Supplementary Materials for "Adjusted Time-varying Population Attributable Hazard in Case-Control Studies"

APPENDIX A: PROOF OF THEOREM 1: STRONG UNIFORM CONSISTENCY OF THE PROPOSED ESTIMATORS

We assume the number of cases n_1 and total number of subjects n satisfy $n_1/n \to \pi_0$ as $n \to \infty$, where $0 < \pi_0 < 1$. In addition we assume the following regularity conditions:

A1. The time t is in a range of $[0, \tau]$ for a constant $\tau > 0$ such that the density of failure time $f_T(t)$, the density of censoring time $f_C(t)$ and their survival functions, $S_T(t)$ and $S_C(t)$ all take positive real values on $[0, \tau]$.

A2. The density of failure time $f_T(t)$ and the density of censoring time $f_C(t)$ are both continuous, uniformly bounded, and have second derivatives on $[0, \tau]$.

A3. Random censoring: the censoring time C is independent of the failure time T, exposure $\mathbf{Z}$ and confounder $\mathbf{U}$ for $t \in [0, \tau]$.

A4. The bandwidth satisfies $h = n^d h_0$ for constants $-1/2 < d < -1/5$ and $h_0 > 0$.

A5. The kernel function $K(\cdot)$ has bounded variation and satisfies the following conditions,

$$\int_{-\infty}^{\infty} K(u)du = 1, \quad \int_{-\infty}^{\infty} K^2(u)du < \infty,$$
$$\int_{-\infty}^{\infty} uK(u)du = 0, \quad \int_{-\infty}^{\infty} u^2K(u)du < \infty.$$

A6. $\mathbf{Z}$ and $\mathbf{U}$ are bounded almost surely and have uniformly bounded total variation on $[0, \tau]$.

Consider n subjects in a case-control study. Let X_i, Δ_i, $\mathbf{Z}_i$ and $\mathbf{U}_i$ be the observed time, the censoring indicator, the exposure and confounders, respectively, for $i = 1, ..., n$. We define the following notation.

$$
\begin{aligned}
A_n(t;\boldsymbol{\beta}) &= \frac{1}{n}\sum_{i=1}^n \Delta_i K_h(t - X_i)\{1 - \exp(-\boldsymbol{\beta}^T\mathbf{Z}_i)\},\\
B_n(t) &= \frac{1}{n}\sum_{i=1}^n \Delta_i K_h(t - X_i),\\
C_n(t) &= \frac{1}{n}\sum_{i=1}^n \Delta_i K_h(t - X_i)\{\exp(-\boldsymbol{\beta}_0^T\mathbf{Z}_i) - \exp(-\hat{\boldsymbol{\beta}}^T\mathbf{Z}_i)\},\\
\mathbf{D}_n(t;\boldsymbol{\beta}) &= \frac{1}{n}\sum_{i=1}^n \Delta_i K_h(t - X_i)\mathbf{Z}_i \exp(-\boldsymbol{\beta}^T\mathbf{Z}_i),\\
\overline{A}_n(t;\boldsymbol{\beta},\boldsymbol{\gamma}) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i)\exp(\boldsymbol{\beta}^T\mathbf{Z}_i + \boldsymbol{\gamma}^T\mathbf{U}_i),\\
\overline{B}_n(t) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i),\\
\overline{C}_n(t) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i)\{\exp(\hat{\boldsymbol{\beta}}^T\mathbf{Z}_i + \hat{\boldsymbol{\gamma}}^T\mathbf{U}_i) - \exp(\boldsymbol{\beta}_0^T\mathbf{Z}_i + \boldsymbol{\gamma}_0^T\mathbf{U}_i)\},\\
\overline{\mathbf{D}}_n(t;\boldsymbol{\theta}) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i)\mathbf{V}_i\exp(\boldsymbol{\theta}^T\mathbf{V}_i),\\
\overline{E}_n(t;\boldsymbol{\gamma}) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i)\exp(\boldsymbol{\gamma}^T\mathbf{U}_i),\\
\overline{F}_n(t) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i)\{\exp(\hat{\boldsymbol{\gamma}}^T\mathbf{U}_i) - \exp(\boldsymbol{\gamma}_0^T\mathbf{U}_i)\},\\
\overline{\mathbf{G}}_n(t;\boldsymbol{\gamma}) &= \frac{1}{n}\sum_{i=1}^n (1-\Delta_i) K_h(t - X_i)\mathbf{U}_i\exp(\boldsymbol{\gamma}^T\mathbf{U}_i).
\end{aligned}
$$

We use P_0 and E_0 to denote the probability and expectation with respect to the target population from which cases and controls are sampled. It is also useful to regard cases and controls as members of a second, hypothetical population of individuals whose disease probability is given by π_0 (Zhao et al. 2017). We use P^* and E^* to denote the probability and expectation with respect to this hypothetical population. Let $p_0 = P_0(T \leq C)$, $\pi_0 = P^*(T \leq C)$. Then the limits of $A_n(t;\boldsymbol{\beta})$, $B_n(t)$, $\mathbf{D}_n(t;\boldsymbol{\theta})$, $\overline{A}_n(t;\boldsymbol{\beta},\boldsymbol{\gamma})$, $\overline{B}_n(t)$, $\overline{\mathbf{D}}_n(t;\boldsymbol{\beta},\boldsymbol{\gamma})$, $\overline{E}_n(t;\boldsymbol{\gamma})$ and $\overline{\mathbf{G}}_n(t;\boldsymbol{\gamma})$ are denoted by

$$
\begin{aligned}
A(t) &= \phi_{adj}(t) f_T(t) S_C(t)\pi_0/p_0,\\
B(t) &= f_T(t) S_C(t)\pi_0/p_0,\\
\mathbf{D}(t;\boldsymbol{\beta}) &= \left\{\int_{\mathcal{Z}} \mathbf{z}\exp(-\boldsymbol{\beta}^T\mathbf{z})dF_{\mathbf{Z}|T}(\mathbf{z}|t)\right\} f_T(t) S_C(t)\pi_0/p_0,\\
\overline{A}(t) &= \psi(t) f_C(t) S_T(t)(1-\pi_0)/(1-p_0),\\
\overline{B}(t) &= f_C(t) S_T(t)(1-\pi_0)/(1-p_0),\\
\overline{\mathbf{D}}(t;\boldsymbol{\theta}) &= \left\{\int_{\mathcal{V}} \mathbf{v}\exp(\boldsymbol{\theta}^T\mathbf{v})dF_{\mathbf{V}|T\geq t}(\mathbf{v})\right\} f_C(t) S_T(t)(1-\pi_0)/(1-p_0),\\
\overline{E}(t) &= \upsilon(t) f_C(t) S_T(t)(1-\pi_0)/(1-p_0),\\
\overline{\mathbf{G}}(t;\boldsymbol{\gamma}) &= \left\{\int_{\mathcal{U}} \mathbf{u}\exp(-\boldsymbol{\gamma}^T\mathbf{u})dF_{\mathbf{U}|T}(\mathbf{u}|t)\right\} f_C(t) S_T(t)(1-\pi_0)/(1-p_0).
\end{aligned}
$$

We first present the following lemmas before proving Theorem 1.

Lemma 1.1. Suppose assumptions A1-A6 are satisfied. Then,

$$
\begin{aligned}
\sup_{t\in[0,\tau]} &\left|\widehat{\varphi}(t;\hat{\boldsymbol{\beta}}) - \varphi(t)\right| \to 0 \ a.s.\\
\sup_{t\in[0,\tau]} &\left|\widehat{\upsilon}(t;\hat{\boldsymbol{\gamma}}) - \upsilon(t)\right| \to 0 \ a.s.
\end{aligned}
$$

as $n \to \infty$.

Proof. The uniform consistency of $B_n(t)$ has been shown in Lemma 1.1 in Zhao et al. (2017). Hence as argued in Lemma 1.2 in Zhao et al. (2017), it suffices to show the consistency of the numerator $A_n(t;\hat{\boldsymbol{\beta}})$. To simplify the notation, unless otherwise specified, the integration with respect to x is from 0 to ∞, integration with respect to $\mathbf{z}$ is on its support $\mathcal{Z}$ and integration with respect to $\mathbf{u}$ is on its support $\mathcal{U}$ through out the proofs in this article.

