The field of values of Jones matrices: classification and special cases

Julio C. Gutiérrez-Vega

Note: Reports are unedited and appear as submitted by the referee. The review history appears in chronological order.

Review History

RSPA-2020-0361.R0 (Original submission)

Review form: Referee 1

Is the manuscript an original and important contribution to its field?
Good

Is the paper of sufficient general interest?
Good

Is the overall quality of the paper suitable?
Good

Can the paper be shortened without overall detriment to the main message?
Yes

Do you think some of the material would be more appropriate as an electronic appendix?
No

Do you have any ethical concerns with this paper?
No

Recommendation?
Accept as is

Comments to the Author(s)
Minor notes:
 Is there a reason for the change of notation referring to the Pauli coefficients from z_m on pg 4 to z_k on pg 11?
 Page 12, after eq 3.27 (line 15): "Now, the number $F(J, S)$ must to lie inside..." remove "to". Maybe break up sentence, too, at "contour. Therefore, ...".

Found ref to be useful: E. A. Jonckheere et al., "Differential topology of numerical range"

Phase sign conventions?

What operations change the eigenvectors but keep constant angular aperture?

Is there another paper to follow? How can one use this from a polarimetric application standpoint?

Review form: Referee 2

Is the manuscript an original and important contribution to its field?
Good

Is the paper of sufficient general interest?
Good

Is the overall quality of the paper suitable?
Acceptable

Can the paper be shortened without overall detriment to the main message?
Yes

Do you think some of the material would be more appropriate as an electronic appendix?
No

Do you have any ethical concerns with this paper?
No

Recommendation?
Major revision is needed (please make suggestions in comments)

Comments to the Author(s)
Manuscript RSPA-2020-0361

In this manuscript the concept of numerical range (field of values) of a matrix is applied in characterizing the Jones matrices occurring in polarization optics. Taking advantage on the stereographic projection of the Poincaré sphere onto the complex plane, the author analyses the mapping between some representative geometrical elements in the numerical range of a Jones matrix and the corresponding elements on the Poincaré sphere and illustrates this connection.

Unfortunately the manuscript is organized in a manner which entertains confusion between the new contributions of the author, on the one hand, and the classical results in the field of linear algebra and the polarization theory, the contributions of previous researchers, and the

previous author's results in this field, on the other hand.

A considerable amount of the paper presents well-known results of linear algebra, without a rigorous and precise indication of the sources: Sect 2. (a), Sect 3. (c) and (d).

The same concerning the standard results of the polarization theory: Sect. 2. (b). Paper references of the manuscript — excepting the own author's ones — are older than 20 years. The author should cover this gap.

Likewise, a considerable amount of the own author's results presented here were published, in slight different forms and contexts, in various OSA journals between 2017 and 2020: Sect. 3 (b), (d) and (f). The author must clarify which is its new original contribution in this manuscript, with respect to the previous papers.

Decision letter (RSPA-2020-0361.R0)

10-Aug-2020

Dear Dr Gutiérrez-Vega

The Editor of Proceedings A has now received comments from referees on the above paper and would like you to revise it in accordance with their suggestions which can be found below (not including confidential reports to the Editor).

Please submit a copy of your revised paper within four weeks - if we do not hear from you within this time then it will be assumed that the paper has been withdrawn. In exceptional circumstances, extensions may be possible if agreed with the Editorial Office in advance.

Please note that it is the editorial policy of Proceedings A to offer authors one round of revision in which to address changes requested by referees. If the revisions are not considered satisfactory by the Editor, then the paper will be rejected, and not considered further for publication by the journal. In the event that the author chooses not to address a referee's comments, and no scientific justification is included in their cover letter for this omission, it is at the discretion of the Editor whether to continue considering the manuscript.

