Supplementary Information for

Cultural evolution of conformity and anti-conformity

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Supporting Information Text

Supplementary Information A: Proof of Result 1

In defining $F_n(p)$ for all $j = k, k+1, \dots, n$ we take $j \geq \frac{n}{2} + 1$ when n is even and $j \geq \frac{n+1}{2}$ when n is odd. Therefore $2j > n$, and so

$$p^j(1-p)^{n-j} - p^{n-j}(1-p)^j = p^{n-j}(1-p)^{n-j}\left[p^{2j-n} - (1-p)^{2j-n}\right]. \tag{A1}$$

It is easily seen that for all $k = 1, 2, \dots,$

$$p^{k+1} - (1-p)^{k+1} = p^k - (1-p)^k - p(1-p)\left[p^{k-1} - (1-p)^{k-1}\right]. \tag{A2}$$

Also

$$p - (1-p) = p^2 - (1-p)^2 = (2p-1). \tag{A3}$$

Therefore, by induction, we deduce that $\left[p^k - (1-p)^k\right]$ for all $k = 1, 2, \dots$ is of the form $(2p-1)$ multiplied by a polynomial in $p(1-p)$. As

$$F_n(p) = \sum_{j=k}^{n} \frac{D(j)}{n}\binom{n}{j}\left[p(1-p)\right]^{n-j}\left[p^{2j-n} - (1-p)^{2j-n}\right], \tag{A4}$$

we conclude that $F_n(p) = (2p-1)G_n(p(1-p))$, and since $F_n(0) = F_n(1) = 0$ we have $G_n(0) = 0$. Note that since $D(n) = D(0) = 0$, $p(1-p)$ is a factor of $F_n(p)$.

Supplementary Information B: Proof of Result 2

Since there is symmetry in the model between the two cultural variants A and B, we analyze only the stability of $p^* = 0$, namely fixation in type B.

Let $p = \varepsilon$. Then from eq. (6), as $D(0) = D(n) = 0$, we have

$$\varepsilon' = \varepsilon + \sum_{j=1}^{n-1} \frac{D(j)}{n}\binom{n}{j}\varepsilon^j(1-\varepsilon)^{n-j}. \tag{B1}$$

Therefore the linear approximation of [B1] in terms of ε is

$$\varepsilon' = \varepsilon + \frac{D(1)}{n}\binom{n}{1}\varepsilon = \varepsilon\big[1 + D(1)\big]. \tag{B2}$$

Hence if $D(1) < 0$, then $p^* = 0$ is locally stable, whereas if $D(1) > 0$, $p^* = 0$ is not locally stable.

For the stability of $p^* = \frac{1}{2}$, we use eq. [A4], written as

$$p' = p + p(1-p)\sum_{j=k}^{n-1} \frac{D(j)}{n}\binom{n}{j}\left[p(1-p)\right]^{n-j-1}\left[p^{2j-n} - (1-p)^{2j-n}\right], \tag{B3}$$

where $k = \frac{n}{2} + 1$ when n is even and $k = \frac{n+1}{2}$ when n is odd.

Let $p = \frac{1}{2} + \varepsilon$, $1 - p = \frac{1}{2} - \varepsilon$, $p(1-p) = \frac{1}{4} - \varepsilon^2$, and $p' = \frac{1}{2} + \varepsilon'$. Then

$$\varepsilon' = \varepsilon + \left(\tfrac{1}{4} - \varepsilon^2\right)\sum_{j=k}^{n-1} \frac{D(j)}{n}\binom{n}{j}\left(\tfrac{1}{4} - \varepsilon^2\right)^{n-j-1}\left[\left(\tfrac{1}{2} + \varepsilon\right)^{2j-n} - \left(\tfrac{1}{2} - \varepsilon\right)^{2j-n}\right]. \tag{B4}$$

Expand $\left(\frac{1}{2} \pm \varepsilon\right)^{2j-n}$. Then since $(2j-n) \geq 1$, up to non-linear terms in ε, we have

$$\left(\tfrac{1}{2} \pm \varepsilon\right)^{2j-n} \simeq \left(\tfrac{1}{2}\right)^{2j-n} \pm (2j-n)\left(\tfrac{1}{2}\right)^{2j-n-1}\varepsilon. \tag{B5}$$

