\mu^C \in (0, 1)$ that satisfy [3] and [4] for the corresponding (γ, ε) parameters. When no $\mu^C \in (0, 1)$ satisfy [3] and [4], the upper bound is 0, since *Grim*0 (where everyone plays D) is always a strict equilibrium.

To see how the $g > l/(1 + l)$ case of Theorem 1 comes from Lemma 2, note that

$$\frac{(1 - \varepsilon)(1 - \mu^C)}{1 - (1 - \varepsilon)\mu^C} \leq 1.$$

Thus, [3] requires $\mu^C > g$. Moreover, combining $\mu^C > g$ with [4] gives $\gamma(1 - \varepsilon)g < l/(1 + l)$. Taking the $(\gamma, \varepsilon) \to (1, 0)$ limit of this inequality gives $g \leq l/(1 + l)$. Thus, when $g > l/(1 + l)$, it follows that $\lim_{(\gamma,\varepsilon)\to(1,0)} \overline{\mu}^C(\gamma, \varepsilon) = 0$.

All that remains is to show that $\lim_{(\gamma,\varepsilon)\to(1,0)} \overline{\mu}^C(\gamma, \varepsilon) = l/(1 + l)$ when $g > l/(1 + l)$. Since $\lim_{\varepsilon\to 0}(1 - \varepsilon)(1 - \mu^C)/(1 - (1 - \varepsilon)\mu^C) = 1$ for any fixed μ^C and $\lim_{(\gamma,\varepsilon)\to(1,0)} 1/(\gamma(1 - \varepsilon)) = 1$, it follows that values of μ^C smaller than, but arbitrarily close to, $l/(1 + l)$ satisfy [3] and [4] in the double limit. Thus, the only difficulty is showing the feasibility of μ^C as a steady-state level of cooperation: because K must be an integer, some values of μ^C cannot be generated by any K, for given values of γ and ε. The following result shows that this "integer problem" becomes irrelevant in the limit. That is, any value of $\mu^C \in (0, 1)$ can be approximated arbitrarily closely by a feasible steady-state share of cooperators for some *GrimK* strategy as $(\gamma, \varepsilon) \to (1, 0)$.

Lemma 3. *Fix any $\mu^C \in (0, 1)$. For all $\Delta > 0$, there exist $\overline{\gamma} < 1$ and $\overline{\varepsilon} > 0$ such that, for all $\gamma > \overline{\gamma}$ and $\varepsilon < \overline{\varepsilon}$, there exists $\hat{\mu}^C$ that satisfies [2] for some K such that $|\hat{\mu}^C - \mu^C| < \Delta$.*

To complete the proof of Theorem 1, we now prove Lemmas 2 and 3.

Proof of Lemma 2. We first establish the feasibility condition of Lemma 2, and then we establish its incentives condition. The feasibility condition comes from the following lemma.

Lemma 4. *In a GrimK equilibrium with cooperator share μ^C, $\mu_k = \beta(\gamma, \varepsilon, \mu^C)^k(1 - \beta(\gamma, \varepsilon, \mu^C))$ for all $k < K$.*

To see why Lemma 4 implies the feasibility condition of Lemma 2, note that

$$\mu^C = \sum_{k=0}^{K-1} \beta(\gamma, \varepsilon, \mu^C)^k(1 - \beta(\gamma, \varepsilon, \mu^C)) = 1 - \beta(\gamma, \varepsilon, \mu^C)^K.$$

Proof of Lemma 4. The inflow into record 0 is $1 - \gamma$, while the outflow from record 0 is $(1 - \gamma(1 - \varepsilon)\mu^C)\mu_0$. Setting these equal gives

$$\mu_0 = \frac{1 - \gamma}{1 - \gamma(1 - \varepsilon)\mu^C} = 1 - \beta(\gamma, \varepsilon, \mu^C).$$

Additionally, for every $0 < k < K$, the inflow into record k is $\gamma(1 - (1 - \varepsilon)\mu^C)\mu_{k-1}$, while the outflow from record k is $(1 - \gamma(1 - \varepsilon)\mu^C)\mu_k$. Setting these equal gives

$$\mu_k = \frac{\gamma(1 - (1 - \varepsilon)\mu^C)}{1 - \gamma(1 - \varepsilon)\mu^C}\mu_{k-1} = \beta(\gamma, \varepsilon, \mu^C)\mu_{k-1}.$$

Combining this with $\mu_0 = 1 - \beta(\gamma, \varepsilon, \mu^C)$ gives $\mu_k = \beta(\gamma, \varepsilon, \mu^C)^k(1 - \beta(\gamma, \varepsilon, \mu^C))$ for $0 \leq k \leq K - 1$. ∎

We now establish the incentive condition of Lemma 2. We will see that the incentive constraint [3] guarantees that a record-0 cooperator plays C against an opponent playing C, and the incentive constraint [4] guarantees that a record-$(K - 1)$ cooperator plays D against an opponent playing D. Record-0 cooperators are the cooperators most tempted to defect against a cooperative opponent and record-$(K - 1)$ cooperators are the cooperators most tempted to cooperate against a defecting opponent, so these constraints guarantee the incentives of all cooperators are satisfied.

Formally, to establish the incentive condition, we rely on the following lemma.

Lemma 5. *In a GrimK equilibrium with cooperator share μ^C,*

$$V_k = \begin{cases} (1 - \beta(\gamma, \varepsilon, \mu^C)^{K-k})\mu^C & \text{if } k < K \\ 0 & \text{if } k \geq K. \end{cases}$$

To derive the incentive condition of Lemma 2 from Lemma 5, note that the expected continuation payoff of a record-0 player from playing C is $(1-\varepsilon)V_0+\varepsilon V_1$, while the expected continuation payoff from playing D is V_1. Thus, a record 0 player strictly prefers to play C against an opponent playing C iff $(1-\varepsilon)\gamma(V_0-V_1)/(1-\gamma)>g$. Combining Lemmas 4 and 5 gives

$$(1-\varepsilon)\frac{\gamma}{1-\gamma}(V_0-V_1)=\frac{1-\varepsilon}{1-(1-\varepsilon)\mu^C}\beta(\gamma,\varepsilon,\mu^C)^K\mu^C=\frac{(1-\varepsilon)(1-\mu^C)}{1-(1-\varepsilon)\mu^C}\mu^C,$$

so [3] follows. Moreover, the expected continuation payoff of a record $K-1$ player from playing C is $(1-\varepsilon)V_{K-1}+\varepsilon V_K$, while the expected continuation payoff from playing D is V_K. Thus, a record $K-1$ player strictly prefers to play D against an opponent playing D iff $(1-\varepsilon)\gamma(V_{K-1}-V_K)/(1-\gamma)<l$. Lemma 5 gives

$$(1-\varepsilon)\frac{\gamma}{1-\gamma}(V_{K-1}-V_K)=\frac{\gamma(1-\varepsilon)\mu^C}{1-\gamma(1-\varepsilon)\mu^C},$$

and setting this to be less than l gives [4].

Proof of Lemma 5. The flow payoff for any record $k\geq K$ is 0, so $V_k=0$ for $k\geq K$. For $k<K$, $V_k=(1-\gamma)\mu^C+\gamma(1-\varepsilon)\mu^C V_k+\gamma(1-(1-\varepsilon)\mu^C)V_{k+1}$, which gives $V_k=(1-\beta(\gamma,\varepsilon,\mu^C))\mu^C+\beta(\gamma,\varepsilon,\mu^C)V_{k+1}$. Combining this with $V_K=0$ gives $V_k=(1-\beta(\gamma,\varepsilon,\mu^C)^{K-k})\mu^C$ for $k<K$. ■

Proof of Lemma 3. The proof first establishes some properties of two functions, $\tilde{K}$ and d, which we now introduce.

Let $\tilde{K}:(0,1)\times(0,1)\times(0,1)\to\mathbb{R}_+$ be the function given by

$$\tilde{K}(\gamma,\varepsilon,\mu^C)=\frac{\ln(1-\mu^C)}{\ln(\beta(\gamma,\varepsilon,\mu^C))}. \quad [5]$$

By construction, $\tilde{K}(\gamma,\varepsilon,\mu^C)$ is the unique $K\in\mathbb{R}_+$ such that $\mu^C=1-\beta(\gamma,\varepsilon,\mu^C)^K$. Let $d:(0,1]\times[0,1)\times(0,1)\to\mathbb{R}$ be the function given by

$$d(\gamma,\varepsilon,\mu^C)=\begin{cases}1+\ln(1-\mu^C)(1-\mu^C)\frac{\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)}{\beta(\gamma,\varepsilon,\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))} & \text{if } \gamma<1\\ 1+\frac{(1-\varepsilon)\ln(1-\mu^C)(1-\mu^C)}{1-(1-\varepsilon)\mu^C} & \text{if } \gamma=1\end{cases}.$$

The μ^C derivative of $\tilde{K}(\gamma,\varepsilon,\mu^C)$ is related to $d(\gamma,\varepsilon,\mu^C)$ by the following lemma.