Note that $A_n(t;\hat{\boldsymbol{\beta}}) = A_n(t;\boldsymbol{\beta}_0) - C_n(t)$. The first term has expectation

$$\begin{aligned}
E^*\{A_n(t;\boldsymbol{\beta}_0)\} &= E^*\{\Delta K_h(t-X)\exp(-\boldsymbol{\beta}_0^T\mathbf{Z})\} \\
&= E^*\{K_h(t-T)\exp(-\boldsymbol{\beta}_0^T\mathbf{Z})|T\le C\}P^*(T\le C) \\
&= \frac{\pi_0}{h}\int\int \exp(-\boldsymbol{\beta}_0^T\mathbf{z})K\left(\frac{t-x}{h}\right) f_{T,\mathbf{Z}|T\le C}(x,\mathbf{z})dxd\mathbf{z} \\
&= \frac{\pi_0}{h}\int\int \exp(-\boldsymbol{\beta}_0^T\mathbf{z})K\left(\frac{t-x}{h}\right) \frac{f_{T|\mathbf{Z}}(x|\mathbf{z})S_C(x)f_{\mathbf{Z}}(\mathbf{z})}{P(T\le C)}dxd\mathbf{z} \\
&= \frac{1}{h}\int K\left(\frac{t-x}{h}\right) f_T(x)S_C(x)\pi_0/p_0\int \exp(-\boldsymbol{\beta}_0^T\mathbf{z})f_{\mathbf{Z}|T}(\mathbf{z}|x)d\mathbf{z}dx, \\
&= \frac{1}{h}\int K\left(\frac{t-x}{h}\right) A(x)dx.
\end{aligned}$$

Using the same arguments as in Lemma 1.1 and 1.2 in Zhao et al. (2017), we can conclude that

$$\sup_{t\in[0,\tau]} |A_n(t;\boldsymbol{\beta}_0) - A(t)| \to 0 \ a.s. \tag{A.1}$$

as $n \to 0$, given $K(.)$ has bounded variation, and $\mathbf{Z}$ is bounded almost surely.

By using Taylor's expansion, the second term $C_n(t)$ can written as

$$\begin{aligned}
C_n(t) &= \frac{1}{n}\sum_{i=1}^{n}\{\exp(-\boldsymbol{\beta}_0^T\mathbf{Z}_i) - \exp(-\hat{\boldsymbol{\beta}}^T\mathbf{Z}_i)\}\Delta_i K_h(t-X_i) \\
&= (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T\mathbf{D}_n(t;\boldsymbol{\beta}_0) + o_p(1).
\end{aligned}$$

Consistency of $\mathbf{D}_n(t;\boldsymbol{\beta})$ can be established similarly as in the proof of Lemma 1.3 in Zhao et al. (2017), by noticing that

$$\begin{aligned}
E^*\{\mathbf{D}_n(t;\boldsymbol{\beta})\} &= E^*\{\mathbf{Z}_i \exp(-\boldsymbol{\beta}^T\mathbf{Z}_i)\Delta_i K_h(t-X_i)\} \\
&= E^*\{\mathbf{Z}_i \exp(-\boldsymbol{\beta}^T\mathbf{Z}_i) K_h(t-X_i)|T \leq C\}P^*(T \leq C) \\
&= \frac{\pi_0}{h}\int\int \{\mathbf{z}\exp(-\boldsymbol{\beta}^T\mathbf{z})\}K\left(\frac{t-x}{h}\right) f_{T,\mathbf{Z}|T\leq C}(x,\mathbf{z})dxd\mathbf{z} \\
&= \frac{1}{h}\int K\left(\frac{t-x}{h}\right)\mathbf{D}(x;\boldsymbol{\beta})dx,
\end{aligned}$$

$\mathbf{D}(t;\boldsymbol{\beta})$ is continuous for t, and $\mathbf{Z}$ is bounded almost surely. By the strong consistency of $\hat{\boldsymbol{\beta}}$ and the continuous mapping theorem, we have

$$\sup_{t\in[0,\tau]} |C_n(t)| \to 0 \ a.s. \tag{A.2}$$

as $n \to 0$. Putting (A.1) and (A.2) together, we conclude that

$$\sup_{t\in[0,\tau]} \left| A_n(t;\hat{\boldsymbol{\beta}}) - A(t) \right| \to 0 \ a.s.$$

as $n \to 0$.

Similarly, by replacing $\boldsymbol{\theta}$ and $\mathbf{V}_i$ with $\boldsymbol{\gamma}$ and $\mathbf{U}_i$ respectively, we can prove $\sup_{t\in[0,\tau]} |\widehat{\upsilon}(t;\hat{\boldsymbol{\gamma}}) - \upsilon(t)| \to 0 \ a.s.$. Lemma 1.1 is proved.

Lemma 1.2. Suppose assumptions A1-A6 are satisfied. Then,

$$\sup_{t\in[0,\tau]} \left| \widehat{\psi}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \psi(t) \right| \to 0 \ a.s.$$

as $n \to \infty$.

Proof. Uniform consistency of $\overline{B}_n(t)$ can be shown following the similar argument of proving the uniform consistency of $B_n(t)$, so it suffices to show the consistency of the numerator $\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$. Note that $\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) = \overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0) + \overline{C}_n(t)$. The expectation of $\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)$ equals

$$
\begin{aligned}
E^*\{\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\} &= E^*\{(1-\Delta)K_h(t-X)\exp(\boldsymbol{\beta}_0^T\mathbf{Z}+\boldsymbol{\gamma}_0^T\mathbf{U})\} \\
&= E^*\{K_h(t-T)\exp(\boldsymbol{\beta}_0^T\mathbf{Z}+\boldsymbol{\gamma}_0^T\mathbf{U})|C\le T\}P^*(C\le T) \\
&= \frac{1-\pi_0}{h}\int\int\int \exp(\boldsymbol{\beta}_0^T\mathbf{z}+\boldsymbol{\gamma}_0^T\mathbf{u})K\left(\frac{t-x}{h}\right) f_{C,\mathbf{Z},\mathbf{U}|C\le T}(x,\mathbf{z},\mathbf{u})dxd\mathbf{z}d\mathbf{u} \\
&= \frac{1-\pi_0}{1-p_0}\frac{1}{h}\int\int\int \exp(\boldsymbol{\beta}_0^T\mathbf{z}+\boldsymbol{\gamma}_0^T\mathbf{u})K\left(\frac{t-x}{h}\right)\frac{f_C(x)S_T(x)f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})}{P(T\le C)}dxd\mathbf{z}d\mathbf{u} \\
&= \frac{1}{h}\int K\left(\frac{t-x}{h}\right)A(x)dx.
\end{aligned}
$$

Using the same arguments as in Lemma 1.1 and 1.2 in Zhao et al. (2017), we conclude that

$$
\sup_{t\in[0,\tau]}\left|\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)-\overline{A}(t)\right| \to 0 \ a.s. \tag{A.3}
$$

as $n\to 0$, given $K(.)$ has bounded variation, and $\mathbf{Z}$ and $\mathbf{U}$ are both bounded almost surely. By using Taylor's expansion, the second term $\overline{C}_n(t)$ can written as

$$
\begin{aligned}
\overline{C}_n(t) &= \frac{1}{n}\sum_{i=1}^n\{\exp(\hat{\boldsymbol{\theta}}^T\mathbf{V}_i)-\exp(\boldsymbol{\theta}_0^T\mathbf{V}_i)\}(1-\Delta_i)K_h(t-X_i) \\
&= (\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)^T\overline{\mathbf{D}}_n(t;\boldsymbol{\theta}_0)+o_p(1).
\end{aligned}
$$

The consistency of $\overline{\mathbf{D}}_n(t;\boldsymbol{\theta})$ can be established similarly as in the proof of Lemma 1.1, and by the strong consistency of $\hat{\boldsymbol{\theta}}$ and the continuous mapping theorem, we conclude that

$$
\sup_{t\in[0,\tau]}\left|\overline{C}_n(t)\right| \to 0 \ a.s. \tag{A.4}
$$

as $n\to 0$. Putting (A.3) and (A.4) together, we conclude that

$$
\sup_{t\in[0,\tau]}\left|\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})-\overline{A}(t)\right| \to 0 \ a.s.
$$

as $n \to 0$. Lemma 1.2 is proved.

Proof of Theorem 1. By applying Lemma 1.1 and Lemma 1.2, and using the continuous mapping theorem, we conclude that $\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}})$ and $\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ are consistent for $\phi_{adj}(t)$ uniformly on $t \in [0,\tau]$. As a weighted sum of $\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}})$ and $\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ with fixed weights, it is easy to prove $\widehat{\phi}_{adjw}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ is uniformly consistent for $\phi_{adj}(t)$ on $t \in [0,\tau]$. Theorem 1 is proved.