So up to non-linear terms in ε,

$$\left(\tfrac{1}{2} + \varepsilon\right)^{2j-n} - \left(\tfrac{1}{2} - \varepsilon\right)^{2j-n} \simeq 2(2j-n)\left(\tfrac{1}{2}\right)^{2j-n-1}\varepsilon. \tag{B6}$$

Thus the linear approximation of [B4] is

$$\varepsilon' = \varepsilon + \frac{\varepsilon}{4}\sum_{j=k}^{n-1} \frac{D(j)}{n}\binom{n}{j}\left(\tfrac{1}{4}\right)^{n-j-1} \cdot 2(2j-n)\left(\tfrac{1}{2}\right)^{2j-n-1}, \tag{B7}$$

or

$$\varepsilon' = \varepsilon\left[1 + \left(\tfrac{1}{2}\right)^{n-2}\sum_{j=k}^{n-1} \frac{D(j)}{n}(nj)(2j-n)\right]. \tag{B8}$$

Therefore, if

$$-2^{n-1} < \sum_{j=k}^{n-1} \frac{D(j)}{n}\binom{n}{j}(2j-n) < 0, \tag{B9}$$

then $p^* = \frac{1}{2}$ is locally stable.

Supplementary Information C: Proof of Result 7

When $\mu = 0$, $\mathbf{L_0}$ in (41) is the diagonal matrix whose N diagonal elements are $(1+s_i)\big[1-D_i(n-1)\big]$ for $i = 1, 2, \ldots, N$. Therefore, these are the N eigenvalues of $\mathbf{M_0}$, and $\rho_\mathbf{0}(0)$, the largest positive eigenvalue, is $\max_{1\le i\le N}\Big\{(1+s_i)\big[1-D_i(n-1)\big]\Big\}$.

When $\mu = \frac{N-1}{N}$, we also have $(1-\mu) = \frac{\mu}{N-1} = \frac{1}{N}$, and (41) reduces to

$$\varepsilon_i' = \frac{1}{N}(1+s_i)\big[1-D_i(n-1)\big]\varepsilon_i + \frac{1}{N}\sum_{j\ne i}(1+s_j)\big[1-D_j(n-1)\big]\varepsilon_j. \qquad \text{[C1]}$$

Therefore, when $\mu = \frac{N-1}{N}$, if $\underline{\varepsilon} = (1, 1, \ldots, 1)$ from [C1] we have

$$\varepsilon_i' = \frac{1}{N}\sum_{j=1}^{N}(1+s_j)\big[1-D_j(n-1)\big]\varepsilon_j \quad \text{for} \quad i = 1, 2, \ldots, N.$$

Hence, in this case $\mathbf{L_0}(1, \ldots, 1) = \frac{1}{N}\sum_{j=1}^{N}(1+s_j)\big[1-D_j(n-1)\big](1, \ldots, 1)$, and by the Perron-Frobenius theorem

$$\rho\left(\tfrac{N-1}{N}\right) = \frac{1}{N}\sum_{j=1}^{N}(1+s_j)\big[1-D_j(n-1)\big]. \qquad \text{[C2]}$$

A similar proof applies to $\rho(0)$ and $\rho(\frac{N-1}{N})$.

Supplementary Information D: Two Populations, No Selection, $\mathbf{D_2} = -\mathbf{D_1}$

We saw in eq. (63) that the recursions (57) had equilibria of the form $(0,0)$, $(1,1)$, and $(\frac{1}{2}, \frac{1}{2})$. However, when $p_2 \ne p_1$ at equilibrium, we have to solve $Q(p_2) = 0$ where $Q(p_2)$ is given by eq. (62). Solving $Q(p_2) = 0$ gives

$$p_2 = \frac{1}{4}\left[(3-2p_1) \pm \sqrt{12p_1(1-p_1)+1}\right]. \qquad \text{[D1]}$$

As we need $0 < p_2 < 1$, we actually have

$$\begin{aligned} 0 < p_1 < \tfrac{1}{2} &\implies p_2 = \frac{1}{4}\left[(3-2p_1) - \sqrt{12p_1(1-p_1)+1}\right] \\ \tfrac{1}{2} < p_1 < 1 &\implies p_2 = \frac{1}{4}\left[(3-2p_1) + \sqrt{12p_1(1-p_1)+1}\right]. \end{aligned} \qquad \text{[D2]}$$