In addition to addressing all of the reviewers' and editor's comments please also ensure that your revised manuscript contains the following sections before the reference list:

• Acknowledgements
• Funding statement

To revise your manuscript, log into http://mc.manuscriptcentral.com/prsa and enter your Author Centre, where you will find your manuscript title listed under "Manuscripts with Decisions." Under "Actions," click on "Create a Revision." Your manuscript number has been appended to denote a revision.

You will be unable to make your revisions on the originally submitted version of the manuscript. Instead, revise your manuscript and upload a new version through your Author Centre.

When submitting your revised manuscript, you will be able to respond to the comments made by the referee(s) and upload a file "Response to Referees" in "Section 6 - File Upload". Please use this to document how you have responded to the comments, and the adjustments you have made. In order to expedite the processing of the revised manuscript, please be as specific as possible in your response to the referee(s).

IMPORTANT: Your original files are available to you when you upload your revised manuscript. Please delete any unnecessary previous files before uploading your revised version.

When revising your paper please ensure that it remains under 28 pages long. In addition, any pages over 20 will be subject to a charge (£150 + VAT (where applicable) per page). Your paper has been ESTIMATED to be 13 pages.

Once again, thank you for submitting your manuscript to Proc. R. Soc. A and I look forward to receiving your revision. If you have any questions at all, please do not hesitate to get in touch.

Yours sincerely
Raminder Shergill
proceedingsa@royalsociety.org

Reviewer(s)' Comments to Author:
Referee: 1

Comments to the Author(s)

Minor notes:
Is there a reason for the change of notation referring to the Pauli coefficients from z_m on pg 4 to z_k on pg 11?
Page 12, after eq 3.27 (line 15): "Now, the number F(J, S) must to lie inside..." remove "to". Maybe break up sentence, too, at "contour. Therefore, ...".

Found ref to be useful: E. A. Jonckheere et al., "Differential topology of numerical range"

Phase sign conventions?

What operations change the eigenvectors but keep constant angular aperture?

Is there another paper to follow? How can one use this from a polarimetric application standpoint?

Referee: 2

Comments to the Author(s)
Manuscript RSPA-2020-0361

In this manuscript the concept of numerical range (field of values) of a matrix is applied in characterizing the Jones matrices occurring in polarization optics. Taking advantage on the stereographic projection of the Poincaré sphere onto the complex plane, the author analyses the mapping between some representative geometrical elements in the numerical range of a Jones matrix and the corresponding elements on the Poincaré sphere and illustrates this connection. Unfortunately the manuscript is organized in a manner which entertains confusion between the new contributions of the author, on the one hand, and the classical results in the field of linear algebra and the polarization theory, the contributions of previous researchers, and the previous author's results in this field, on the other hand.
A considerable amount of the paper presents well-known results of linear algebra, without a rigorous and precise indication of the sources: Sect 2. (a), Sect 3. (c) and (d).
The same concerning the standard results of the polarization theory: Sect. 2. (b). Paper references of the manuscript — excepting the own author's ones — are older than 20 years. The author should cover this gap.
Likewise, a considerable amount of the own author's results presented here were published, in slight different forms and contexts, in various OSA journals between 2017 and 2020: Sect. 3 (b),

(d) and (f). The author must clarify which is its new original contribution in this manuscript, with respect to the previous papers.

Board Member:
Comments to Author(s):
Both reviews are positive. However, reviewer 2 notes some confusion with regard to references, and a need to clarify/emphasise the novel results that the author presents. If the author revises the paper with these suggestions, I believe that it will make a more influential paper and satisfy reviewer 2's concerns.

Author's Response to Decision Letter for (RSPA-2020-0361.R0)

See Appendix A.

RSPA-2020-0361.R1 (Revision)

Review form: Referee 1

Is the manuscript an original and important contribution to its field?
Good

Is the paper of sufficient general interest?
Good

Is the overall quality of the paper suitable?
Good

Can the paper be shortened without overall detriment to the main message?
Yes

Do you think some of the material would be more appropriate as an electronic appendix?
No

Do you have any ethical concerns with this paper?
No

Recommendation?
Accept with minor revision (please list in comments)

Comments to the Author(s)
I believe the author to have addressed my previous minor points.