Lemma 6. $\tilde{K}:(0,1)\times(0,1)\times(0,1)\to\mathbb{R}_+$ *is differentiable in* μ^C *with derivative given by*

$$\frac{\partial\tilde{K}}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)=-\frac{d(\gamma,\varepsilon,\mu^C)}{(1-\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}.$$

Proof of Lemma 6. From [5], it follows that $\tilde{K}(\gamma,\varepsilon,\mu^C)$ is differentiable in μ^C with derivative given by

$$\begin{aligned}\frac{\partial\tilde{K}}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)&=-\frac{\frac{\ln(\beta(\gamma,\varepsilon,\mu^C))}{1-\mu^C}+\frac{\ln(1-\mu^C)\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)}{\beta(\gamma,\varepsilon,\mu^C)}}{\ln(\beta(\gamma,\varepsilon,\mu^C))^2}\\&=-\frac{1+\ln(1-\mu^C)(1-\mu^C)\frac{\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)}{\beta(\gamma,\varepsilon,\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}}{(1-\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}\\&=-\frac{d(\gamma,\varepsilon,\mu^C)}{(1-\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}.\end{aligned}$$ ■

The following two lemmas concern properties of $d(\gamma,\varepsilon,\mu^C)$ that will be useful for the proof of Lemma 3.

Lemma 7. $d:(0,1]\times[0,1)\times(0,1)\to\mathbb{R}$ *is well-defined and continuous.*

Proof of Lemma 7. Since $\beta(\gamma,\varepsilon,\mu^C)$ is differentiable and only takes values in $(0,1)$, it follows that $d(\gamma,\varepsilon,\mu^C)$ is well-defined. Moreover, since $\beta(\gamma,\varepsilon,\mu^C)$ is continuously differentiable for all $\mu^C\in(0,1)$, $d(\gamma,\varepsilon,\mu^C)$ is continuous for $\gamma<1$. All that remains is to check that $d(\gamma,\varepsilon,\mu^C)$ is continuous for $\gamma=1$.

First, note that $d(1,\varepsilon,\mu^C)$ is continuous in (ε,μ^C). Thus, we need only check the limit in which γ approaches 1, but never equals 1. Note that

$$\begin{aligned}\frac{\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)}{\beta(\gamma,\varepsilon,\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}&=-\frac{\frac{\gamma(1-\varepsilon)(1-\gamma)}{(1-\gamma(1-\varepsilon)\mu^C)^2}}{\beta(\gamma,\varepsilon,\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}\\&=-\Big(\frac{\gamma(1-\varepsilon)}{\beta(\gamma,\varepsilon,\mu^C)(1-\gamma(1-\varepsilon)\mu^C)}\Big)\Big(\frac{1-\beta(\gamma,\varepsilon,\mu^C)}{\ln(\beta(\gamma,\varepsilon,\mu^C))}\Big).\end{aligned} \quad [6]$$

It is clear that

$$\lim_{\substack{(\tilde{\gamma},\tilde{\varepsilon},\tilde{\mu}^C)\to(1,\varepsilon,\mu^C)\\ \tilde{\gamma}\neq 1}} \frac{\tilde{\gamma}(1-\tilde{\varepsilon})}{\beta(\tilde{\gamma},\tilde{\varepsilon},\tilde{\mu}^C)(1-\tilde{\gamma}(1-\tilde{\varepsilon})\tilde{\mu}^C)} = \frac{1-\varepsilon}{(1-(1-\varepsilon)\mu^C)} \tag{7}$$

for all $(\varepsilon,\mu^C)\in[0,1)\times(0,1)$. For γ close to 1,

$$\ln(\beta(\gamma,\varepsilon,\mu^C)) = \beta(\gamma,\varepsilon,\mu^C) - 1 + O((\beta(\gamma,\varepsilon,\mu^C)-1)^2).$$

Thus,

$$\lim_{\substack{(\tilde{\gamma},\tilde{\varepsilon},\tilde{\mu})\to(1,\varepsilon,\mu^C)\\ \tilde{\gamma}\neq 1}} \frac{1-\beta(\tilde{\gamma},\tilde{\varepsilon},\tilde{\mu}^C)}{\ln(\beta(\tilde{\gamma},\tilde{\varepsilon},\tilde{\mu}^C))} = -1 \tag{8}$$

for all $(\varepsilon,\mu^C)\in[0,1)\times(0,1)$. Equations 6, 7, and 8 together imply that $d(\gamma,\varepsilon,\mu^C)$ is continuous for $\gamma=1$. ■

Lemma 8. *$d(1,0,\mu^C)$ has precisely one zero in $\mu^C\in(0,1)$, and the zero is located at $\mu^C = 1-1/e$.*

Proof of Lemma 8. This follows from the fact that $d(1,0,\mu^C) = 1+\ln(1-\mu^C)$. ■

With these preliminaries established, we now present the proof of Lemma 3.

Completing the Proof of Lemma 3. Fix some $\tilde{\mu}^C\in(0,1)$ such that $\tilde{\mu}^C\neq 1-1/e$. Lemma 8 says $d(1,0,\tilde{\mu}^C)\neq 0$. Because of this and the continuity of d, there exist some $\lambda>0$ and some $\delta>0$, $\overline{\gamma}'<1$, and $\overline{\varepsilon}>0$ such that $|d(\gamma,\varepsilon,\mu^C)|>\lambda$ for all $\gamma>\overline{\gamma}'$, $\varepsilon<\overline{\varepsilon}$, and $|\mu^C-\tilde{\mu}^C|<\delta$.

Additionally, note that $\lim_{\gamma\to 1}\inf_{(\varepsilon,\mu^C)\in(0,\overline{\varepsilon})\times(\mu^C-\delta,\mu^C+\delta)}\beta(\gamma,\varepsilon,\mu^C)=1$. Together these facts imply that there exists some $\overline{\gamma}<1$ such that

$$\left|\frac{\partial\tilde{K}}{\partial\mu^C}(\gamma,\varepsilon,\mu^C)\right| = \left|\frac{d(\gamma,\varepsilon,\mu^C)}{(1-\mu^C)\ln(\beta(\gamma,\varepsilon,\mu^C))}\right| > \frac{2}{\min\{\delta,\Delta\}}$$

and $\tilde{K}(\gamma,\varepsilon,\mu^C)\geq 1$ for all $\gamma>\overline{\gamma}$, $\varepsilon<\overline{\varepsilon}$, and $|\mu^C-\tilde{\mu}^C|<\delta$. It thus follows that

$$\sup_{|\mu^C-\tilde{\mu}^C|\leq\min\{\delta,\Delta\}} |\tilde{K}(\gamma,\varepsilon,\mu^C)-\tilde{K}(\gamma,\varepsilon,\tilde{\mu}^C)|>1$$

for all $\gamma>\overline{\gamma}$, $\varepsilon<\overline{\varepsilon}$. Hence, there exists some $\hat{\mu}^C$ within Δ of $\tilde{\mu}^C$ and some non-negative integer $\hat{K}$ such that $\tilde{K}(\gamma,\varepsilon,\hat{\mu}^C)=\hat{K}$, which implies that $\hat{\mu}^C$ is feasible since $\hat{\mu}^C = 1-\beta(\gamma,\varepsilon,\hat{\mu}^C)^{\hat{K}}$. ■

Limit Cooperation under Trigger Strategies

We characterize the maximum level of cooperation that the class of *trigger strategies* can achieve in the $(\gamma,\varepsilon)\to(1,0)$ limit. Recall that this is the class of strategies that satisfy the following properties: (i) The set of all possible records can be partitioned into two classes, "good records" G and "bad records" B. (ii) Partners cooperate if and only if they both have good records: $s(k,k')=C$ for all pairs $(k,k')\in G\times G$, and $s(k,k')=D$ for all other pairs (k,k'). (iii) The class B is absorbing: if $k\in B$, then every record k' that can be reached starting at record k is also in B. As with *GrimK*, let $\mu^C=\sum_{k\in G}\mu_k$ denote the steady-state share of cooperators in a trigger strategy equilibrium, and let $\overline{\overline{\mu}}^C(\gamma,\varepsilon)$ be the maximal share of cooperators in any trigger strategy equilibrium when the survival probability is γ and the noise level is ε.

Theorem 9.

$$\lim_{(\gamma,\varepsilon)\to(1,0)}\overline{\overline{\mu}}^C(\gamma,\varepsilon) = \begin{cases}\frac{l}{1+l} & \text{if } g<\frac{l}{1+l}\\ 0 & \text{if } g>\frac{l}{1+l}\end{cases}.$$

This result shows that the maximum level of cooperation in the double limit achieved by strategies in the *GrimK* class equals that of the broader trigger strategy class. Since every *GrimK* strategy is a trigger strategy, the maximum level of cooperation achieved by trigger strategies weakly exceeds the maximum level achieved by *GrimK* strategies. Thus, it suffices to show that $\limsup_{(\gamma,\varepsilon)\to(1,0)}\overline{\overline{\mu}}^C(\gamma,\varepsilon)\leq l/(1+l)$ when $g<l/(1+l)$ and $\limsup_{(\gamma,\varepsilon)\to(1,0)}\overline{\overline{\mu}}^C(\gamma,\varepsilon)=0$ when $g>l/(1+l)$. This is a consequence of the following two lemmas.