To classify these possible equilibria we have to use [D2] in (58a) and solve for p_1 subject to $0 < p_1 < 1$. In fact, substituting [D2] into (58a) and using $D_2 = -D_1$ with (60), we must solve the equation

$$F(p_1) = D_1p_1(1-p_1)(2p_1-1)(1-2\mu) - \mu p_1 + \frac{\mu}{4}\left[3-2p_1 \pm \sqrt{12p_1(1-p_1)+1}\right], \qquad \text{[D3]}$$

where $0 < p_1 < 1$. Observe that

$$\begin{aligned} F(0) &= \frac{\mu}{4}[3\pm 1] > 0, \\ F\big(\tfrac{1}{2}\big) &= -\frac{\mu}{2} + \frac{\mu}{4}[2\pm 2], \\ F(1) &= -\mu + \frac{\mu}{4}[1\pm 1]. \end{aligned} \qquad \text{[D4]}$$

Hence, if $p_2 = \frac{1}{4}\left[(3-2p_1) - \sqrt{12p_1(1-p_1)+1}\right]$ as $F(0) > 0$ and $F\big(\frac{1}{2}\big) < 0$, and there exists p_1^* with $0 < p_1^* < \frac{1}{2}$ such that $F(p_1^*) = 0$ in accordance with [D2], in which case also $0 < p_2^* < \frac{1}{2}$. If $p_2 = \frac{1}{4}\left[(3-2p_1) + \sqrt{12p_1(1-p_1)+1}\right]$, then as $F\big(\frac{1}{2}\big) > 0$ and $F(1) < 0$ we have a solution p_1^* of $F(p_1) = 0$ satisfying $\frac{1}{2} < p_1^* < 1$ following [D2], and also $\frac{1}{2} < p_2^* < 1$. Therefore, in addition to $(0,0)$, $\big(\frac{1}{2}, \frac{1}{2}\big)$, $(1,1)$ two more polymorphic equilibria exist: (p_1^*, p_2^*) with $0 < p_1^*, p_2^* < \frac{1}{2}$, and (p_1^{**}, p_2^{**}) with $\frac{1}{2} < p_1^{**}, p_2^{**} < 1$. Simulations have shown that these equilibria can be stable.

Supplementary Information E: Two Populations, No Selection, $\mathbf{D_2 = D_1}$

If $D_2 = D_1$, then from (59),

$$p_1(1-p_1)(2p_1-1) = -p_2(1-p_2)(2p_2-1). \tag{E1}$$

Let $z = 1 - p_2$, then [E1] becomes

$$p_1(1-p_1)(2p_1-1) = z(1-z)(2z-1). \tag{E2}$$

Using the above analysis, we conclude that either $1 - p_2 = z = p_1$ or

$$(1-p_2) = z = \frac{1}{4}\left[(3-2p_1) \pm \sqrt{12p_1(1-p_1)+1}\right] \tag{E3}$$

with the same specification as in eq. [D2].

Observe that if $p_1 = 0$ then from (58) $z = 1$ and $p_2 = 0$, and when $p_1 = 1$ then $z = 0$ and $p_2 = 1$. Thus $(0,0)$ and $(1,1)$ are possible equilibria. If $p_2 = 1 - p_1$ $(z = p_1)$, using (58a) and [E1], we have

$$D_1 p_1(1-p_1)(2p_1-1)(1-2\mu) + \mu(1-2p_1) = 0. \tag{E4}$$

Hence, if $p_1 = \frac{1}{2}$, then also $p_2 = \frac{1}{2}$ and we have the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Otherwise

$$D_1 p_1(1-p_1)(1-2\mu) = \mu \tag{E5}$$

and

$$p_1(1-p_1) = \frac{\mu}{D_1(1-2\mu)}. \tag{E6}$$

We assume that $0 < \mu < \frac{1}{2}$. Solutions to (E5) exist if $0 < \frac{\mu}{D_1(1-2\mu)} < \frac{1}{4}$ or if $D_1 > \frac{4\mu}{1-2\mu}$. Since we also require $-2 < D_1 < 1$, we must have $\frac{4\mu}{1-2\mu} < 1$ or $\mu < \frac{1}{6}$. Thus we can have two polymorphic equilibria $(p_1^*, 1-p_1^*)$ or $(1-p_1^*, p_1^*)$, both satisfying [E6] and $p_2^* = 1 - p_1^*$, provided $\mu < \frac{1}{6}$.