APPENDIX B: PROOF OF THEOREM 2: ASYMPTOTIC NORMALITY OF THE PROPOSED ESTIMATORS

Proof. Let $t \in [0,\tau]$. The asymptotic normality of $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \phi_{adj}(t)\}$ can be obtained using the same argument as in the proof of Theorem 2 in Zhao et al. (2017). Specifically, we have

$$\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \phi_{adj}(t)\} = \sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\} + \sqrt{nh}\{\widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0) - \phi_{adj}(t)\}$$

where the first part vanishes and the second part converges to a normal distribution as $n \to 0$.

Now we prove the asymptotic normality of $\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$. Write

$$\begin{aligned}\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})-\phi_{adj}(t)\} &= -\sqrt{nh}\{\widehat{\psi}^{-1}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\widehat{\upsilon}(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\}\\ &= -\sqrt{nh}\left\{\frac{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})}{\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})}-\psi^{-1}(t)\upsilon(t)\right\}\\ &= -\sqrt{nh}\left\{\frac{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})}{\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})}\right\}.\end{aligned}$$

The consistency of $\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ has been shown in the proof of Lemma 1.2, so it suffices to study the asymptotic normality of $\sqrt{nh}\{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\}$. Using the results in the previous proofs, we note that

$$\begin{aligned}&\sqrt{nh}\{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\}\\ =\ &\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0)-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\}+\sqrt{nh}\{\overline{F}_n(t)-\psi^{-1}(t)\upsilon(t)\overline{C}_n(t)\}\\ =\ &\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0)-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\}+o_p(1),\end{aligned}$$

so the asymptotic normality of $\sqrt{nh}\{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\}$ only depends on $\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0)-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\}$. Let

$$e_{n,i}(t)=\sqrt{\frac{h}{n}}\{\exp(\boldsymbol{\gamma}_0^T\mathbf{U}_i)-\psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{Z}_i+\boldsymbol{\gamma}_0\mathbf{U}_i)\}(1-\Delta_i)K_h(t-X_i),$$

so that $\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0)-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\}=\sum_{i=1}^n e_{n,i}(t)$. We will set forth the Lyapunov' conditions. Let $g(\mathbf{u},\mathbf{z};t)=\exp(\boldsymbol{\gamma}_0^T\mathbf{u})-\psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{z}+\boldsymbol{\gamma}_0^T\mathbf{u})$. Then we have

$$\begin{aligned}
E^*\{e_{n,i}(t)\} &= \sqrt{\frac{h}{n}} E^* \left[\{\exp(\boldsymbol{\gamma}_0^T \mathbf{U}) - \psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T \mathbf{Z} + \boldsymbol{\gamma}_0^T \mathbf{U})\}(1-\Delta)K_h(t-X) \right] \\
&= \frac{1-\pi_0}{\sqrt{nh}} \int\int\int g(\mathbf{u},\mathbf{z};t) K\left(\frac{t-x}{h}\right) f_{C,\mathbf{Z},\mathbf{U}|C\le T}(x,\mathbf{z},\mathbf{u}) dx d\mathbf{z} d\mathbf{u} \\
&= \frac{1-\pi_0}{\sqrt{nh}(1-p_0)} \int\int\int g(\mathbf{u},\mathbf{z};t) K\left(\frac{t-x}{h}\right) f_C(x) S_{T|\mathbf{Z},\mathbf{U}}(x|\mathbf{z},\mathbf{u}) f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u}) dx d\mathbf{z} d\mathbf{u} \\
&= \frac{1-\pi_0}{1-p_0}\sqrt{\frac{h}{n}} \int\int\int g(\mathbf{u},\mathbf{z};t) K(y) f_C(t-yh) S_{T|\mathbf{Z},\mathbf{U}}(t-yh|\mathbf{z},\mathbf{u}) f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u}) dy d\mathbf{z} d\mathbf{u} \\
&\quad \textit{by letting } y = \frac{t-x}{h}, \\
&= \frac{1-\pi_0}{1-p_0}\sqrt{\frac{h}{n}} \left\{ \int\int g(\mathbf{u},\mathbf{z};t) f_C(t) S_{T|\mathbf{Z},\mathbf{U}}(t|\mathbf{z},\mathbf{u}) f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u}) d\mathbf{z} d\mathbf{u} + O(h^2) \right\} \\
&\quad \textit{by the Taylor's expansion}, \\
&= \frac{1-\pi_0}{1-p_0}\sqrt{\frac{h}{n}} O(h^2) = O(\sqrt{h^5/n}).
\end{aligned}$$

The last step comes from the fact that

$$\begin{aligned}
&\int\int g(\mathbf{u},\mathbf{z};t) f_C(t) S_{T|\mathbf{Z},\mathbf{U}}(t|\mathbf{z},\mathbf{u}) f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u}) d\mathbf{z} d\mathbf{u} \\
= &\int\int \{\exp(\boldsymbol{\gamma}_0^T \mathbf{u}) - \psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T \mathbf{z} + \boldsymbol{\gamma}_0^T \mathbf{u})\} f_C(t) S_{T|\mathbf{Z},\mathbf{U}}(t|\mathbf{z},\mathbf{u}) f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u}) d\mathbf{z} d\mathbf{u} \\
= &f_C(t) S_T(t) \int\int \{\exp(\boldsymbol{\gamma}_0^T \mathbf{u}) - \psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T \mathbf{z} + \boldsymbol{\gamma}_0^T \mathbf{u})\} f_{\mathbf{Z},\mathbf{U}|T\ge t}(\mathbf{z},\mathbf{u}) d\mathbf{z} d\mathbf{u} \\
= &f_C(t) S_T(t) \{\upsilon(t) - \psi^{-1}(t)\upsilon(t)\cdot\psi(t)\} = 0.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
E^*\{e_{n,i}^2(t)\} &= \frac{h}{n}E^*\left[\{\exp(\boldsymbol{\gamma}_0^T\mathbf{U}) - \psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{Z}+\boldsymbol{\gamma}_0^T\mathbf{U})\}^2(1-\Delta)^2K_h^2(t-X)\right] \\
&= \frac{1-\pi_0}{nh}\int\int\int g^2(\mathbf{u},\mathbf{z};t)K^2\left(\frac{t-x}{h}\right)f_{C,\mathbf{Z},\mathbf{U}|C\le T}(x,\mathbf{z},\mathbf{u})dxd\mathbf{z}d\mathbf{u} \\
&= \frac{1-\pi_0}{nh(1-p_0)}\int\int\int g^2(\mathbf{u},\mathbf{z};t)K^2\left(\frac{t-x}{h}\right)f_C(x)S_{T|\mathbf{Z},\mathbf{U}}(x|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})dxd\mathbf{z}d\mathbf{u} \\
&= \frac{1-\pi_0}{n(1-p_0)}\int\int\int g^2(\mathbf{u},\mathbf{z};t)K^2(y)f_C(t-yh)S_{T|\mathbf{Z},\mathbf{U}}(t-yh|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})dyd\mathbf{z}d\mathbf{u} \\
&\quad \textit{by letting } y=\frac{t-x}{h}, \\
&= \frac{1-\pi_0}{n(1-p_0)}\left\{\int\int g^2(\mathbf{u},\mathbf{z};t)f_C(t)S_{T|\mathbf{Z},\mathbf{U}}(t|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})d\mathbf{z}d\mathbf{u}\int K^2(y)dy + O(h)\right\} \\
&\quad \textit{by the Taylor's expansion},
\end{aligned}
$$

and

$$
\begin{aligned}
E^*\{e_{n,i}^3(t)\} &= \left(\frac{h}{n}\right)^{3/2}E^*\left[\{\exp(\boldsymbol{\gamma}_0^T\mathbf{U}) - \psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{Z}+\boldsymbol{\gamma}_0^T\mathbf{U})\}^3(1-\Delta)^3K_h^3(t-X)\right] \\
&= \frac{1-\pi_0}{1-p_0}\left(\frac{1}{nh}\right)^{3/2}\int\int\int g^3(\mathbf{u},\mathbf{z};t)K^3\left(\frac{t-x}{h}\right)f_{C,\mathbf{Z},\mathbf{U}|C\le T}(x,\mathbf{z},\mathbf{u})dxd\mathbf{z}d\mathbf{u} \\
&= \frac{1-\pi_0}{1-p_0}\left(\frac{1}{nh}\right)^{3/2}\int\int\int g^3(\mathbf{u},\mathbf{z};t)K^3\left(\frac{t-x}{h}\right)f_C(x)S_{T|\mathbf{Z},\mathbf{U}}(x|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})dxd\mathbf{z}d\mathbf{u} \\
&= \frac{1-\pi_0}{n\sqrt{nh}(1-p_0)}\int\int\int g^3(\mathbf{u},\mathbf{z};t)K^3(y)f_C(t-yh)S_{T|\mathbf{Z},\mathbf{U}}(t-yh|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})dyd\mathbf{z}d\mathbf{u} \\
&\quad \textit{by letting } y=\frac{t-x}{h}, \\
&= \frac{1-\pi_0}{n\sqrt{nh}(1-p_0)}\left\{\int\int g^3(\mathbf{u},\mathbf{z};t)f_C(t)S_{T|\mathbf{Z},\mathbf{U}}(t|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})d\mathbf{z}d\mathbf{u}\int K^3(y)dy + O(h)\right\} \\
&\quad \textit{by the Taylor's expansion.}
\end{aligned}
$$