Addressing the layout of the proceeding sections in the introduction is a welcomed addition and encourage the author to consider this format where applicable in the future. I felt the direct prompt helped to establish the author's intended key takeaways from the paper, which is helpful to keep the introduction nearby while reading through the article.

"This article complements these works in the sense that it establishes a complete, unified, and consistent description of the Jones matrices (including all the special cases) applying the concept

of the field of values. " - I find this statement to be true of the crux of what the author is intending to convey in the paper with a generalized application space. It seems to be more about the process of applying the general principles.

"This without losing sight that this article has new results by itself. " - Regarding which sentence paragraph? It seems vague, maybe unnecessary. Is the author referencing the list preceding this statement?

Also, I think the author and reader could be benefited from adding to the special cases of the FoV section. We see that the special cases are general types of matrices but not types of optical operations. Or, point to the previous work such as the appendix of "Defective Jones matrices: geometric phase and passivity condition" for example. Providing a corresponding example to the optical transformation that accompanies the example matrix.

I understand the other reviewer's comments regarding existing work, it seems that the author has worked to address that to bring this topic into a more general application perspective including the inverse problem of numerical range of Jones matrix formalism and its connection to the Poincare sphere. This is addressed specifically for defective Jones matrices and orthotransmission states in the author's previous work.

In order to keep the paper mostly self contained, I see the length as reasonable.

I have marked this as "accept with minor revisions" on this iteration in case more discussion is prompted, and is/was probably the more appropriate selection.

Review form: Referee 2

Is the manuscript an original and important contribution to its field?
Good

Is the paper of sufficient general interest?
Good

Is the overall quality of the paper suitable?
Good

Can the paper be shortened without overall detriment to the main message?
Yes

Do you think some of the material would be more appropriate as an electronic appendix?
No

Do you have any ethical concerns with this paper?
No

Recommendation?
Accept as is

Comments to the Author(s)
Manuscript RSPA-2020-0361.R1

The last paragraph of the Introduction responds to my suggestions. It constitutes now a model of how situate and emphasize the new original results of a paper in the context of the

previous works, including the author's one.

    As a recommendation for the future works of the author:
    Even from the first reading I have remarked the final sentence of the Introduction:
    "The theory described in this paper can be directly applied to physical systems where linear transformations are described by 2x2 complex matrices,   e. g., two-level spinor systems.' Indeed these results have a much larger applicability than that in polarization optics. Having in view that Jones matrix formalism is a possible representation of the Lorentz group, the results obtained by the author, in this and in the previous papers, could be applied to a series of problems which shear the underlying mathematics of the Lorentz group (in specific problems of various fields of physics: polarization optics, multilayers, interferometry, laser cavity optics, geometrical optics, quantum optics, etc.)
    A reach bibliography on this subject can be found, e. g. in:

J.Opt. Soc. Amer., B 33 (5), 898 (2016)
    and a good introduction in this field in the older:

    This is one of the lines on which the author could enlarge his perspective in order to pass from incremental results based on a very specific mathematics to fundamental results in the field.

Decision letter (RSPA-2020-0361.R1)

04-Sep-2020

Dear Dr Gutiérrez-Vega,

On behalf of the Editor, I am pleased to inform you that your Manuscript RSPA-2020-0361.R1 entitled "The field of values of Jones matrices: Classification and special cases" has been accepted for publication subject to minor revisions in Proceedings A. Please find the referees' comments below.

The reviewer(s) have recommended publication, but also suggest some minor revisions to your manuscript. Therefore, I invite you to respond to the reviewer(s)' comments and revise your manuscript. Please note that we have a strict upper limit of 28 pages for each paper. Please endeavour to incorporate any revisions while keeping the paper within journal limits. Please note that page charges are made on all papers longer than 20 pages. If you cannot pay these charges you must reduce your paper to 20 pages before submitting your revision. Your paper has been ESTIMATED to be 13 pages. We cannot proceed with typesetting your paper without your agreement to meet page charges in full should the paper exceed 20 pages when typeset. If you have any questions, please do get in touch.