To check the local stability of these equilibria, the linear approximation of our transformation (57a) and (57b) near $(p_1^*, 1-p_1^*)$ is given by

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} (1-\mu)A & \mu A \\ \mu A & (1-\mu)A \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \tag{E7}$$

where

$$A = 1 + D_1\left[6p_1^* - 6\left(p_1^*\right)^2 - 1\right]. \tag{E8}$$

Using [E6], [E8] reduces to

$$A = 1 - D_1 + \frac{6\mu}{1-2\mu}. \tag{E9}$$

The eigenvalues of the matrix in [E7] determining the stability of either $(p_1^*, 1-p_1^*)$ or $(1-p_1^*, p_1^*)$ are A and $(1-2\mu)A$. Since $A > 0$ and $D_1 < 1$, the largest eigenvalue is A and it is smaller than 1 if $D_1 > \frac{6\mu}{1-2\mu}$. Thus we need $\mu < \frac{1}{8}$ for these equilibria to be stable.

To sum up, when $\mu < \frac{1}{8}$ we have two polymorphic equilibria both of which are stable when $D_1 > \frac{6\mu}{1-2\mu}$ and unstable if $\frac{4\mu}{1-2\mu} < D_1 < \frac{6\mu}{1-2\mu}$. An example with two such stable equilbria is shown in Figure S4. When $0 < D_1 < \frac{4\mu}{1-2\mu}$ these equilibria do not exist.

Supplementary Information F: Two Populations: interaction of migration, selection, and conformity

Consider the case $s = 0.25$, $D_1 = 0.21$, $D_2 = 0.18$ with three role models. Here $s/(1+s) = 0.2$ so that in the absence of migration, in subpopulation 1, $p_1 = 0$ and $p_1 = 1$ are both stable, while in subpopulation 2, only $p_2 = 1$ is stable. Including migration entails that $(p_1, p_2) = (0,0)$ becomes unstable and $(p_1, p_2) = (1,1)$ is locally stable.

Another interesting example of the effect of migration sets $s = 0.25$, $D_1 = -0.3$, and $D_2 = 0.05$. In this case, population 1 would maintain a polymorphism in the absence of population 2 while population 2 would fix on $p_2 = 1$ in the absence of population 1. The fixation of B, namely $(p_1, p_2) = (0,0)$ is unstable for all $\mu \in \left[0, \frac{1}{2}\right]$, but fixation of A, namely $(p_1, p_2) = (1,1)$, is stable for $\mu > 0.0438$. This suggests that if $0 < \mu < 0.0438$, the two-population system is able to maintain a polymorphism, but greater migration (larger μ) results in the loss of type B from both populations. The delicacy of the interaction between s, D_1, and D_2 in determining the evolutionary dynamics is exemplified by changing $D_1 = -0.3$ to $D_1 = -0.5$, in which case the threshold for stability of $(p_1, p_2) = (1,1)$ changes from $\mu > 0.0438$ to $\mu > 0.3529$. If D_1 and D_2 are kept at -0.3 and 0.05, respectively, but s is reduced from 0.25 to 0.1, then fixation in A, $(p_1, p_2) = (1,1)$, is unstable for all $\mu \in \left[0, \frac{1}{2}\right]$, and fixation in B, $(p_1, p_2) = (0,0)$, is also unstable for all $\mu \in \left[0, \frac{1}{2}\right]$, suggesting that the polymorphism is stable for all legitimate values of μ.