Given that

$$\begin{aligned}\sum_{i=1}^{n} E^*\{e_{n,i}(t) - E^*e_{n,i}(t)\}^3 &= nE^*\{e_{n,i}(t)^3\} - 3nE^*\{e_{n,i}(t)^2\}E^*\{e_{n,i}(t)\} + 2n\left[E^*\{e_{n,i}(t)\}\right]^3 \\ &= O(1/\sqrt{nh}) + 3nO(1/n)\cdot O(\sqrt{h^5/n}) + 2nO((h/n)^{3/2}) \\ &= O(1/\sqrt{nh}),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{n} Var^*\{e_{n,i}(t)\} = nE^*\{e_{n,i}^2(t)\} - n[E^*\{e_{n,i}(t)\}]^2 \\ =\ & \frac{(1-\pi_0)}{(1-p_0)}\int\int g^2(\mathbf{u},\mathbf{z};t)f_C(t)S_{T|\mathbf{Z},\mathbf{U}}(t|\mathbf{z},\mathbf{u})f_{\mathbf{Z},\mathbf{U}}(\mathbf{z},\mathbf{u})d\mathbf{z}d\mathbf{u}\int K^2(y)dy + O(h) + O(h^5) \\ =\ & \sigma_E^2(t) + O(h),\end{aligned}$$

where

$$\sigma_E^2(t) = \overline{B}(t)\int K^2(u)du\int\int\{\exp(\boldsymbol{\gamma}_0^T\mathbf{u}) - \psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{z} + \boldsymbol{\gamma}_0^T\mathbf{u})\}^2 f_{\mathbf{Z},\mathbf{U}|T\geq t}(\mathbf{z},\mathbf{u})d\mathbf{z}d\mathbf{u},$$

the Lyapunov's condition satisfies by verifying

$$\frac{\sum_{i=1}^n E^*\{e_{n,i}(t) - E^*e_{n,i}(t)\}^3}{[\sum_{i=1}^n Var^*\{e_{n,i}(t)\}]^{3/2}} = O(1/\sqrt{nh}) \to 0$$

as $\sqrt{nh} \to \infty$. Then by Lyapunov's Central Limit theorem and Slutsky's theorem, we conclude that

$$\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0) - \psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\} \to_d N(0,\sigma_E^2(t)),$$

and thereafter

$$\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\} \to_d N(0,\sigma_-^2(t)),$$

where $\sigma_-^2(t) = \sigma_E^2(t)\overline{A}^{-2}(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)$.

Write

$$\begin{aligned}
& \sqrt{nh}\{\widehat{\phi}_{adjw}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\} \\
= \; & \pi_0\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \phi_{adj}(t)\} + (1-\pi_0)\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\}. \qquad \text{(A.5)} \\
= \; & \pi_0[\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\} + \sqrt{nh}\{\widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0) - \phi_{adj}(t)\}] \qquad \text{(A.6)} \\
& -(1-\pi_0)\left[\frac{\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0) - \psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\} + \sqrt{nh}\{\overline{F}_n(t) - \psi^{-1}(t)\upsilon(t)\overline{C}_n(t)\}}{\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})}\right].
\end{aligned}$$

The asymptotic normality of the two parts in (A.5) has been studied. Now we study the covariance between them. We have shown in the previous proofs that the asymptotic normality of $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \phi_{adj}(t)\}$ is determined by $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0) - \phi_{adj}(t)\}$ which only relies on the cases, and that of $\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\}$ is determined by $\sqrt{nh}\{\overline{E}_n(t;\boldsymbol{\gamma}_0) - \psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\}$ which only relies the controls. Therefore they are asymptotically independent. By multivariate central limit theorem,

$$\sqrt{nh}\{\widehat{\phi}_{adjw}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\} \to_d N(0,\sigma_w^2(t)),$$

where

$$\sigma_w^2(t) = \pi_0^2\sigma_+^2(t) + (1-\pi_0)^2\sigma_-^2(t).$$

Theorem 2 is proved.

APPENDIX C: PROOF OF THEOREM 3: VARIANCE ESTIMATORS FOR THE PROPOSED ESTIMATORS

Proof. Using results in the proofs of Theorem 1 and Theorem 2 and with similar arguments, it is straightforward to show that $\widehat{\sigma}_{+}^{*2}(t;h)$, $\widehat{\sigma}_{-}^{*2}(t;h)$ and $\widehat{\sigma}_{w}^{*2}(t;h)$ are uniformly consistent for $\sigma_{+}^{2}(t)$, $\sigma_{-}^{2}(t)$ and $\sigma_{w}^{2}(t)$. Below we show how to derive the variance estimators under finite sampling $\widehat{\sigma}_{+}^{*2}(t;h)$, $\widehat{\sigma}_{-}^{*2}(t;h)$, and $\widehat{\sigma}_{w}^{*2}(t;h)$.

Note that in equation (A.6), we have

$$\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\} = \sqrt{h}\{\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T\}\mathbf{D}_n(t;\boldsymbol{\beta}_0)/B_n(t) + O_p(\sqrt{h/n}),$$

in which the first leading term goes to 0 in $O(\sqrt{h})$ and under finite sampling does not converge to 0 quickly. So the variance under finite sampling would be approximately $\sigma_{+}^{2}(t)$ plus a correction term from the variance of $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\}$ and the covariance between $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\}$ and $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0) - \phi_{adj}(t)\}$.
Using similar arguments in the proof of Theorem 2, it is straightforward to see that the limiting variance of $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\}$ equals to $\sigma_C^2(t;h)B^{-2}(t)$ and the limiting covariance between $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0)\}$ and $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\boldsymbol{\beta}_0) - \phi_{adj}(t;\boldsymbol{\beta}_0)\}$ is $\sigma_{AC}(t;h)B^{-2}(t)$, where

$$\begin{aligned}
\sigma_C^2(t;h) &= h\mathbf{D}(t;\boldsymbol{\beta}_0)^T I^{-1}(\boldsymbol{\beta}_0)\mathbf{D}(t;\boldsymbol{\beta}_0), \\
\sigma_{AC}(t;h) &= h\mathbf{D}(t;\boldsymbol{\beta}_0)^T E\left[\Delta_i K_h(t - X_i)\{1 - \phi_{adj}(t) - exp(-\boldsymbol{\beta}_0\mathbf{Z}_i)\}\mathbf{l}_{\boldsymbol{\beta}}(X_i)\right],
\end{aligned}$$

where $\mathbf{l}_{\boldsymbol{\beta}}(X_i)$ is the efficient influence function for $\boldsymbol{\beta}$ in the logistic model. We substitute the parameters and expectations with respective (empirical) estimators to obtain $\widehat{\sigma}_{+}^{*2}(t;h)$.