Lemma 10. *In any trigger strategy equilibrium, $\gamma(1-\varepsilon)\mu^C<l/(1+l)$.*

Lemma 11. *In any trigger strategy equilibrium, $\mu^C>g$.*

To see that Theorem 9 follows from Lemmas 10 and 11, note that $\gamma(1-\varepsilon)\mu^C<l/(1+l)$ implies that $\mu^C\leq l/(1+l)$ in the $(\gamma,\varepsilon)\to(1,0)$ limit. Thus, $\limsup_{(\gamma,\varepsilon)\to(1,0)}\overline{\overline{\mu}}^C(\gamma,\varepsilon)\leq l/(1+l)$. Moreover, combining $\mu^C\leq l/(1+l)$ with $\mu^C>g$ implies that $\limsup_{(\gamma,\varepsilon)\to(1,0)}\overline{\overline{\mu}}^C(\gamma,\varepsilon)=0$ when $g>l/(1+l)$.

We now present the proofs of Lemma 10 and 11.

Proof of Lemma 10. Let k be a cooperator record. It must be that if a player with current record k is recorded as playing C, their next period record would also be a cooperator record. Otherwise, the player with record k would be better-off always playing D.

We can use this to obtain a lower bound on the value functions at cooperator records. Let $\underline{V}^G := \inf_{k \in G} V_k$ be the infimum of the value functions at cooperator records. We will show that

$$\underline{V}^G \geq \frac{1-\gamma}{1-\gamma(1-\varepsilon)}(\mu^C(1+l)-l). \tag{9}$$

The reason for this is that it must be suboptimal for a player with a cooperator record k to play C against D, so V_k must satisfy

$$\begin{aligned} V_k &> (1-\gamma)(\mu^C(1+l)-l) + \gamma(1-\varepsilon)V_{r_k(C)} + \gamma\varepsilon V_{r_k(D)}, \\ &> (1-\gamma)(\mu^C(1+l)-l) + \gamma(1-\varepsilon)V_{r_k(C)}, \end{aligned}$$

where the second inequality follows from the fact that $V_{k'} \geq 0$ for all $k' \in \mathbb{N}$, which implies $V_{r_k(D)} \geq 0$. Thus,

$$V_k > (1-\gamma)(\mu^C(1+l)-l) + \gamma(1-\varepsilon)\underline{V}^G$$

for all cooperator records $r \in G$, which likewise implies

$$\underline{V}^G \geq (1-\gamma)(\mu^C(1+l)-l) + \gamma(1-\varepsilon)\underline{V}^G.$$

Solving this for $\underline{V}^G$ gives

$$\underline{V}^G \geq \frac{1-\gamma}{1-\gamma(1-\varepsilon)}(\mu^C(1+l)-l),$$

so we conclude that the expression in [9] does indeed give a lower bound for $\underline{V}^G$.

Let k' be a cooperator record at which a player will transition to defector status if they are recorded as playing D. There must be such a record in any equilibrium with cooperation, as otherwise every player would always play D. A necessary condition for record k' players to prefer to rather play D rather than C against D is

$$-(1-\gamma)l + \gamma(1-\varepsilon)V_{r_{k'}(C)} < 0.$$

Since $V_{r_{k'}(C)} \geq \underline{V}^G$, it follows that

$$-(1-\gamma)l + \gamma(1-\varepsilon)\underline{V}^G < 0,$$

which by [9] implies

$$-(1-\gamma)l + \gamma(1-\varepsilon)\frac{1-\gamma}{1-\gamma(1-\varepsilon)}(\mu^C(1+l)-l) < 0.$$

Solving this inequality gives

$$\gamma(1-\varepsilon)\mu^C < \frac{\ell}{1+\ell}.$$

■

Proof of Lemma 11. For any cooperator record k, we have

$$V_k = (1-\gamma)\mu^C + \gamma(1-\varepsilon)\mu^C V_{r_k(C)} + \gamma(1-(1-\varepsilon)\mu^C)V_{r_k(D)}. \tag{10}$$

The condition for a record k preciprocator to prefer playing C rather than D against C is

$$(1-\varepsilon)\gamma(V_{r_k(C)} - V_{r_k(D)}) > (1-\gamma)g. \tag{11}$$

Combining [10] and [11] gives

$$\frac{1-\varepsilon}{1-(1-\varepsilon)\mu^C}\left(\mu^C - V_r - \frac{\gamma}{1-\gamma}(V_k - V_{r_k(C)})\right) > g. \tag{12}$$

Let $\overline{V}^G := \sup_{k \in G} V_k$ be the supremum of the value functions at cooperator records. Since [12] holds for all cooperator records $k \in G$ and $V_{r_k(C)} \leq \overline{V}^G$, we have

$$\frac{1-\varepsilon}{1-(1-\varepsilon)\mu^C}\left(\mu^C - \overline{V}^G\right) \geq g. \tag{13}$$

The expected lifetime payoff of a newborn player is $V_0 = \left(\mu^C\right)^2$, so $\overline{V}^G \geq (\mu^C)^2$. Combining this with [13] gives

$$\frac{(1-\varepsilon)(1-\mu^C)}{1-(1-\varepsilon)\mu^C}\mu^C \geq g,$$

which implies $\mu^C > g$, since $(1-\varepsilon)(1-\mu^C)/(1-(1-\varepsilon)\mu^C) < 1$. ■

Convergence of $GrimK$ Strategies

We now derive a key stability property of $GrimK$ strategies. Fix an arbitrary initial record distribution $\mu^0 \in \Delta(\mathbb{N})$. When all individuals use $GrimK$ strategies, the population share with record k at time t, μ_k^t, evolves according to

$$\begin{aligned} \mu_0^{t+1} &= 1-\gamma+\gamma(1-\varepsilon)\mu^{C,t}\mu_0^t, \\ \mu_k^{t+1} &= \gamma(1-(1-\varepsilon)\mu^{C,t})\mu_{k-1}^t+\gamma(1-\varepsilon)\mu^{C,t}\mu_k^t \text{ for } 0<k<K, \end{aligned} \qquad [14]$$

where $\mu^{C,t} = \sum_{k=0}^{K-1} \mu_k^t$.

Fixing K, we say that distribution μ *dominates* (or is *more favorable than*) distribution $\tilde{\mu}$ if, for every $k<K$, $\sum_{\tilde{k}=0}^{k} \mu_{\tilde{k}} \geq \sum_{\tilde{k}=0}^{k} \tilde{\mu}_{\tilde{k}}$; that is, if for every $k<K$ the share of the population with record no worse than k is greater under distribution μ than under distribution $\tilde{\mu}$. Under the $GrimK$ strategy, let $\bar{\mu}$ denote the steady state with the largest share of cooperators, and let $\underline{\mu}$ denote the steady state with the smallest share of cooperators.

Theorem 12.

1. *If μ^0 dominates $\bar{\mu}$, then $\lim_{t\to\infty}\mu^t = \bar{\mu}$.*

2. *If μ^0 is dominated by $\underline{\mu}$, then $\lim_{t\to\infty}\mu^t = \underline{\mu}$.*

Let $x_k = \sum_{\tilde{k}=0}^{k} \mu_{\tilde{k}}$ denote the share of the population with record no worse than k. From Equation 14, it follows that

$$\begin{aligned} x_0^{t+1} &= 1-\gamma+\gamma(1-\varepsilon)x_{K-1}^t x_0^t, \\ x_k^{t+1} &= 1-\gamma+\gamma x_{k-1}^t+\gamma(1-\varepsilon)x_{K-1}^t(x_k^t-x_{k-1}^t) \text{ for } 0<k<K. \end{aligned} \qquad [15]$$

To see this, note that $x_0 = \mu_0$ and $x_{K-1} = \mu^C$, so rewriting the first line in Equation 14 gives the first line in Equation 15. Additionally, for $0<k<K$, Equation 14 gives

$$\begin{aligned} x_k^{t+1} = \sum_{\tilde{k}\leq k}\mu_{\tilde{k}}^{t+1} &= 1-\gamma+\gamma\sum_{\tilde{k}\leq k-1}\mu_{\tilde{k}-1}^t+\gamma(1-\varepsilon)\mu^{C,t}\mu_k^t, \\ &= 1-\gamma+\gamma x_{k-1}^t+\gamma(1-\varepsilon)x_{K-1}^t(x_k^t-x_{k-1}^t). \end{aligned}$$

Lemma 13. *The update map in Equation 15 is weakly increasing: If $(x_0^t,...,x_{K-1}^t) \geq (\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$, then $(x_0^{t+1},...,x_{K-1}^{t+1}) \geq (\tilde{x}_0^{t+1},...,\tilde{x}_{K-1}^{t+1})$.*

Proof of Lemma 13. The right-hand side of the first line in Equation 15 depends only on the product of x_0^t and x_{K-1}^t, and it is strictly increasing in this product. The right-hand side of the second line in Equation 15 depends only on x_{k-1}^t, x_k^t, and x_{K-1}^t, and, holding fixed any two of these variables, it is weakly increasing in the third variable. ■

Proof of Theorem 12. We prove the first statement of Theorem 12. A similar argument handles the second statement. Let $(\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$ denote the time-path corresponding to the highest possible initial conditions, i.e. $(\tilde{x}_0^0,...,\tilde{x}_{K-1}^0) = (1,...,1)$. By Lemma 13, $(\tilde{x}_0^{t+1},...,\tilde{x}_{K-1}^{t+1}) \leq (\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$ for all t. Thus, it follows that $\lim_{t\to\infty}(\tilde{x}_0^t,...,\tilde{x}_{K-1}^t) = \inf_t(\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$, so in particular $\lim_{t\to\infty}(\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$ exists. Since the update rules in Equation 15 are continuous, it follows that $\lim_{t\to\infty}(\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$ must be a steady state of the system. By Lemma 13 and the fact that $(\overline{x}_0,...,\overline{x}_{K-1})$ is the steady state with the highest share of cooperators, it follows that $\lim_{t\to\infty}(\tilde{x}_0^t,...,\tilde{x}_{K-1}^t) = (\overline{x}_0,...,\overline{x}_{K-1})$.