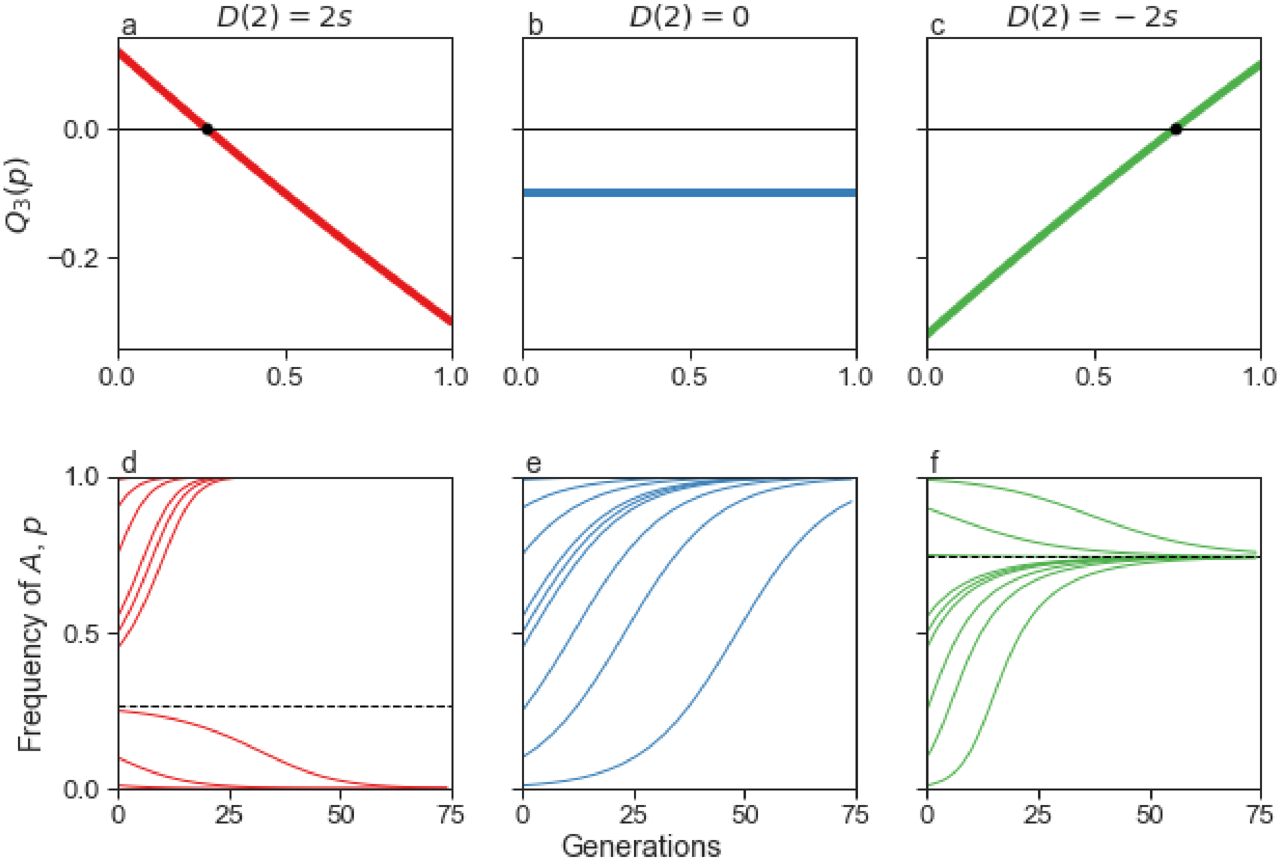

Fig. S1. Frequency-dependent bias with $n = 3$. The top row $\mathbf{(a\text{–}c)}$ shows the shape of $Q_3(p) =$ in eq. (25) for three values of $v = D(2)$, the conformity coefficient: $\mathbf{(a)}$ conformity bias with $D(2) = 2s > s/(1 + s)$, which gives $p^* < \frac{1}{2}$, $\mathbf{(b)}$ unbiased transmission with $D(2) = 0$, $\mathbf{(c)}$ anti-conformity bias with $D(2) = -2s$, which gives $p^* > \frac{1}{2}$. The circles mark the value of the polymorphic equilibrium p^* (which solves $Q_3(p) = 0$), if it exists. The bottom row $\mathbf{(d\text{–}f)}$ shows the frequency of variant A over time, with the dashed line denoting p^*. Different lines are for different initial frequencies of A. It can be seen that either type A or type B goes to fixation, with a larger domain of attraction for the favorable variant A when $D(2) = 2s$; that A goes to fixation when $D(2) = 0$ regardless of initial conditions; and that $p^* > \frac{1}{2}$ is globally stable when $D(2) = -2s$, such that a polymorphism is maintained over the long term. Here phenotype A has a selective advantage of $s = 0.1$.