Similarly we can estimate the variance of $\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}};\hat{\boldsymbol{\gamma}})$ with finite sampling correction. Recall that

$$\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})-\phi_{adj}(t)\} \;=-\sqrt{nh}\{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\}\overline{A}_n^{-1}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}).$$

Let

$$\begin{aligned}
\widetilde{E}_n(t;\boldsymbol{\beta},\boldsymbol{\gamma}) &= \overline{E}_n(t;\boldsymbol{\gamma})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\boldsymbol{\beta},\boldsymbol{\gamma})\\
&= \frac{1}{n}\sum_{i=1}^{n}(1-\Delta_i)K_h(t-X_i)\{\exp(\boldsymbol{\gamma}^T\mathbf{U}_i)-\psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}^T\mathbf{Z}_i+\boldsymbol{\gamma}^T\mathbf{U}_i)\},\\
\widetilde{F}_n(t) &= \widetilde{E}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})-\widetilde{E}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)=\overline{F}_n(t)-\psi^{-1}(t)\upsilon(t)\overline{C}_n(t).
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{nh}\{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\} &= \sqrt{nh}\widetilde{E}_n(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)+\sqrt{nh}\widetilde{F}_n(t)\\
&= \sum_{i=1}^{n}e_{n,i}(t)+\sum_{i=1}^{n}f_{n,i}(t)+o_p(1),
\end{aligned}$$

where

$$\begin{aligned}
e_{n,i}(t) &= \sqrt{\frac{h}{n}}(1-\Delta_i)K_h(t-X_i)\{\exp(\boldsymbol{\gamma}_0^T\mathbf{U}_i)-\psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{Z}_i+\boldsymbol{\gamma}_0^T\mathbf{U}_i)\},\\
f_{n,i}(t) &= \sqrt{\frac{h}{n}}\{\mathbf{l}_{\boldsymbol{\gamma}}(X_i)^T\overline{\mathbf{G}}(t;\boldsymbol{\gamma}_0)-\psi^{-1}(t)\upsilon(t)\mathbf{l}_{\boldsymbol{\theta}}(X_i)^T\overline{\mathbf{D}}(t;\boldsymbol{\theta}_0)\}.
\end{aligned}$$

Therefore, the limiting variance of $\sqrt{nh}\{\overline{E}_n(t;\hat{\boldsymbol{\gamma}})-\psi^{-1}(t)\upsilon(t)\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})\}/\overline{A}_n(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ under finite sampling is $\{\sigma_E^2(t)+\sigma_F^2(t;h)+2\sigma_{EF}(t;h)\}/\overline{A}^2(t)$, where

$$\begin{aligned}
\sigma_E^2(t) &= \overline{B}(t)\int K^2(u)du\int\int\{\exp(\boldsymbol{\gamma}_0^T\mathbf{u})-\psi^{-1}(t)\upsilon(t)\exp(\boldsymbol{\beta}_0^T\mathbf{z}+\boldsymbol{\gamma}_0^T\mathbf{u})\}^2 f_{\mathbf{Z},\mathbf{U}|T\geq t}(\mathbf{z},\mathbf{u})d\mathbf{z}d\mathbf{u},\\
\sigma_F^2(t;h) &= hE\{\mathbf{l}_{\boldsymbol{\gamma}}(X_i)^T\overline{\mathbf{G}}(t;\boldsymbol{\gamma}_0)-\psi^{-1}(t)\upsilon(t)\mathbf{l}_{\boldsymbol{\theta}}(X_i)^T\overline{\mathbf{D}}(t;\boldsymbol{\theta}_0)\}^2,\\
\sigma_{EF}(t;h) &= nE\{e_{n,i}(t)f_{n,i}(t)\}.
\end{aligned}$$

We substitute the parameters and expectations with respective (empirical) estimators to obtain $\widehat{\sigma}_{-}^{*2}(t;h)$.

To obtain the limiting variance of $\widehat{\phi}_{adjw}(t;\hat{\boldsymbol{\beta}};\hat{\boldsymbol{\gamma}})$ under finite sampling, write

$$\begin{aligned} & \sqrt{nh}\{\widehat{\phi}_{adjw}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\} \\ = \; & \pi_0\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \phi_{adj}(t)\} + (1-\pi_0)\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\}. \end{aligned}$$

The variances of these two terms under finite sampling have been studied. Write the components into asymptotic *i.i.d.* forms

$$\begin{aligned} \sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}}) - \phi_{adj}(t)\} &= \sum_{i=1}^{n} a_{n,i}(t)B^{-1}(t) + \sum_{i=1}^{n} c_{n,i}(t)B^{-1}(t) + o_p(1), \\ \sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}}) - \phi_{adj}(t)\} &= -\sum_{i=1}^{n} e_{n,i}(t)\overline{A}^{-1}(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0) - \sum_{i=1}^{n} f_{n,i}(t)\overline{A}^{-1}(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0) + o_p(1), \end{aligned}$$

where $a_{n,i}(t) = \sqrt{h/n}\{1-\phi_{adj}(t)-\exp(-\boldsymbol{\beta}_0^T\mathbf{Z}_i)\}\Delta_i K_h(t-X_i)$ and $c_{n,i}(t) = \sqrt{h/n}\mathbf{l}_{\boldsymbol{\beta}}(X_i)^T\mathbf{D}(t)$. Then we obtain the covariance between $\sqrt{nh}\{\widehat{\phi}_{adj+}(t;\hat{\boldsymbol{\beta}})-\phi_{adj}(t)\}$ and $\sqrt{nh}\{\widehat{\phi}_{adj-}(t;\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})-\phi_{adj}(t)\}$ as

$$\begin{aligned} -\Big\{ & \sigma_{AF}(t)\overline{A}^{-1}(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)B^{-1}(t) + \sigma_{CE}(t)\overline{A}^{-1}(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)B^{-1}(t) \\ & + \sigma_{CF}(t)\overline{A}^{-1}(t;\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)B^{-1}(t)\Big\}, \end{aligned}$$

where $\sigma_{AF}(t) = nE\{a_{n,i}(t)f_{n,i}(t)\}$, $\sigma_{CE}(t) = nE\{c_{n,i}(t)e_{n,i}(t)\}$ and $\sigma_{CF}(t) = nE\{c_{n,i}(t)f_{n,i}(t)\}$. We substitute the parameters and expectations with respective (empirical) estimators to obtain $\widehat{\sigma}_{w}^{*2}(t;h)$.

APPENDIX D: SUPPLEMENTARY RESULTS FOR THE SIMULATION STUDIES AND REAL DATA EXAMPLE

The additional results for the simulations and real data example are presented in Table A.1-A.9 and Figure A.1.

APPENDIX E: THE TRUE VALUES OF $\phi_{adj}(t)$ AND $\Phi_{adj}(t)$ UNDER VARIOUS PARAMETER SPECIFICATIONS

To check the sensitivity of the proposed $\phi_{adj}(t)$ against the rare disease assumption, we calculated differences between $\phi_{adj}(t)$ and $\Phi_{adj}(t)$ numerically under a wide range of scenarios. The results show that when disease probability $\Pr(T < 70)$ is below 5%, the approximation is mostly satisfied with absolute difference below 0.05. When the disease probability increases, additional parameter restrictions need to be imposed to maintain a good approximation.

We consider the Cox model $\lambda(t|Z,U) = \lambda_0(t)\exp(\beta Z + \gamma U)$ with a Weibull baseline hazard $\lambda_0(t) = (\nu/\eta)(t/\eta)^{\nu-1}$, a dichotomous exposure Z and a dichotomous confounder U. Let $P_Z = \Pr\{Z = 1\}$, $P_U = \Pr\{U = 1\}$, and $P_{ZU} = \Pr\{Z = 1, U = 1\}$ at baseline. Specifically, we set parameters as the combination of the following values: $\eta = 90, 180, 360, 720$, $\nu = 2$, $\beta = 0.5, 1, 2, 4, 8$, $\gamma = 0.5, 1, 2, 4, 8$, $P_Z = 0.05, 0.1, 0.2, 0.4, 0.8$, $P_U = 0.05, 0.1, 0.2, 0.4, 0.8$, and $D' = -1, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, 1$, where D' is the scaled correlation used for association of two binary variables (genetic variants) (Slatkin 2008) and here we use it for the association of Z and U, because it has a full range from -1 to 1. With the inclusion constraint of $P_Z + P_U - P_{ZU} \leq 1$ and $\Phi_{adj}(t) \geq 0$ at selected ages, a total of 17,581 scenarios are considered. For each scenario, we calculated $\phi_{adj}(t)$, $\Phi_{adj}(t)$, and the difference $\phi_{adj}(t) - \Phi_{adj}(t)$ at $t = 30, 50, 70$, respectively.

A total of 2,167 scenarios have the disease probability below 5%. Among them, only 48 scenarios have absolute difference above 0.05. There scenarios have the feature of $\{\beta \leq 2, \gamma = 4, P_U = 0.05\}$. For the remaining 2119 scenarios, we generated the scatter plots for difference $\phi_{adj}(t) - \Phi_{adj}(t)$ versus $\Phi_{adj}(t)$ in Figure A.2, in which each point represents one scenario. From Figure A.2, the vast majority of scenarios have absolute difference very close to 0.