Now, fix some $(x_0^0,...,x_{K-1}^0) \geq (\overline{x}_0,...,\overline{x}_{K-1})$. By Lemma 13,

$$(\overline{x}_0,...,\overline{x}_{K-1}) \leq (x_0^t,...,x_{K-1}^t) \leq (\tilde{x}_0^t,...,\tilde{x}_{K-1}^t)$$

for all t, so it follows that $\lim_{t\to\infty}(x_0^t,...,x_{K-1}^t) = (\overline{x}_0,...,\overline{x}_{K-1})$. ■

Evolutionary Analysis

We have so far analyzed the efficiency of $GrimK$ equilibrium steady states (Theorem 1) and convergence to such steady states when all players use the $GrimK$ strategy (Theorem 12). To further examine the robustness of $GrimK$ strategies, we now perform two types of evolutionary analysis. In the next subsection, we show that, when $g < l/(1+l)$, there are sequences of $GrimK$ equilibria that obtain the maximum cooperator share of $l/(1+l)$ as $(\gamma,\varepsilon)\to(1,0)$ that are robust to invasion by a small mass of mutants who follow any other $GrimK'$ strategy, such as *Always Defect* (i.e., $Grim0$). In the following subsection, we report simulations of the evolutionary dynamic when a $GrimK$ steady state is invaded by mutants playing another $GrimK'$ strategy.

Steady-State Robustness. We consider the following notion of steady-state robustness.

Definition 1. *A GrimK equilibrium with share of cooperators μ^C is **steady-state robust to mutants** if, for every $K' \neq K$ and $\alpha > 0$, there exists some $\overline{\delta} > 0$ such that when the share of players playing GrimK is $1-\delta$ and the share of players playing GrimK$'$ is δ with $\delta < \overline{\delta}$, then*

- *There is a steady state where the fraction of players playing GrimK that are cooperators, $\tilde{\mu}^C$, satisfies $|\tilde{\mu}^C - \mu^C| < \alpha$, and*
- *It is strictly optimal to play GrimK.*

We show that, whenever strategic complementarities are strong enough to support a cooperative *GrimK* equilibrium, there is a sequence of *GrimK* equilibria that are robust to mutants and attains the maximum cooperation level of $l/(1+l)$ when expected lifespans are long and noise is small.

Theorem 14. *Suppose that $g < l/(1+l)$. There is a family of GrimK equilibria giving a share of cooperators $\mu^C(\gamma,\varepsilon)$ for parameters γ, ε such that:*

1. *$\lim_{(\gamma,\varepsilon)\to(1,0)} \mu^C(\gamma,\varepsilon) = l/(1+l)$, and*
2. *There is some $\overline{\gamma} < 1$ and $\overline{\varepsilon} > 0$ such that, when $\gamma > \overline{\gamma}$ and $\varepsilon < \overline{\varepsilon}$, the GrimK equilibrium with share of cooperators $\overline{\mu}^C(\gamma,\varepsilon)$ is steady-state robust to mutants.*

Proof. We assume that $K' < K$; the proof for $K' > K$ is analogous. Fix some $g < \tilde{\mu}^C < l/(1+l)$ satisfying $\tilde{\mu}^C \neq 1 - 1/e$. By Lemmas 2 and 3, we know that there exists some family of *GrimK* equilibria with share of cooperators $\tilde{\mu}^C(\gamma,\varepsilon)$ such that $\lim_{(\gamma,\varepsilon)\to(1,0)} \tilde{\mu}^C(\gamma,\varepsilon) = \tilde{\mu}^C$. Fix some γ, ε, and consider the modified environment where share $1-\delta$ of the players use the *GrimK* strategy corresponding to $\tilde{\mu}^C(\gamma,\varepsilon)$ and share δ of the players use some other *GrimK$'$*.

Let μ_K^K denote the share of the players playing *GrimK* that have record less than K, let $\mu_{K'}^K$ be the share of *GrimK* players with record less than K', and let $\mu_{K'}^{K'}$ be the share of the players playing *GrimK$'$* that have record less than K'. Then in an steady state we have

$$\begin{aligned}
\mu_K^K &= 1 - \beta(\gamma,\varepsilon,(1-\delta)\mu_K^K + \delta\mu_K^{K'})^K,\\
\mu_{K'}^K &= 1 - \beta(\gamma,\varepsilon,(1-\delta)\mu_K^K + \delta\mu_K^{K'})^{K'},\\
\mu_K^{K'} &= 1 - \gamma^{K-K'}\beta(\gamma,\varepsilon,(1-\delta)\mu_{K'}^K + \delta\mu_{K'}^{K'})^{K'},\\
\mu_{K'}^{K'} &= 1 - \beta(\gamma,\varepsilon,(1-\delta)\mu_{K'}^K + \delta\mu_{K'}^{K'})^{K'}.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
f_K^K(\gamma,\varepsilon,\mu_K^K,\mu_{K'}^K,\mu_K^{K'},\mu_{K'}^{K'}) &:= \mu_K^K + \beta(\gamma,\varepsilon,(1-\delta)\mu_K^K + \delta\mu_K^{K'})^K - 1 = 0,\\
f_{K'}^K(\gamma,\varepsilon,\mu_K^K,\mu_{K'}^K,\mu_K^{K'},\mu_{K'}^{K'}) &:= \mu_{K'}^K + \beta(\gamma,\varepsilon,(1-\delta)\mu_K^K + \delta\mu_K^{K'})^{K'} - 1 = 0,\\
f_K^{K'}(\gamma,\varepsilon,\mu_K^K,\mu_{K'}^K,\mu_K^{K'},\mu_{K'}^{K'}) &:= \mu_K^{K'} + \gamma^{K-K'}\beta(\gamma,\varepsilon,(1-\delta)\mu_{K'}^K + \delta\mu_{K'}^{K'})^{K'} - 1 = 0,\\
f_{K'}^{K'}(\gamma,\varepsilon,\mu_K^K,\mu_{K'}^K,\mu_K^{K'},\mu_{K'}^{K'}) &:= \mu_{K'}^{K'} + \beta(\gamma,\varepsilon,(1-\delta)\mu_{K'}^K + \delta\mu_{K'}^{K'})^{K'} - 1 = 0.
\end{aligned} \tag{16}$$

Note that $\mu_K^K = \tilde{\mu}^C(\gamma,\varepsilon)$, $\mu_{K'}^K = 1 - \beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))^{K'}$, $\mu_K^{K'} = 1 - \gamma^{K-K'}\beta(\gamma,\varepsilon,1-\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))^{K'})^{K'}$, $\mu_{K'}^{K'} = 1 - \beta(\gamma,\varepsilon,1-\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))^{K'})^{K'}$ solves [16] when $\delta = 0$. The partial derivatives of the left-hand side of [16] evaluated at this point are given by

$$\begin{aligned}
&\begin{bmatrix}
\frac{\partial f_K^K}{\partial \mu_K^K} & \frac{\partial f_K^K}{\partial \mu_{K'}^K} & \frac{\partial f_K^K}{\partial \mu_K^{K'}} & \frac{\partial f_K^K}{\partial \mu_{K'}^{K'}}\\
\frac{\partial f_{K'}^K}{\partial \mu_K^K} & \frac{\partial f_{K'}^K}{\partial \mu_{K'}^K} & \frac{\partial f_{K'}^K}{\partial \mu_K^{K'}} & \frac{\partial f_{K'}^K}{\partial \mu_{K'}^{K'}}\\
\frac{\partial f_K^{K'}}{\partial \mu_K^K} & \frac{\partial f_K^{K'}}{\partial \mu_{K'}^K} & \frac{\partial f_K^{K'}}{\partial \mu_K^{K'}} & \frac{\partial f_K^{K'}}{\partial \mu_{K'}^{K'}}\\
\frac{\partial f_{K'}^{K'}}{\partial \mu_K^K} & \frac{\partial f_{K'}^{K'}}{\partial \mu_{K'}^K} & \frac{\partial f_{K'}^{K'}}{\partial \mu_K^{K'}} & \frac{\partial f_{K'}^{K'}}{\partial \mu_{K'}^{K'}}
\end{bmatrix}\\
=&\begin{bmatrix}
1 + K\beta^{K-1}\frac{\partial\beta}{\partial\mu^C} & 0 & 0 & 0\\
K'\beta^{K'-1}\frac{\partial\beta}{\partial\mu^C} & 1 & 0 & 0\\
0 & \gamma^{K-K'}K'\beta^{K'-1}\frac{\partial\beta}{\partial\mu^C} & 1 & 0\\
0 & K'\beta^{K'-1}\frac{\partial\beta}{\partial\mu^C} & 0 & 1
\end{bmatrix}.
\end{aligned} \tag{17}$$