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When submitting your revised manuscript, you will be able to respond to the comments made by the referee(s) and upload a file "Response to Referees" in "Section 6 - File Upload". You can use this to document any changes you make to the original manuscript. In order to expedite the processing of the revised manuscript, please be as specific as possible in your response to the referee(s).

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Once again, thank you for submitting your manuscript to Proceedings A and I look forward to receiving your revision. If you have any questions at all, please do not hesitate to get in touch.

Best wishes
Raminder Shergill
proceedingsa@royalsociety.org
Proceedings A

Reviewer(s)' Comments to Author:

Referee: 2

Comments to the Author(s)
Manuscript RSPA-2020-0361.R1

The last paragraph of the Introduction responds to my suggestions. It constitutes now a model of how situate and emphasize the new original results of a paper in the context of the previous works, including the author's one.

As a recommendation for the future works of the author:
Even from the first reading I have remarked the final sentence of the Introduction:
"The theory described in this paper can be directly applied to physical systems where linear transformations are described by 2x2 complex matrices, e. g., two-level spinor systems.'
Indeed these results have a much larger applicability than that in polarization optics. Having in view that Jones matrix formalism is a possible representation of the Lorentz group, the results obtained by the author, in this and in the previous papers, could be applied to a series of problems which shear the underlying mathematics of the Lorentz group (in specific problems of various fields of physics: polarization optics, multilayers, interferometry, laser cavity optics, geometrical optics, quantum optics, etc.)
A reach bibliography on this subject can be found, e. g. in:

J.Opt. Soc. Amer., B 33 (5), 898 (2016)
    and a good introduction in this field in the older:

This is one of the lines on which the author could enlarge his perspective in order to pass from incremental results based on a very specific mathematics to fundamental results in the field.

Referee: 1

Comments to the Author(s)
I believe the author to have addressed my previous minor points.

Addressing the layout of the proceeding sections in the introduction is a welcomed addition and encourage the author to consider this format where applicable in the future. I felt the direct prompt helped to establish the author's intended key takeaways from the paper, which is helpful to keep the introduction nearby while reading through the article.

"This article complements these works in the sense that it establishes a complete, unified, and consistent description of the Jones matrices (including all the special cases) applying the concept of the field of values. " - I find this statement to be true of the crux of what the author is intending

to convey in the paper with a generalized application space. It seems to be more about the process of applying the general principles.

"This without losing sight that this article has new results by itself. " - Regarding which sentence paragraph? It seems vague, maybe unnecessary. Is the author referencing the list preceding this statement?

Also, I think the author and reader could be benefited from adding to the special cases of the FoV section. We see that the special cases are general types of matrices but not types of optical operations. Or, point to the previous work such as the appendix of "Defective Jones matrices: geometric phase and passivity condition" for example. Providing a corresponding example to the optical transformation that accompanies the example matrix.

I understand the other reviewer's comments regarding existing work, it seems that the author has worked to address that to bring this topic into a more general application perspective including the inverse problem of numerical range of Jones matrix formalism and its connection to the Poincare sphere. This is addressed specifically for defective Jones matrices and orthotransmission states in the author's previous work.

In order to keep the paper mostly self contained, I see the length as reasonable.

I have marked this as "accept with minor revisions" on this iteration in case more discussion is prompted, and is/was probably the more appropriate selection.

Board Member
Comments to Author(s):
I am satisfied that the author has modified the paper sufficiently to satisfy the concerns of both reviewers.

Author's Response to Decision Letter for (RSPA-2020-0361.R1)

See Appendix B.

Decision letter (RSPA-2020-0361.R2)

09-Sep-2020

Dear Dr Gutiérrez-Vega

I am pleased to inform you that your manuscript entitled "The field of values of Jones matrices: Classification and special cases" has been accepted in its final form for publication in Proceedings A.