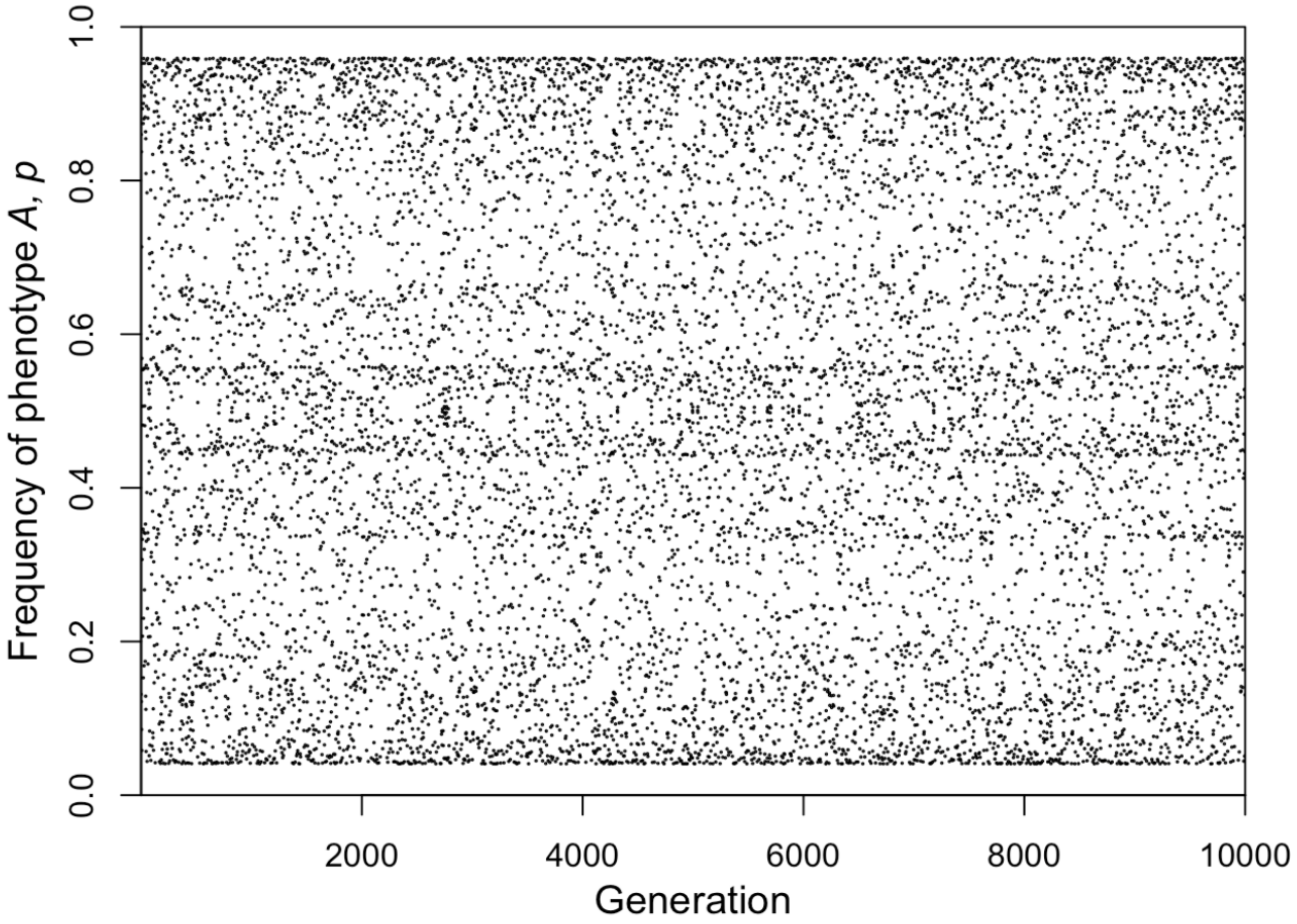

Fig. S2. The frequency of A, p from Eq. [7], is plotted over time. The parameters are $n = 14$ and $D(j) = -j + 0.00001$.