Because $\tilde{\mu}^C(\gamma,\varepsilon) = 1 - \beta(\gamma,\varepsilon,\mu^C(\gamma,\varepsilon))^K$ and $K = \ln(1-\tilde{\mu}^C(\gamma,\varepsilon))/\ln(\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)))$,

$$\begin{aligned} &1 + K\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))^{K-1}\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)) \\ =&1 + \ln(1-\tilde{\mu}^C(\gamma,\varepsilon))(1-\tilde{\mu}^C(\gamma,\varepsilon))\frac{\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))}{\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))\ln(\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)))}. \end{aligned}$$

Recall that

$$\beta(\gamma,\varepsilon,\mu^C) = \frac{\gamma(1-(1-\varepsilon)\mu^C)}{1-\gamma(1-\varepsilon)\mu^C} = 1 - \frac{1-\gamma}{1-\gamma(1-\varepsilon)\mu^C}.$$

Thus, $\lim_{(\gamma,\varepsilon)\to(1,0)}\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)) = 1$. Hence, it follows that for high γ and small ε, $\ln(\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))) = -(1-\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))) + O(1-\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))^2$. Moreover,

$$\begin{aligned} \frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)) &= -\frac{(1-\gamma)\gamma(1-\varepsilon)}{(1-\gamma(1-\varepsilon)\mu^C(\gamma,\varepsilon))^2} \\ &= -\frac{\gamma(1-\varepsilon)}{1-\gamma(1-\varepsilon)\tilde{\mu}^C(\gamma,\varepsilon)}(1-\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))). \end{aligned}$$

Combining these results gives us

$$\lim_{(\gamma,\varepsilon)\to(1,0)}\frac{\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))}{\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))\ln(\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)))} = \frac{1}{1-\tilde{\mu}^C}.$$

Since $\lim_{(\gamma,\varepsilon)\to(1,0)}\ln(1-\tilde{\mu}^C(\gamma,\varepsilon))(1-\tilde{\mu}^C(\gamma,\varepsilon)) = \ln(1-\tilde{\mu}^C)(1-\tilde{\mu}^C)$, it further follows that

$$\lim_{(\gamma,\varepsilon)\to(1,0)} 1 + K\beta(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon))^{K-1}\frac{\partial\beta}{\partial\mu^C}(\gamma,\varepsilon,\tilde{\mu}^C(\gamma,\varepsilon)) = 1 + \ln(1-\tilde{\mu}). \quad [18]$$

Since $\tilde{\mu} \neq 1 - 1/e$, we have $1 + \ln(1-\tilde{\mu}) \neq 0$. Thus, using [18], we conclude that the determinant of the matrix of partial derivatives in [17] is non-zero, and so can appeal to the implicit function theorem to conclude that for sufficiently high γ and small ε, for each $K' \neq K$ and $\alpha > 0$, there is some $\delta_1 > 0$ such that when the share of players playing $GrimK$ is $1-\delta$ and the share of players playing $GrimK'$ is δ with $\delta < \delta_1$, there is a steady state where the fraction of players using $GrimK$ that are cooperators, $\mu^{C'}$, is such that $|\mu^{C'} - \tilde{\mu}^C(\gamma,\varepsilon)| < \alpha$. Additionally, because the $GrimK$ equilibrium with share of cooperators $\tilde{\mu}^C(\gamma,\varepsilon)$ is a strict equilibrium where players have uniformly strict incentives to play according to $GrimK$ at every own record and partner record, it follows that there is some $0 < \overline{\delta} < \delta_1$ such that, when the share of players playing $GrimK$ is $1-\delta$ and the share of players playing $GrimK'$ is δ with $\delta < \overline{\delta}$, there is a steady state with share of cooperators $\mu^{C'}$ such that $|\mu^{C'} - \tilde{\mu}^C(\gamma,\varepsilon)| < \alpha$ where it is strictly optimal to play $GrimK$.

■

Dynamics. We performed a simulation to capture dynamic evolution. We considered a population initially playing the $Grim5$ equilibrium with steady-state share of cooperators of $\mu^C \approx 0.8998$ when $\gamma = 0.9, \varepsilon = 0.1, g = 0.4, l = 2.8$ that is infected with a mutant population playing $Grim1$ at $t = 0$. The initial share of the population that played $Grim5$ was .95, and the complementary share of 0.05 played $Grim1$. At $t = 0$, all of the $Grim1$ mutants had record 0, while the record shares of the $Grim5$ population were proportional to those in the original steady state. At period t, the players match, observe each others' records (but not what population their opponent belongs to), and then play as their strategy dictates. We denote the average payoff of the $Grim5$ players and $Grim1$ players at period t by $\pi^{Grim5,t}$ and $\pi^{Grim1,t}$, respectively.

The evolution of the system from period $t-1$ to t was driven by the average payoffs and sizes of the two populations at $t-1$. In particular, at any period $t > 0$, the share of the newborn players that belonged to the $Grim5$ population ($\mu^{NGrim5,t}$) was proportional to the product of $\mu^{Grim5,t-1}$ and $\pi^{Grim5,t-1}$, and similarly the share of the $1-\gamma$ newborn players that belonged to the $Grim1$ population ($\mu^{NGrim1,t}$) was proportional to the product of $\mu^{Grim1,t-1}$ and $\pi^{Grim1,t-1}$. Formally,

$$\begin{aligned} \mu^{NGrim5,t} &= \frac{\mu^{Grim5,t-1}\pi^{Grim5,t-1}}{\mu^{Grim5,t-1}\pi^{Grim5,t-1} + \mu^{Grim1,t-1}\pi^{Grim1,t-1}}(1-\gamma) \\ \mu^{NGrim1,t} &= \frac{\mu^{Grim1,t-1}\pi^{Grim1,t-1}}{\mu^{Grim5,t-1}\pi^{Grim5,t-1} + \mu^{Grim1,t-1}\pi^{Grim1,t-1}}(1-\gamma). \end{aligned}$$

Fig. S1 presents the results of this simulation. **Fig. S1a** depicts the evolution of the share of players that use $Grim5$ and are cooperators (i.e. have record $k < 5$). Initially, this share is below the steady-state value of ≈ 0.8998, and is decreasing as the $Grim1$ mutants obtain high payoffs relative to the normal $Grim5$ players on average. However, the share of cooperator $Grim5$ players eventually begins to increase and approaches its steady-state value as the mutants die out.

The reason the mutants eventually die out is that their payoffs eventually decline, as depicted in **Fig. S1b**. The tendency of the $Grim1$ players to defect means that they tend to move to high records relatively quickly, and so while they initially receive a high payoff from defecting against cooperators, this advantage is short lived.

We found similar results when the mutant population plays *Grim*9 rather than *Grim*1, although the average payoff in the mutant population never exceeded that in the normal population. And we again found similar results when a population initially playing the *Grim*8 equilibrium with steady-state share of cooperators of $\mu^C \approx 0.613315$ and $\gamma = 0.95, \varepsilon = 0.05, g = 0.5, l = 4$ is infected with a mutant population playing *Grim*3 at $t = 0$, and for when it is infected with a mutant population playing *Grim*13.

Public Goods

Our analysis so far has taken the basic unit of social interaction to be the standard 2-player prisoner's dilemma. However, there are important social interactions that involve many players: the management of the commons and other public resources is a leading example (67–70). Such multiplayer public goods games have been the subject of extensive theoretical and experimental research (48, 71–75). Here we show that a simple variant of *GrimK* strategies can support positive robust cooperation in the multiplayer public goods game when there is sufficient strategic complementarity.

We use the same model as considered so far, except that now in each period the players randomly match in groups of size n, for some fixed integer $n \geq 2$. All players in each group simultaneously decide whether to *Contribute* (C) or *Not Contribute* (D). If exactly x of the n players in the group contribute, each group member receives a benefit of $f(x) \geq 0$, where $f : \mathbb{N} \to \mathbb{R}_+$ is a strictly increasing function with $f(0) = 0$. In addition, each player who contributes incurs a private cost of $c > 0$. This coincides with the 2-player PD when $n = 2$, $f(1) = 1 + g$, $f(2) = l + 2 + g$, and $c = l + 1 + g$.

For each $x \in \{0, \ldots, n-1\}$, let $\Delta(x) = f(x+1) - f(x)$ denote the marginal benefit to each member when there is an additional contribution. Assume that $\Delta(x) < c < n\Delta(x)$ for each $x \in \{0, \ldots, n-1\}$. This assumption makes the public good game an n-player PD, in that D is the selfishly optimal action while everyone playing C is socially optimal.

We consider the same record system as in the 2-player PD: Newborns have record 0. If a player plays D, their record increases by 1. If a player plays C, their record increases by 1 with probability $\varepsilon > 0$, and remains constant with probability $1 - \varepsilon$.

As in the 2-player PD, we find that a key determinant of the prospects for robust cooperation is the degree of strategic complementarity or substitutability in the social dilemma. In the public good game, we say that the interaction exhibits *strategic complementarity* if $\Delta(x)$ is increasing in x (i.e., contributing is more valuable when more partners contribute), and exhibits *strategic substitutability* if $\Delta(x)$ is decreasing in x.