Our Production Office will be in contact with you in due course. You can expect to receive a proof of your article soon. Please contact the office to let us know if you are likely to be away from e-mail in the near future. If you do not notify us and comments are not received within 5 days of sending the proof, we may publish the paper as it stands.

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For tips on promoting your accepted paper see our blog post: https://royalsociety.org/blog/2020/07/promoting-your-latest-paper-and-tracking-your-results/

On behalf of the Editor of Proceedings A, we look forward to your continued contributions to the Journal.

Sincerely,
Raminder Shergill
proceedingsa@royalsociety.org

Appendix A

We are submitting the revisions of the paper (ID RSPA-2020-0361) "The field of values of Jones matrices: Classification and special cases". We appreciate the editorial work done on our work, especially in these complicated times of epidemics and confinement.

Below you will find the responses to each of the issues and questions of the reviewers.

RESPONSES TO THE REVIEWERS

BLACK TEXT: Reviewer comments

RED TEXT: Author responses

REVIEWER 1

1. Minor notes:

Is there a reason for the change of notation referring to the Pauli coefficients from z_m on pg 4 to z_k on pg 11?

- **It was a typo in page 11, I have corrected it. Additionally, I have included the range of the indices m and k just after Eq. (2.8) to avoid confusions.**

Page 12, after eq 3.27 (line 15): "Now, the number F(J, S) must to lie inside..." remove "to". Maybe break up sentence, too, at "contour. Therefore, ...".

- **Thanks for the suggestions, I have implemented them.**

Found ref to be useful: E. A. Jonckheere et al., "Differential topology of numerical range"

- **Thanks for the suggestion, I was not aware about this paper. I have added it to the list of references.**

Phase sign conventions?

- **I added after Eq. (3.27) the sign conventions typically adopted for the phase.**

What operations change the eigenvectors but keep constant angular aperture?

- **Mathematically can be obtained with unitary similarity transformations (i.e. rotations) acting on the overall Jones matrix. Optically, can be produced with lossless phase retarders.**

Is there another paper to follow? How can one use this from a polarimetric application standpoint?.

- **Polarimetry applications are highly related to the phase shifts of the optical system. In particular, the angular aperture of the FoV is related to geometric phase, as described in Sect. 3(f).**
- **I have included some references to emphasize this point, at the end of Sect. 3.**

****

REVIEWER 2

In this manuscript the concept of numerical range (field of values) of a matrix is applied in characterizing the Jones matrices occurring in polarization optics. Taking advantage on the stereographic projection of the Poincaré sphere onto the complex plane, the author analyses the mapping between some representative geometrical elements in the numerical range of a Jones matrix and the corresponding elements on the Poincaré sphere and illustrates this connection.

Unfortunately the manuscript is organized in a manner which entertains confusion between the new contributions of the author, on the one hand, and the classical results in the field of linear algebra and the polarization theory, the contributions of previous researchers, and the previous author's results in this field, on the other hand.

- **I have rewritten the last paragraph of the introduction to make explicit the contributions of the paper. As the reviewer pointed out, I have included some known results from the matrix theory and Pauli matrices. The idea is to establish needed notation and make the paper relatively self-contained. These results are written mostly in Sect. 2. I have added some extra references to cite these concepts correctly.**

A considerable amount of the paper presents well-known results of linear algebra, without a rigorous and precise indication of the sources: Sect 2. (a), Sect 3. (c) and (d).

- **Reviewer is right, I have included some references in the mentioned sections to cite properly known results.**

The same concerning the standard results of the polarization theory: Sect. 2. (b). Paper references of the manuscript — excepting the own author's ones — are older than 20 years. The author should cover this gap.

- **I have included some recent references [Refs. 18-21] to cover the gap mentioned by the reviewer.**

Likewise, a considerable amount of the own author's results presented here were published, in slight different forms and contexts, in various OSA journals between 2017 and 2020: Sect. 3 (b), (d) and (f). The author must clarify which is its new original contribution in this manuscript, with respect to the previous papers.