A total of 2,121 scenarios have the disease probability within the range of 5% to 10%. Among them, 180 scenarios having absolute difference above 0.05. There scenarios have the feature of $\{\beta = 0.05, \gamma \geq 4, P_U \leq 0.1\}$, $\{1 \leq \beta \leq 2, \gamma \geq 2, P_U \leq 0.1\}$, $\{\beta = 4, \gamma \geq 2, P_Z \leq 0.1, P_U \leq 0.2\}$ or $\{\beta = 8, \gamma = 8, P_Z = 0.05, P_U = 0.05\}$.

A total of 2,269 scenarios have the disease probability within the range of 10% to 20%. Among them, 321 scenarios having absolute difference above 0.05. There scenarios have the feature of $\{\beta = 0.05, \gamma \geq 4, P_U \leq 0.2\}$, $\{1 \leq \beta \leq 2, \gamma \geq 2, P_U \leq 0.4\}$, $\{\beta = 4, \gamma \geq 1, P_Z \leq 0.4, P_U \leq 0.4\}$ or $\{\beta = 8, \gamma \geq 2, P_Z \leq 0.1\}$.

In practice, if the disease probability is common, one should examine the parameter estimates. Generally speaking, if the effects of confounders and/or exposure are large, there might be a discrepancy between $\phi_{adj}(t)$ and $\Phi_{adj}(t)$ and one should be cautious to use $\phi_{adj}(t)$ as an alternative.

References

Slatkin, M. (2008), 'Linkage disequilibrium—understanding the evolutionary past and mapping the medical future', *Nature Reviews Genetics* **9**(6), 477.

Zhao, W., Chen, Y. Q. & Hsu, L. (2017), 'On estimation of time-dependent attributable fraction from population-based case-control studies', *Biometrics* **73**(3), 866–875.

Table A.1: Summary statistics for $\hat{\phi}_{adjw}(t;\hat{\beta},\hat{\gamma})$ with different bandwidth. h_{CV}: bandwidth from cross-validation. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: empirical standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals.

Scenario I: binary U												
	Age(yrs)	h_{CV}	h=1	h=2	h=3	h=4	h=5	h=6	h=7	h=8	h=9	h=10
Bias	30	0.000	-0.002	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
	50	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
	70	0.000	-0.002	-0.003	-0.002	-0.002	-0.001	-0.001	-0.001	0.000	0.000	0.000
ESD	30	0.025	0.044	0.035	0.032	0.030	0.028	0.028	0.027	0.026	0.026	0.026
	50	0.025	0.046	0.036	0.032	0.030	0.029	0.028	0.027	0.026	0.026	0.026
	70	0.028	0.063	0.047	0.040	0.036	0.034	0.032	0.031	0.030	0.029	0.028
ASE	30	0.026	0.043	0.035	0.031	0.029	0.028	0.027	0.027	0.026	0.026	0.026
	50	0.026	0.046	0.036	0.032	0.030	0.029	0.028	0.027	0.026	0.026	0.026
	70	0.028	0.062	0.047	0.040	0.036	0.034	0.032	0.031	0.030	0.029	0.028
CR(%)	30	95.4	94.2	95.0	94.8	94.7	95.0	95.3	95.3	95.2	95.0	95.2
pointwise	50	95.1	94.0	94.8	95.1	94.8	94.8	94.3	94.5	94.8	94.6	94.5
	70	95.4	93.6	94.5	94.7	94.9	95.4	95.3	95.6	95.1	95.1	95.4
Scenario II: continuous U												
	Age(yrs)	h_{CV}	h=1	h=2	h=3	h=4	h=5	h=6	h=7	h=8	h=9	h=10
Bias	30	0.000	-0.001	-0.001	0.000	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
	50	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	0.000	0.000	0.000	0.000	0.000
	70	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	0.000	0.000	0.000	0.000
ESD	30	0.026	0.046	0.035	0.032	0.030	0.028	0.028	0.027	0.026	0.026	0.026
	50	0.026	0.045	0.035	0.031	0.029	0.028	0.027	0.026	0.026	0.025	0.025
	70	0.028	0.057	0.043	0.037	0.034	0.032	0.031	0.030	0.029	0.028	0.028
ASE	30	0.026	0.045	0.036	0.032	0.030	0.029	0.028	0.027	0.027	0.026	0.026
	50	0.026	0.045	0.036	0.032	0.030	0.029	0.028	0.027	0.026	0.026	0.026
	70	0.027	0.056	0.043	0.037	0.034	0.032	0.031	0.029	0.029	0.028	0.027
CR(%)	30	94.3	93.9	95.0	95.9	95.3	95.3	95.1	94.9	95.0	95.0	95.2
pointwise	50	94.9	94.5	94.8	95.5	95.8	95.7	95.9	95.7	95.5	95.6	95.6
	70	94.7	93.8	95.5	95.1	94.8	94.5	94.7	94.6	94.7	95.0	95.5

Table A.2: Estimates and the 95% confidence intervals of $\phi_{adj}(t)$ and the model-based adjusted PAF for the colorectal cancer case-control study

	History of diabetes		Obesity	
Parameter	Estimate	95% CI	Estimate	95% CI
$\hat{\phi}_{adjw}(50)$	0.020	0.005, 0.035	0.070	0.005, 0.035
$\hat{\phi}_{adjw}(60)$	0.025	0.012, 0.038	0.050	0.012, 0.038
$\hat{\phi}_{adjw}(70)$	0.028	0.014, 0.042	0.044	0.014, 0.042
$\hat{\phi}_{adjw}(80)$	0.028	0.014, 0.042	0.037	0.014, 0.042
$\hat{\phi}_{adjw}(90)$	0.025	0.008, 0.041	0.032	0.008, 0.041
$\widehat{PAF}_{adj}$	0.027	-0.002, 0.056	0.043	0.002, 0.083
	Year-since-quit-smoking		Pack year	
Parameter	Estimate	95% CI	Estimate	95% CI
$\hat{\phi}_{adjw}(50)$	0.058	-0.032, 0.149	0.151	0.076, 0.226
$\hat{\phi}_{adjw}(60)$	0.045	-0.024, 0.114	0.151	0.079, 0.223
$\hat{\phi}_{adjw}(70)$	0.037	-0.020, 0.094	0.153	0.081, 0.226
$\hat{\phi}_{adjw}(80)$	0.030	-0.017, 0.078	0.144	0.075, 0.213
$\hat{\phi}_{adjw}(90)$	0.023	-0.015, 0.061	0.137	0.067, 0.207
$\widehat{PAF}_{adj}$	0.037	-0.029, 0.104	0.150	0.074, 0.226