We first show that with strategic substitutability the unique strict equilibrium is *Never Contribute*. This generalizes our finding that *Always Defect* is the unique strict equilibrium in the 2-player PD when $g \geq l$.

Theorem 15. *For any $n \geq 2$, if the public good game exhibits strategic substitutability, the unique strict equilibrium is* Never Contribute.

Proof. Suppose n players who all have the same record k meet. By symmetry, either they all contribute or none of them contribute. In the former case, contributing is optimal for a record-k player when all partners contribute, so by strategic substitutability contributing is also optimal for a record-k player when a smaller number of partners contribute. Thus, a record-k player contributes regardless of their partners' records. In the latter case, not contributing is optimal for a record-k player when no partners contribute, so by strategic substitutability not contributing is also optimal for a record-k player when a larger number of partners contribute.

We have established that, for each k, record-k players do not condition their behavior on their opponents' records. Hence, the distribution of future opposing actions faced by any player is independent of their record. This implies that not contributing is always optimal. ∎

We now turn to the case of strategic complementarity and consider the following simple generalization of *GrimK* strategies: Records $k < K$ are considered to be "good," while records $k \geq K$ are considered "bad." When n players meet, they all contribute if all of their records are good; otherwise, none of them contribute.

For *GrimK* strategies to form an equilibrium, two incentive constraints must be satisfied: First, a player with record 0 (the "safest" good record) must want to contribute in a group with $n-1$ other good-record players. Second, a player with record $K-1$ (the "most fragile" good record) must not want to contribute in a group where no one else contributes.

We let $g = c - \Delta(n-1)$ and $l = c - \Delta(0)$. Note that

$$V_0 = (1-\gamma)(\mu^C)^{n-1}(f(n) - c) + \gamma(1-\varepsilon)(\mu^C)^{n-1}V_0 + \gamma(1 - (1-\varepsilon)(\mu^C)^{n-1})V_1,$$

which gives

$$(1-\varepsilon)\frac{\gamma}{1-\gamma}(V_0 - V_1) = \frac{1-\varepsilon}{1-(1-\varepsilon)(\mu^C)^{n-1}}((\mu^C)^{n-1}(f(n)-c) - V_0).$$

By a similar argument to Lemma 5, it can be established that $V_0 = \mu^C(\mu^C)^{n-1}(f(n) - c)$. We thus find that the cooperation constraint for a record 0 player is

$$\frac{1-\varepsilon}{1-(1-\varepsilon)(\mu^C)^{n-1}}(1-\mu^C)(\mu^C)^{n-1}(f(n)-c) > g. \quad [19]$$

In addition,

$$V_{K-1} = (1-\gamma)(\mu^C)^{n-1}(f(n) - c) + \gamma(1-\varepsilon)(\mu^C)^{n-1}V_{K-1}$$

gives

$$(1-\varepsilon)\frac{\gamma}{1-\gamma}V_{K-1} = \frac{\gamma(1-\varepsilon)}{1-\gamma(1-\varepsilon)(\mu^C)^{n-1}}(\mu^C)^{n-1}(f(n)-c).$$

Thus, the defection constraint for a record $K-1$ player is

$$\frac{\gamma(1-\varepsilon)}{1-\gamma(1-\varepsilon)(\mu^C)^{n-1}}(\mu^C)^{n-1}(f(n)-c) < l,$$

which gives

$$(\mu^C)^{n-1} < \frac{1}{\gamma(1-\varepsilon)}\frac{l}{f(n)-c+l} \Leftrightarrow \mu^C < \left(\frac{1}{\gamma(1-\varepsilon)}\right)^{\frac{1}{n-1}}\left(\frac{l}{f(n)-c+l}\right)^{\frac{1}{n-1}}. \quad [20]$$

This gives $\mu^C \leq (l/(f(n)-c+l))^{1/(n-1)}$ in the $(\gamma,\varepsilon)\to(1,0)$ limit.

Moreover, in the limit where $\varepsilon \to 0$, [19] gives

$$\frac{1-\mu^C}{1-(\mu^C)^{n-1}}(\mu^C)^{n-1}(f(n)-c) \geq g \Leftrightarrow \frac{1}{\sum_{m=0}^{n-2}(\mu^C)^m}(\mu^C)^{n-1}(f(n)-c) \geq g.$$

Note that $(\mu^C)^{n-1}/\sum_{m=0}^{n-2}(\mu^C)^m$ is increasing in μ^C. Thus, this inequality, along with the previous upper bound for μ^C, puts the following requirement on the parameters:

$$\frac{1-\left(\frac{l}{f(n)-c+l}\right)^{\frac{1}{n-1}}}{\frac{f(n)-c}{f(n)-c+l}}\frac{l}{f(n)-c+l}(f(n)-c) \geq g,$$

which simplifies to

$$g \leq \left(1-\left(\frac{l}{f(n)-c+l}\right)^{\frac{1}{n-1}}\right)l. \quad [21]$$

So far we have established [21], which is a necessary condition on the g, l parameters for any cooperation to be sustainable with *GrimK* strategies in the $(\gamma,\varepsilon)\to(1,0)$ limit. We can further characterize the maximum limit share of cooperators in *GrimK* equilibria using very similar arguments as those in Lemmas 2 and 3.

Theorem 16.

$$\lim_{(\gamma,\varepsilon)\to(1,0)} \overline{\mu}_n^C(\gamma,\varepsilon) = \begin{cases} \left(\frac{l}{f(n)-c+l}\right)^{\frac{1}{n-1}} & \textit{if } g < \left(1-\left(\frac{l}{f(n)-c+l}\right)^{\frac{1}{n-1}}\right)l \\ 0 & \textit{if } g > \left(1-\left(\frac{l}{f(n)-c+l}\right)^{\frac{1}{n-1}}\right)l \end{cases}.$$

Theorem 16 shows that *GrimK* strategies can support robust social cooperation in the n-player public goods game in much the same manner as in the 2-player PD. To see how this result reduces to Theorem 1 in the 2-player PD, note that $f(2)-c=1$, so $(l/(f(n)-c+l))^{1/(n-1)} = l/(1+l)$ when $n=2$.

In the 2-player PD, we found that the class of *GrimK* strategies could achieve the same level of cooperation as a more general class of trigger strategies in the limit where $(\gamma,\varepsilon)\to(1,0)$. We note that such a result holds here as well for the class of trigger strategies that satisfy: (i) The set of all possible records can be partitioned into two classes, "good records" G and "bad records" B. (ii) When n players meet, they all contribute if all of their records are good and none of them contribute if any one of them has a bad record. (iii) The class B is absorbing: if $k \in B$, then every record k' that can be reached starting at record k is also in B.

Appendix

Convergence Matlab Files.

```
% Parameters
gamma    = 0.8;
epsilon  = 0.02;
T        = 100; % Time periods

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Grim1
k = 1;

% Initialize Cooperator Share Arrays
```

```
cooperator_share_high    = zeros(T,1);         % Highest trajectory
cooperator_share_steady  = 0.248359*ones(T,1); % Steady state
cooperator_share_low     = zeros(T,1);         % Lowest trajectory

% Initialize Period Share Distribution Arrays
share_distribution_high    = zeros(k,1);
share_distribution_high(1) = 1;           % Highest trajectory
share_distribution_low     = zeros(k,1); % Lowest trajectory

% Iterate Over Time Periods
for t = 1:T
    % Highest Trajectory
    cooperator_share_high(t) = sum(share_distribution_high); % Compute cooperator share
    share_distribution_high  = update_grim_k(gamma,epsilon,k,...
        share_distribution_high); % Update period share distribution

    % Lowest Trajectory
    cooperator_share_low(t)  = sum(share_distribution_low); % Compute cooperator share
    share_distribution_low   = update_grim_k(gamma,epsilon,k,...
        share_distribution_low); % Update period share distribution
end

% Format Figure
t          = 0:T-1;
dimensions = [0,0,10,6];
figure('units','inch','position',dimensions)
hold on
plot(t,cooperator_share_high,'-*','linewidth',2);
plot(t,cooperator_share_steady,'-*','linewidth',2);
plot(t,cooperator_share_low,'-*','linewidth', 2);
hold off
set(gca,'TickLabelInterpreter','latex');
set(gca,'FontSize',32,'FontWeight','bold');
xlabel('Time ($t$)', 'Interpreter', 'latex');
yl = ylabel('Share of Cooperators ($\mu^{C}$)', 'Interpreter', 'latex');
yl.Position(1) = yl.Position(1) + abs(yl.Position(1) * 0.4);
yl.Position(2) = yl.Position(2) - abs(yl.Position(2) * 0.1);
ylim([0, 1]);
xlim([0,30]);
legend({'Highest Trajectory','Steady State','Lowest Trajectory'},...
    'Location','northeast','Interpreter','latex');
set(gcf,'color','w');
hold off

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Grim2
k = 2;

% Initialize Cooperator Share Arrays
cooperator_share_high          = zeros(T,1);         % Highest trajectory
cooperator_share_high_steady   = .985542*ones(T,1); % Highest steady state
cooperator_share_middle_steady = .918367*ones(T,1); % Middle stead state
cooperator_share_low_steady    = .647111*ones(T,1); % Lowest steady state
cooperator_share_low           = zeros(T,1);         % Lowest trajectory

% Initialize Period Share Distribution Arrays
share_distribution_high    = zeros(k,1);
share_distribution_high(1) = 1;           % Highest trajectory
share_distribution_low     = zeros(k,1); % Lowest trajectory

% Iterate Over Time Periods
```

```
for t = 1:T
    % Highest Trajectory
    cooperator_share_high(t) = sum(share_distribution_high); % Compute cooperator share
    share_distribution_high  = update_grim_k(gamma,epsilon,k,...
        share_distribution_high); % Update period share distribution

    % Lowest Trajectory
    cooperator_share_low(t)  = sum(share_distribution_low); % Compute cooperator share
    share_distribution_low   = update_grim_k(gamma,epsilon,k,...
        share_distribution_low); % Update period share distribution
end

% Format Figure
t          = 0:T-1;
dimensions = [0,0,10,6];
figure('units','inch','position',dimensions)
hold on
plot(t,cooperator_share_high,'-*','linewidth',2);
plot(t,cooperator_share_high_steady,'-*','linewidth',2);
plot(t,cooperator_share_middle_steady,'-*','linewidth',2);
plot(t,cooperator_share_low_steady,'-*','linewidth',2);
plot(t,cooperator_share_low,'-*','linewidth', 2);
hold off
set(gca,'TickLabelInterpreter','latex');
set(gca,'FontSize',32,'FontWeight','bold');
xlabel('Time ($t$)', 'Interpreter', 'latex');
yl = ylabel('Share of Cooperators ($\mu^{C}$)', 'Interpreter', 'latex');
yl.Position(1) = yl.Position(1) + abs(yl.Position(1) * 0.4);
yl.Position(2) = yl.Position(2) - abs(yl.Position(2) * 0.1);
ylim([0, 1]);
xlim([0,30]);
legend({'Highest Trajectory','Highest Steady State','Middle Steady State',...
    'Lowest Steady State','Lowest Trajectory'},'Location','southeast',...
    'Interpreter','latex');
set(gcf,'color','w');
hold off
```

```
function updated_share_distribution = update_grim_k(gamma,epsilon,k,...
    share_distribution)

% Initialize Updated Share Distribution Array
updated_share_distribution = zeros(k,1);

% Update Share Distribution Array
updated_share_distribution(1,1) = 1 - gamma + ...
    gamma*(1 - epsilon)*sum(share_distribution)*share_distribution(1);

if k>1
    for i = 2:k
    updated_share_distribution(i,1) = ...
        gamma*(1-(1-epsilon)*sum(share_distribution))*share_distribution(i-1)...
        + gamma*(1-epsilon)*sum(share_distribution)*share_distribution(i);
    end
end

end
```