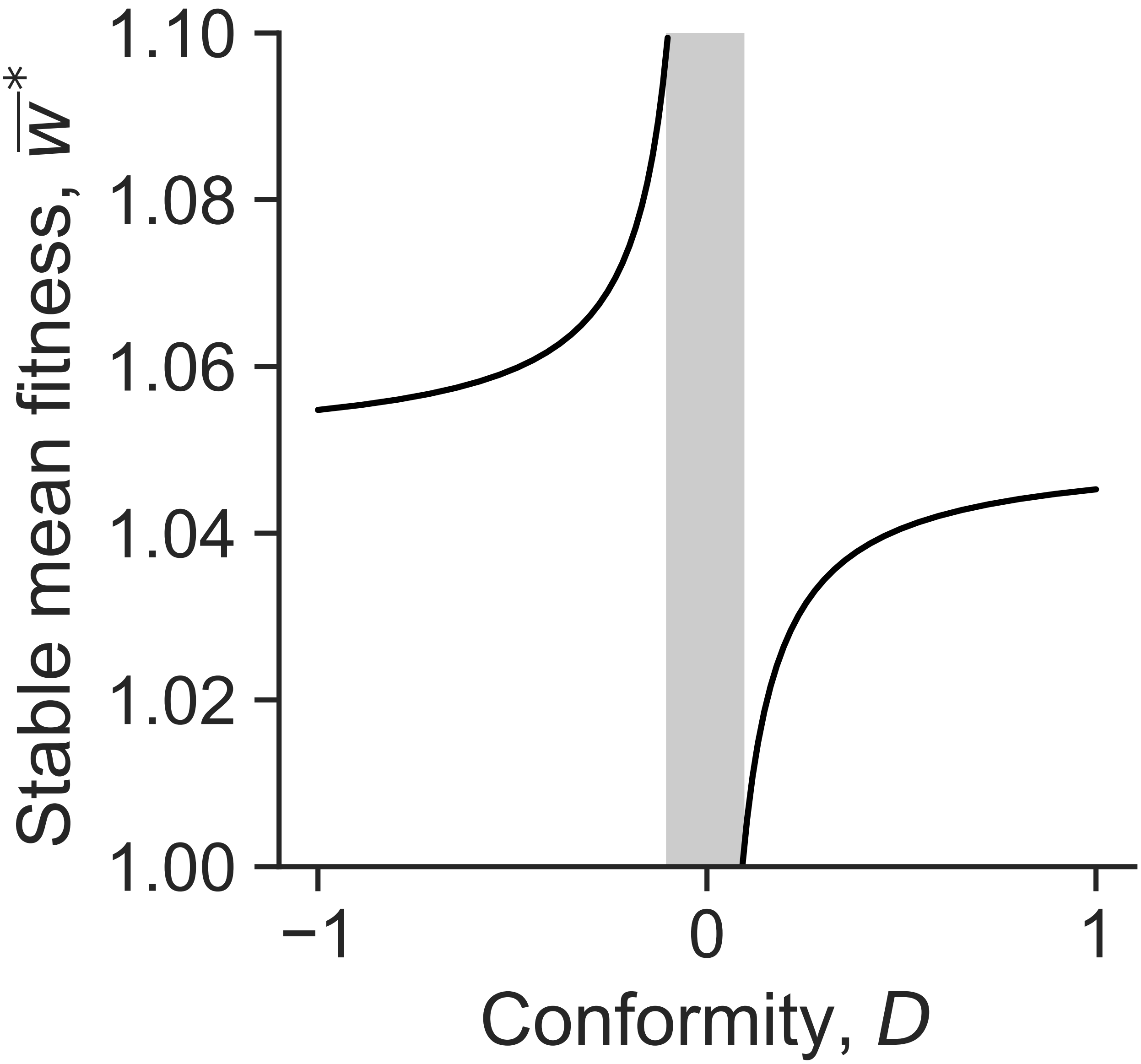

Fig. S3. Mean fitness increases with frequency-dependent bias. The figure shows the population mean fitness W (eq. 24) at the protected polymorphism p^* (the solution to eq. 25, if it exists) as a function of the coefficients of conformity v. In the shaded area $s < v < s/(1+s)$ and p^* does not exist. Here, $s = 0.1$.