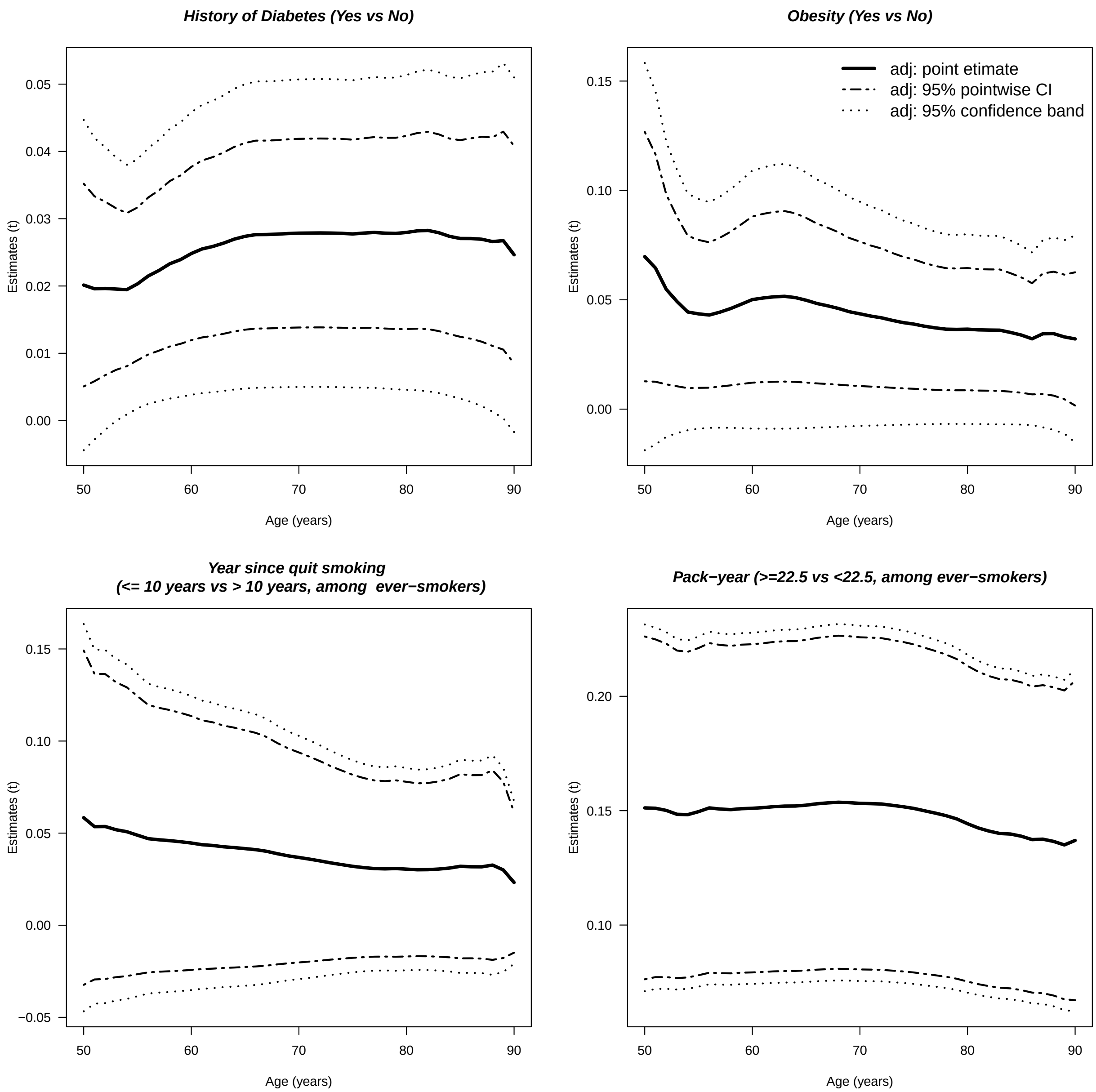

Figure A.1: The simultaneous confidence bands calculated based on bootstrap with 200 replicates and the pointwise confidence intervals of $\phi_{adj}(t)$ for GECCO study

Table A.3: Summary statistics of the $\phi_{adj}(t)$ estimators under scenario I and II for 70% censoring and equal number of cases and controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: empirical standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

	Scenario I: binary U					Scenario II: continuous U			
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	30	0.509	0.507			30	0.512	0.510	
	50	0.491	0.486			50	0.497	0.492	
	70	0.462	0.450			70	0.473	0.463	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	30	-0.001	-0.003	-0.002	Bias	30	0.000	-0.003	-0.001
	50	-0.001	-0.003	-0.002		50	-0.001	-0.002	-0.001
	70	-0.001	-0.002	-0.001		70	-0.001	-0.001	-0.001
ESD	30	0.027	0.030	0.025	ESD	30	0.028	0.034	0.027
	50	0.026	0.032	0.025		50	0.027	0.032	0.026
	70	0.028	0.037	0.027		70	0.028	0.036	0.027
ASE	30	0.027	0.031	0.026	ASE	30	0.028	0.032	0.027
	50	0.027	0.030	0.025		50	0.027	0.031	0.026
	70	0.028	0.034	0.026		70	0.027	0.033	0.026
CR(%)	30	96.0	95.4	95.9	CR(%)	30	95.2	94.8	95.4
pointwise	50	95.6	95.2	95.2	pointwise	50	95.6	94.9	95.7
	70	94.6	93.3	94.0		70	94.3	94.7	95.2
CR(%)	20:70	94.8	94.9	95.0	CR(%)	20:70	93.9	94.4	95.6

Table A.4: Summary statistics of the $\phi_{adj}(t)$ estimators under scenario I and II for 90% censoring and equal number of cases and controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: empirical standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

	Scenario I: binary U					Scenario II: continuous U			
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	30	0.517	0.516			30	0.518	0.518	
	50	0.513	0.511			50	0.515	0.513	
	70	0.506	0.503			70	0.509	0.507	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	30	-0.001	-0.004	-0.001	Bias	30	-0.002	-0.004	-0.003
	50	0.000	-0.001	-0.001		50	-0.001	-0.003	-0.001
	70	0.000	-0.002	-0.001		70	-0.001	-0.003	-0.001
ESD	30	0.029	0.036	0.027	ESD	30	0.030	0.036	0.028
	50	0.027	0.032	0.027		50	0.028	0.032	0.027
	70	0.027	0.032	0.026		70	0.028	0.033	0.026
ASE	30	0.029	0.033	0.027	ASE	30	0.030	0.035	0.028
	50	0.027	0.030	0.026		50	0.028	0.031	0.026
	70	0.028	0.031	0.026		70	0.027	0.031	0.026
CR(%)	30	94.2	93.3	94.2	CR(%)	30	95.6	94.2	93.7
pointwise	50	94.7	93.7	94.6	pointwise	50	94.2	94.8	94.5
	70	95.2	94.0	95.3		70	94.4	94.4	94.5
CR(%)	20:70	94.4	93.9	94.3	CR(%)	20:70	94.5	93.4	93.5

Table A.5: Summary statistics of the $\phi_{adj}(t)$ estimators under scenario I and II for 95% censoring and equal number of cases and controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: empirical standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

	Scenario I: binary U					Scenario II: continuous U			
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	30	0.517	0.516			30	0.518	0.518	
	50	0.513	0.511			50	0.515	0.513	
	70	0.506	0.503			70	0.509	0.507	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	30	0.001	0.000	0.000	Bias	30	0.000	-0.001	-0.001
	50	0.000	-0.001	-0.001		50	0.000	-0.001	-0.001
	70	0.000	-0.002	0.000		70	0.001	-0.002	0.000
ESD	30	0.028	0.031	0.026	ESD	30	0.028	0.032	0.027
	50	0.026	0.031	0.026		50	0.028	0.031	0.027
	70	0.029	0.039	0.028		70	0.031	0.035	0.028
ASE	30	0.027	0.030	0.026	ASE	30	0.028	0.031	0.027
	50	0.027	0.030	0.026		50	0.027	0.031	0.026
	70	0.030	0.035	0.028		70	0.029	0.034	0.027
CR(%)	30	93.7	95.1	94.8	CR(%)	30	94.0	95.0	94.3
pointwise	50	95.8	94.7	96.3	pointwise	50	95.2	93.9	94.8
	70	94.9	94.5	94.1		70	93.2	95.8	94.4
CR(%)	20:70	93.9	93.5	93.9	CR(%)	20:70	94.1	92.3	94.0

Table A.6: Summary statistics of the $\phi_{adj}(t)$ estimators under scenario I and II for 80% censoring and 1000 controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: empirical standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

Scenario I: binary U					Scenario II: continuous U				
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	30	0.509	0.507			30	0.512	0.510	
	50	0.491	0.486			50	0.497	0.492	
	70	0.462	0.450			70	0.473	0.463	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	30	0.000	-0.002	-0.001	Bias	30	-0.001	-0.003	-0.001
	50	0.000	-0.004	-0.001		50	-0.001	-0.003	-0.001
	70	0.000	-0.005	-0.001		70	-0.001	-0.005	-0.001
ESD	30	0.029	0.038	0.028	ESD	30	0.030	0.038	0.029
	50	0.029	0.042	0.028		50	0.029	0.040	0.029
	70	0.031	0.053	0.031		70	0.033	0.050	0.031
ASE	30	0.030	0.036	0.029	ASE	30	0.030	0.037	0.029
	50	0.029	0.037	0.029		50	0.030	0.037	0.029
	70	0.032	0.049	0.031		70	0.032	0.045	0.030
CR(%)	30	97.3	94.7	96.5	CR(%)	30	95.6	93.6	94.9
pointwise	50	95.8	94.2	95.8	pointwise	50	95.3	94.6	95.0
	70	96.5	94.2	95.4		70	95.3	95.0	95.0
CR(%)	20:70	96.0	95.1	95.4	CR(%)	20:70	96.0	94.6	95.3

Table A.7: Summary statistics of the $\phi_{adj}(t)$ estimators under scenario I and II for 80% censoring and 4000 controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: empirical standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