Evolutionary Dynamics Matlab Files.

```
% Parameters
gamma    = 0.9;
epsilon = 0.1;
g        = 0.4;
l        = 2.8;
T        = 100; % Time periods

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Normal - Grim5, Mutant - Grim1
k_normal                 = 5;
k_mutant                 = 1;
k                        = max(k_normal,k_mutant);

% Initialize Normal Cooperator Share Arrays
normal_cooperator_shares        = zeros(T,1);          % Time series
normal_cooperator_shares_steady = 0.899754*ones(T,1); % Steady state

% Initialize Normal Total Share Array
normal_total_share        = zeros(T,1); % Time series
period_normal_total_share = 0.95;         % Period value

% Initialize Normal Share Distribution Arrays
normal_share_distribution        = zeros(T,k); % Time series
period_normal_share_distribution = zeros(1,k); % Period value

% Set inital normal share distribution to be proportional to steady state
% distribution
for i=1:k_normal

    period_normal_share_distribution(1,i) = ...
        period_normal_total_share*beta(gamma,epsilon,0.899754)^(i-1)...
        *(1-beta(gamma,epsilon,0.899754));

end

if k>k_normal

    for i=k_normal+1:k

        period_normal_share_distribution(1,i) = ...
            period_normal_total_share*beta(gamma,epsilon,0.899754)^(k_normal)...
            *gamma^(i-k_normal-1)*(1-gamma);

    end

end

% Initialize Normal Average Payoff Array
normal_payoff = zeros(T,1);

% Initialize Mutant Total Share Array
mutant_total_share        = zeros(T,1);                    % Time series
period_mutant_total_share = 1-period_normal_total_share; % Period value

% Initialize Mutant Share Distribution Arrays
mutant_share_distribution              = zeros(T,k);                   % Time series
period_mutant_share_distribution       = zeros(1,k);                   % Period value
period_mutant_share_distribution(1,1) = period_mutant_total_share; % Initial mutants have
    record 0

% Initialize Mutant Average Payoff Array
```

```
mutant_payoff = zeros(T,1);

% Iterate Over Time Periods
for t = 1:T
    % Update Shares
    normal_total_share(t,1)          = period_normal_total_share;
    normal_share_distribution(t,:) = period_normal_share_distribution(1,:);
    normal_cooperator_shares(t)     = sum(period_normal_share_distribution(1,1:k_normal));
    mutant_total_share(t,1)          = period_mutant_total_share;
    mutant_share_distribution(t,:) = period_mutant_share_distribution(1,:);

    % Compute Period Payoffs
    [period_normal_payoff,period_mutant_payoff] = ...
        payoffs_general(g,l,k_normal,k_mutant,period_normal_total_share,...
        period_normal_share_distribution,period_mutant_total_share,...
        period_mutant_share_distribution);

    % Update Payoff Time Series
    normal_payoff(t,1) = period_normal_payoff;
    mutant_payoff(t,1) = period_mutant_payoff;

    % Compute Updated Period Shares
    [period_normal_total_share,period_normal_share_distribution(1,:),...
        period_mutant_total_share,period_mutant_share_distribution(1,:)]...
        = dynamic_update_general(gamma,epsilon,k_normal,k_mutant,...
        period_normal_total_share,period_normal_share_distribution(1,:),...
        period_mutant_total_share,period_mutant_share_distribution(1,:),...
        period_normal_payoff,period_mutant_payoff);

end


% Format Figures
t = 0:T-1;
dimensions = [0,0,10,6];

figure('units','inch','position',dimensions)
hold on
plot(t,normal_cooperator_shares,'-*','linewidth', 1);
plot(t,normal_cooperator_shares_steady,'-*','linewidth', 1);
set(gca,'TickLabelInterpreter','latex');
set(gca,'FontSize',24,'FontWeight','bold');
xlabel('Time ($t$)', 'Interpreter', 'latex');
yl = ylabel('Share of Normal Cooperators', 'Interpreter', 'latex');
yl.Position(1) = yl.Position(1) + abs(yl.Position(1) * 0.4);
yl.Position(2) = yl.Position(2) + abs(yl.Position(2) * 0.05);
ylim([.8, 1]);
xlim([0,60]);
legend({'$Grim5$ Cooperators','Steady State'},'Location','northeast',...
    'Interpreter','latex');
set(gcf,'color','w');
hold off

figure('units','inch','position',dimensions)
hold on
plot(t,normal_payoff,'-*','linewidth', 1);
plot(t,mutant_payoff,'-*','linewidth', 1);
set(gca,'TickLabelInterpreter','latex');
set(gca,'FontSize',24,'FontWeight','bold');
xlabel('Time ($t$)', 'Interpreter', 'latex');
yl = ylabel('Average Payoffs', 'Interpreter', 'latex');
```

```
yl.Position(1) = yl.Position(1) + abs(yl.Position(1) * 0.25);
yl.Position(2) = yl.Position(2) + abs(yl.Position(2) * 0.2);
ylim([0, 1.5]);
xlim([0,60]);
legend({'$Grim5$ Players','$Grim1$ Players'},'Location','northeast',...
    'Interpreter','latex');
set(gcf,'color','w');
hold off


function f = beta(gamma,epsilon,cooperator_share)

f = gamma*(1-(1-epsilon)*cooperator_share)/(1-gamma*(1-epsilon)*cooperator_share);

end

function [ratio_normal,ratio_mutant] = ...
    proper_ratios_general(period_normal_total_share,period_mutant_total_share,...
    period_normal_payoff,period_mutant_payoff)

if (period_normal_payoff>0) && (period_mutant_payoff>0)
    ratio_normal = period_normal_total_share*period_normal_payoff/...
        (period_normal_total_share*period_normal_payoff + ...
        period_mutant_total_share*period_mutant_payoff);
    ratio_mutant = period_mutant_total_share*period_mutant_payoff/...
        (period_normal_total_share*period_normal_payoff + ...
        period_mutant_total_share*period_mutant_payoff);
end

if (period_normal_payoff>0) && (period_mutant_payoff<=0)
    ratio_normal = 1;
    ratio_mutant = 0;
end

if (period_normal_payoff<=0) && (period_mutant_payoff>0)
    ratio_normal = 0;
    ratio_mutant = 1;
end

if (period_normal_payoff<=0) && (period_mutant_payoff<=0)
    ratio_normal = period_normal_total_share/(period_normal_total_share + ...
        period_mutant_total_share);
    ratio_mutant = period_mutant_total_share/(period_normal_total_share + ...
        period_mutant_total_share);
end

end

function [period_normal_payoff,period_mutant_payoff] = payoffs_general(g,l,...
    k_normal,k_mutant,period_normal_total_share,period_normal_share_distribution,...
    period_mutant_total_share,period_mutant_share_distribution)

normal_cooperator_share = sum(period_normal_share_distribution(1,1:k_normal));
mutant_cooperator_share = sum(period_mutant_share_distribution(1,1:k_mutant));

if k_normal>k_mutant
    % Compute the Share of Mutant Players Misperceived by Normal Players
    misperceived_mutant_share = ...
        sum(period_mutant_share_distribution(1,k_mutant+1:k_normal));

    % Compute "Total Population Payoffs"
```

```
    total_normal_payoff = normal_cooperator_share*((normal_cooperator_share...
        +mutant_cooperator_share)*1 - misperceived_mutant_share*l);
    total_mutant_payoff = mutant_cooperator_share*(normal_cooperator_share...
        +mutant_cooperator_share)*1 + ...
        misperceived_mutant_share*normal_cooperator_share*(1+g);

end

if k_mutant>k_normal
    % Compute the Share of Normal Players Misperceived by Mutant Players
    misperceived_normal_share = sum(period_normal_share_distribution(1,k_normal+1:k_mutant)
        );

    % Compute "Total Population Payoffs"
    total_normal_payoff = normal_cooperator_share*(normal_cooperator_share...
        +mutant_cooperator_share)*1 + misperceived_normal_share*mutant_cooperator_share*(1+
            g);
    total_mutant_payoff = mutant_cooperator_share*...
        ((normal_cooperator_share + mutant_cooperator_share)*1 - ...
        misperceived_normal_share*l);

end

% Compute Average Payoffs
period_normal_payoff = total_normal_payoff/period_normal_total_share;
period_mutant_payoff = total_mutant_payoff/period_mutant_total_share;
end

function [updated_period_normal_total_share,updated_period_normal_share_distribution,...
        updated_period_mutant_total_share,updated_period_mutant_share_distribution] = ...
        dynamic_update_general(gamma,epsilon,k_normal,k_mutant,...
        period_normal_total_share,period_normal_share_distribution,...
        period_mutant_total_share,period_mutant_share_distribution,...
        period_normal_payoff,period_mutant_payoff)

k = max(k_normal,k_mutant);

% Compute Ratios of Incoming Players that are Normal or Mutant
[ratio_normal,ratio_mutant] = ...
    proper_ratios_general(period_normal_total_share,...
    period_mutant_total_share,period_normal_payoff,period_mutant_payoff);

% Compute Updated Total Share of Normal and Mutant Players
updated_period_normal_total_share = gamma*period_normal_total_share + (1-gamma)*
    ratio_normal;
updated_period_mutant_total_share = gamma*period_mutant_total_share + (1-gamma)*
    ratio_mutant;

% Initialize Updated Share Distribution Arrays
updated_period_normal_share_distribution = zeros(1,k);
updated_period_mutant_share_distribution = zeros(1,k);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute Updated Period Normal Share Distribution

% Compute Share of Players Perceived as Cooperators by Normal Players
mu_c = sum(period_normal_share_distribution(1,1:k_normal)) + ...
    sum(period_mutant_share_distribution(1,1:k_normal));

% Computed Updated Normal Shares
updated_period_normal_share_distribution(1,1) = ...
    gamma*(1-epsilon)*mu_c*period_normal_share_distribution(1,1) + ...
```

```
    (1-gamma)*ratio_normal;

for i=2:k_normal
    updated_period_normal_share_distribution(1,i) = ...
        gamma*(1-(1-epsilon)*mu_c)*period_normal_share_distribution(1,i-1) + ...
        gamma*(1-epsilon)*mu_c*period_normal_share_distribution(1,i);
end

if k_mutant>k_normal
    updated_period_normal_share_distribution(1,k_normal+1) = ...
        gamma*(1-(1-epsilon)*mu_c)*period_normal_share_distribution(1,k_normal);

if k_mutant>k_normal+1
    for i=k_normal+2:k_mutant
        updated_period_normal_share_distribution(1,i) = ...
            gamma*period_normal_share_distribution(1,i-1);
    end
end

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute Updated Period Mutant Share Distribution

% Compute Share of Players Perceived as Cooperators by Normal Players
mu_c = sum(period_normal_share_distribution(1,1:k_mutant)) + ...
    sum(period_mutant_share_distribution(1,1:k_mutant));

% Computed Updated Mutant Shares
updated_period_mutant_share_distribution(1,1) = ...
    gamma*(1-epsilon)*mu_c*period_mutant_share_distribution(1,1) + ...
    (1-gamma)*ratio_mutant;

for i=2:k_mutant
    updated_period_mutant_share_distribution(1,i) = ...
        gamma*(1-(1-epsilon)*mu_c)*period_mutant_share_distribution(1,i-1) + ...
        gamma*(1-epsilon)*mu_c*period_mutant_share_distribution(1,i);
end

if k_normal>k_mutant
    updated_period_mutant_share_distribution(1,k_mutant+1) = ...
        gamma*(1-(1-epsilon)*mu_c)*period_mutant_share_distribution(1,k_mutant);
end

if k_normal>k_mutant+1
    for i=k_mutant+2:k_normal
        updated_period_mutant_share_distribution(1,i) = ...
            gamma*period_mutant_share_distribution(1,i-1);
    end
end

end
```