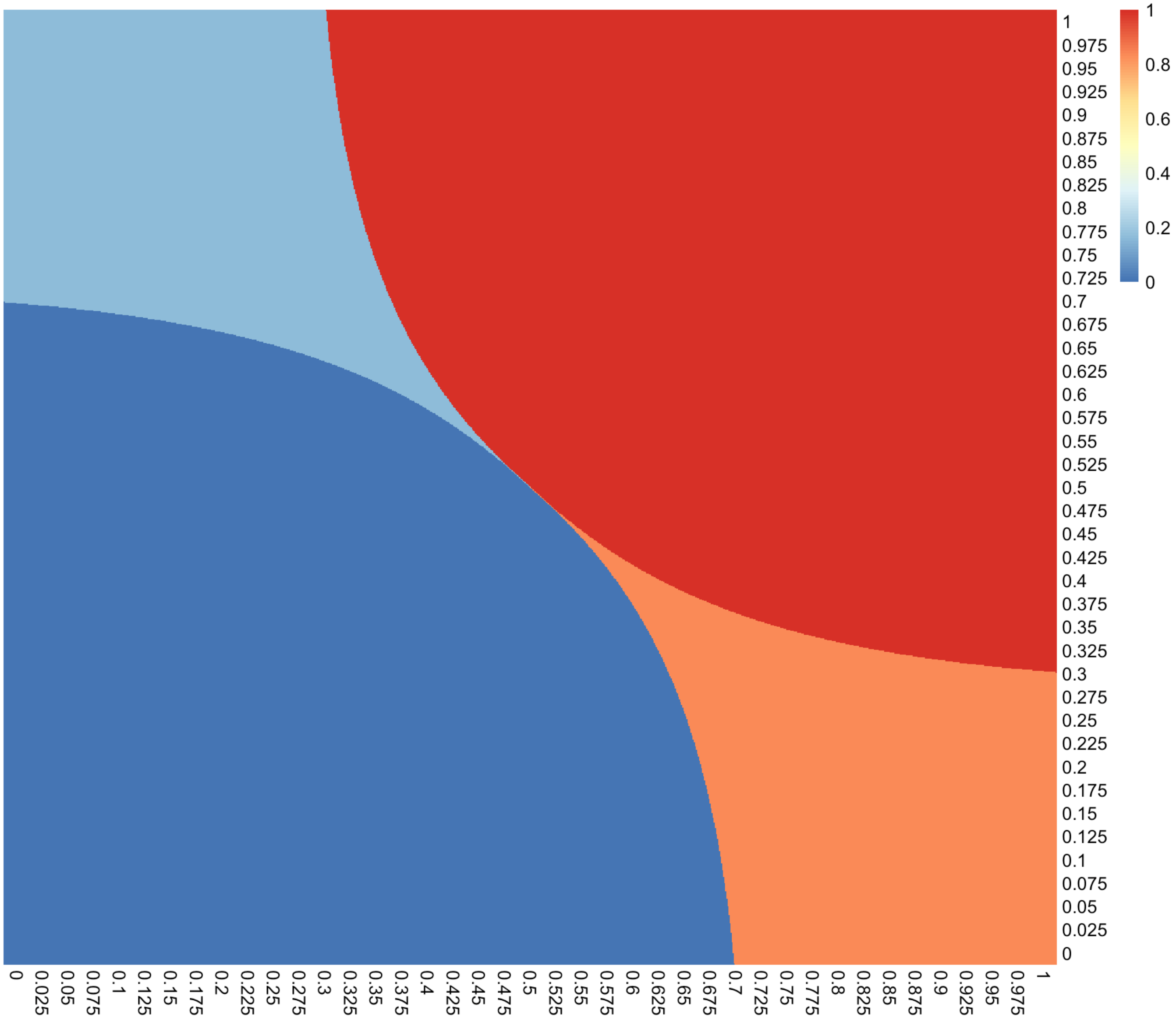

Fig. S4. There are two populations with equal conformity parameters $D_1 = D_2 = 0.4$ and $n = 3$. There is no selection ($s_1 = s_2 = 0$). Migration occurs between the two populations at rate $\mu = 0.05$. The x-axis is the initial frequency of variant A in population 1 and the y-axis is the initial frequency of variant A in population 2. The colors correspond to the frequency of variant A at equilibrium given the initial conditions. Red corresponds to fixation of A, dark blue corresponds to loss of A, and orange and light blue correspond to the polymorphic solutions of eq. (64).

Table S1. Supplementary Table S1: Properties of $\boldsymbol{\phi_n}$

n	Lower Bound for $\boldsymbol{\phi_n}$	Dynamics Near Lower Bound
3	0	Converges to $p = \frac{1}{2}$
4	-0.5	Converges to $p = \frac{1}{2}$
5	-1.25	2-generation cycles
6	-1.5	2-generation cycles
7	-1.968750	2-generation cycles
8	-2.062500	2-generation cycles
9	-2.390625	4-generation cycles
10	-2.421875	Chaos
11	-2.685547	Chaos
12	-2.695312	10-generation cycles
13	-2.926270	Chaos
14	-2.929199	Chaos
15	-3.140259	Chaos
16	-3.141113	Chaos
17	-3.337952	70-generation cycles
18	-3.338196	Chaos
19	-3.523796	Chaos
20	-3.523865	Chaos