Scenario I: binary U					Scenario II: continuous U				
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	30	0.509	0.507			30	0.512	0.510	
	50	0.491	0.486			50	0.497	0.492	
	70	0.462	0.450			70	0.473	0.463	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	30	0.000	-0.001	-0.001	Bias	30	-0.002	-0.002	-0.002
	50	-0.002	-0.001	-0.001		50	-0.001	-0.002	-0.001
	70	-0.001	-0.004	-0.002		70	-0.002	-0.003	-0.002
ESD	30	0.025	0.027	0.024	ESD	30	0.026	0.027	0.025
	50	0.026	0.027	0.024		50	0.026	0.028	0.025
	70	0.031	0.034	0.027		70	0.029	0.033	0.027
ASE	30	0.025	0.026	0.024	ASE	30	0.026	0.026	0.024
	50	0.025	0.026	0.024		50	0.025	0.026	0.024
	70	0.029	0.031	0.026		70	0.028	0.030	0.026
CR(%)	30	94.7	93.9	94.6	CR(%)	30	94.0	94.6	93.6
pointwise	50	93.9	95.0	95.0	pointwise	50	93.7	93.6	93.6
	70	94.3	93.7	93.7		70	94.0	92.2	93.8
CR(%)	20:70	93.2	93.7	93.7	CR(%)	20:70	93.6	92.8	93.5

Table A.8: Summary statistics of the estimators from simulated datasets based on real data for 1250 controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: sampling standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

Diabetes as the exposure					Obesity as the exposure				
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	40	.0269	.0269			40	.0408	.0408	
	60	.0266	.0266			60	.0407	.0407	
	80	.0253	.0252			80	.0403	.0401	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	40	-0.0002	-0.0003	-0.0002	Bias	40	-0.0008	-0.0006	-0.0007
	60	-0.0002	-0.0003	-0.0002		60	-0.0006	-0.0005	-0.0006
	80	-0.0002	0.0002	-0.0001		80	-0.0005	-0.0010	-0.0006
ESD	40	0.0094	0.0106	0.0090	ESD	40	0.0213	0.0217	0.0213
	60	0.0083	0.0087	0.0083		60	0.0212	0.0212	0.0212
	80	0.0087	0.0101	0.0085		80	0.0211	0.0211	0.0210
ASE	40	0.0095	0.0107	0.0093	ASE	40	0.0209	0.0211	0.0208
	60	0.0087	0.0089	0.0086		60	0.0207	0.0208	0.0207
	80	0.0090	0.0102	0.0088		80	0.0207	0.0207	0.0205
CR(%)	40	95.1	92.9	95.1	CR(%)	40	94.5	95.1	94.8
pointwise	60	95.4	96.4	96.1	pointwise	60	94.7	94.4	94.4
	80	95.6	94.0	96.5		80	94.3	94.7	94.4
CR(%)	40:80	94.4	93.5	93.9	CR(%)	40:80	93.1	93.1	93.8

Table A.9: Summary statistics of the estimators from simulated datasets based on real data for 5000 controls. Bias: absolute difference between the true value of $\phi_{adj}(t)$ and the mean of the point estimator. ESD: sampling standard deviation. ASE: mean of asymptotic-based standard error estimates. CR pointwise: coverage rate of 95% pointwise confidence intervals. CR: coverage rate of 95% simultaneous confidence bands.

	Diabetes as the exposure					Obesity as the exposure			
	Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$			Age	$\phi_{adj}(t)$	$\Phi_{adj}(t)$	
	40	.0269	.0269			40	.0408	.0408	
	60	.0266	.0266			60	.0407	.0407	
	80	.0253	.0252			80	.0403	.0401	
	Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$		Age	$\widehat{\phi}_{adj+}$	$\widehat{\phi}_{adj-}$	$\widehat{\phi}_{adjw}$
Bias	40	-0.0005	-0.0004	-0.0004	Bias	40	-0.0006	-0.0005	-0.0005
	60	-0.0004	-0.0002	-0.0003		60	-0.0006	-0.0005	-0.0005
	80	0.0000	-0.0003	-0.0003		80	-0.0004	-0.0006	-0.0005
ESD	40	0.0077	0.0072	0.0072	ESD	40	0.0159	0.0157	0.0157
	60	0.0067	0.0066	0.0065		60	0.0156	0.0156	0.0156
	80	0.0078	0.0067	0.0066		80	0.0156	0.0153	0.0153
ASE	40	0.0076	0.0071	0.0069	ASE	40	0.0152	0.0151	0.0151
	60	0.0066	0.0065	0.0065		60	0.0150	0.0150	0.0150
	80	0.0072	0.0067	0.0066		80	0.0151	0.0149	0.0148
CR(%)	40	93.8	94.7	95.0	CR(%)	40	94.5	94.4	94.4
pointwise	60	95.2	95.2	95.3	pointwise	60	94.2	94.5	94.4
	80	93.3	94.5	94.5		80	94.5	93.9	94.8
CR(%)	40:80	94.1	93.6	93.1	CR(%)	40:80	93.8	93.1	93.9

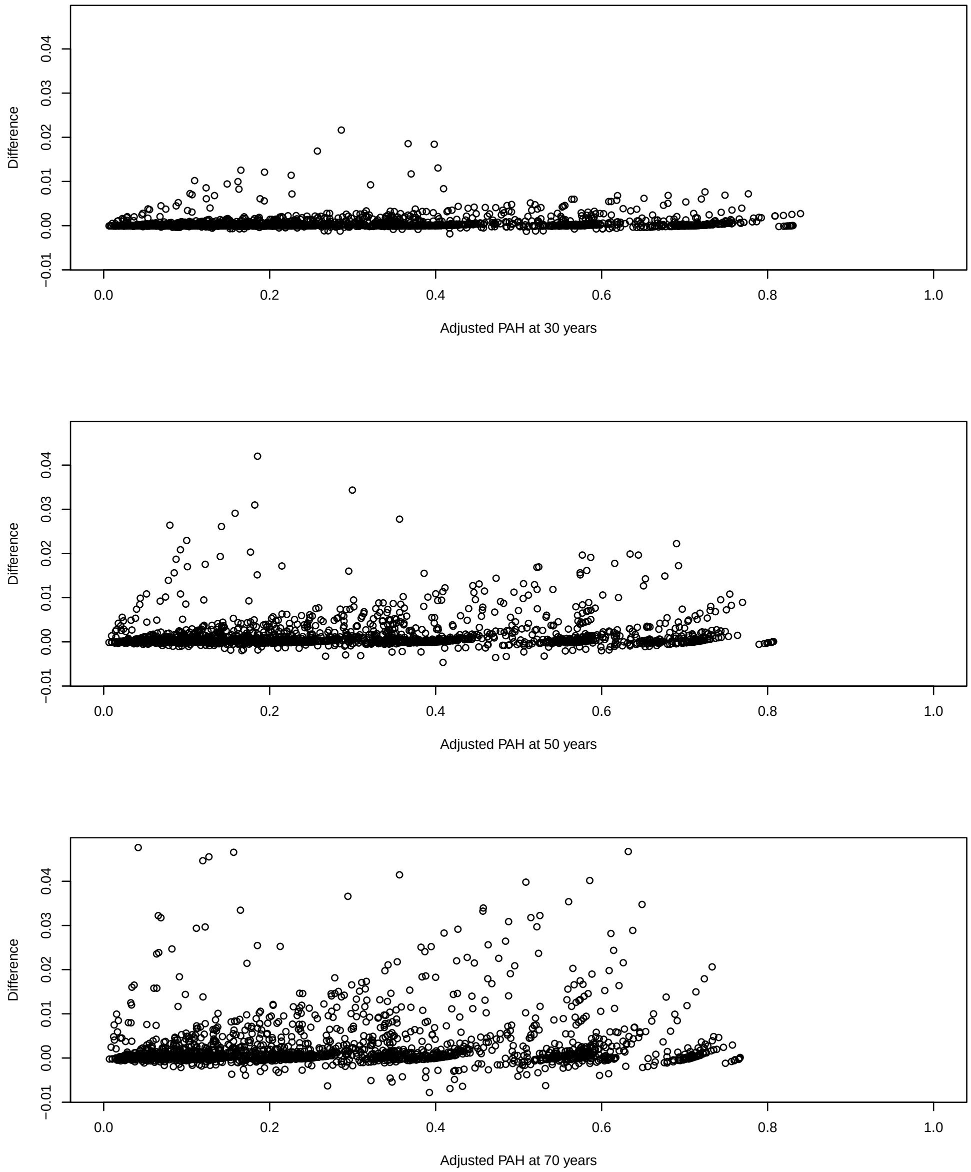

Figure A.2: The difference for the 2119 scenarios when the disease probability $pr\{T \leq 70\} \leq 5\%$. In each sub-graph, the horizontal axis is $\Phi_{adj}(t)$ and the vertical axis is the difference $\phi_{adj}(t) - \Phi_{adj}(t)$ at $t = 30, 50, 70$.