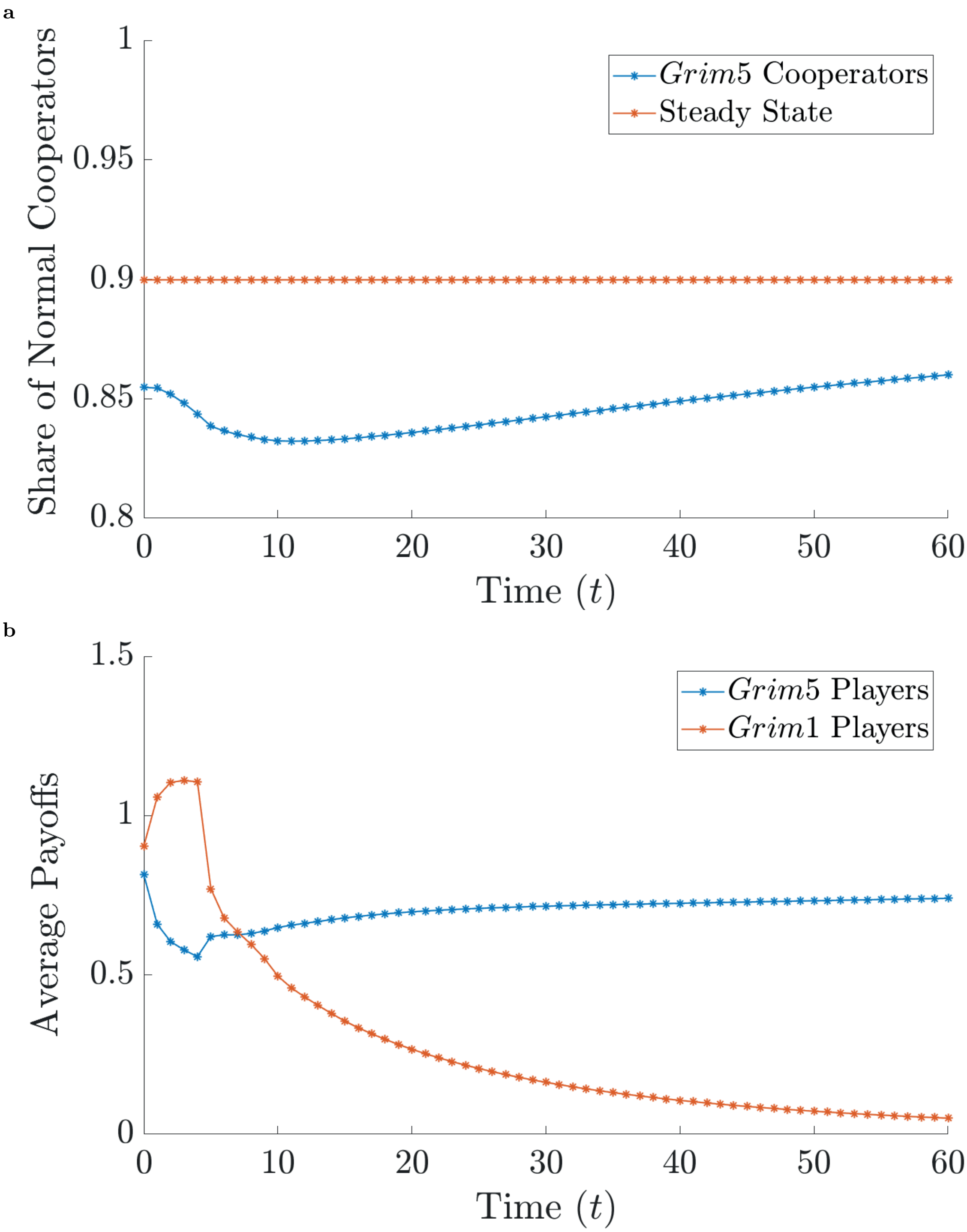

Fig. S1. Evolutionary dynamics. a, The blue curve depicts the evolution of the share of players that use $Grim5$ and are cooperators (i.e. have some record $k < 5$). **b**, The average payoffs in the normal $Grim5$ population (blue curve) and in the mutant $Grim1$ population (red curve).

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