- **I have rewritten the last paragraph of the introduction to clarify the contribution of this work with respect to previous works.**

  Thanks for the suggestions, I consider that the reviewer's comments helped to improve the manuscript.

****

BOARD MEMBER:

Comments to Author(s):

Both reviews are positive. However, reviewer 2 notes some confusion with regard to references, and a need to clarify/emphasise the novel results that the author presents. If the author revises the paper with these suggestions, I believe that it will make a more influential paper and satisfy reviewer 2's concerns.

- **Thanks for the positive opinion. I implemented the reviewer's suggestions.**

ADDITIONAL CHANGES

- Some minor orthographic typos has been corrected.
- Funding section has been revised.
- Acknowledgements section is not necessary in this paper.

Best regards

Dr. Julio C. Gutiérrez-Vega

Appendix B

Dear Dr. Raminder Shergill

Editors,   PRSA

We are submitting the revisions of the paper (ID RSPA-2020-0361) "The field of values of Jones matrices: Classification and special cases". We appreciate the editorial work done on our work, especially in these complicated times of epidemics and confinement.

Below you will find the responses to each of the issues and questions of the reviewers.

RESPONSES TO THE REVIEWERS

BLACK TEXT: Reviewer comments

RED TEXT: Author responses

REVIEWER 1

I believe the author to have addressed my previous minor points.

Addressing the layout of the proceeding sections in the introduction is a welcomed addition and encourage the author to consider this format where applicable in the future. I felt the direct prompt helped to establish the author's intended key takeaways from the paper, which is helpful to keep the introduction nearby while reading through the article.

- **I added the layout of the sections in the introduction, as requested by the reviewer. Additionally, I split the original section 3 (that was very long) into 2 sections.**

"This article complements these works in the sense that it establishes a complete, unified, and consistent description of the Jones matrices (including all the special cases) applying the concept of the field of values. " - I find this statement to be true of the crux of what the author is intending to convey in the paper with a generalized application space. It seems to be more about the process of applying the general principles.

- **Yes, I agree with the reviewer.**

"This without losing sight that this article has new results by itself. " - Regarding which sentence paragraph? It seems vague, maybe unnecessary. Is the author referencing the list preceding this statement?

- **Yes, I agree with the reviewer, the sentence seems vague and confusing, and it is not strictly necessary because I addressed the contributions of the paper in the preceding paragraph. So, I have removed the sentence.**

Also, I think the author and reader could be benefited from adding to the special cases of the FoV section. We see that the special cases are general types of matrices but not types of optical operations. Or, point to the previous work such as the appendix of "Defective Jones matrices: geometric phase and passivity condition" for example. Providing a corresponding example to the optical transformation that accompanies the example matrix.

- **Thanks for the suggestions, I have added the typical optical realizations for each corresponding special case of the Jones matrices.**

I understand the other reviewer's comments regarding existing work, it seems that the author has worked to address that to bring this topic into a more general application perspective including the inverse problem of numerical range of Jones matrix formalism and its connection to the Poincare sphere. This is addressed specifically for defective Jones matrices and orthotransmission states in the author's previous work.

- **Thanks**

In order to keep the paper mostly self contained, I see the length as reasonable.

- **Thanks**

I have marked this as "accept with minor revisions" on this iteration in case more discussion is prompted, and is/was probably the more appropriate selection.

- **Thanks**

****

REVIEWER 2

The last paragraph of the Introduction responds to my suggestions. It constitutes now a model of how situate and emphasize the new original results of a paper in the context of the previous works, including the author's one.

- **Thanks**

As a recommendation for the future works of the author:

Even from the first reading I have remarked the final sentence of the Introduction:

"The theory described in this paper can be directly applied to physical systems where linear transformations are described by 2x2 complex matrices, e. g., two-level spinor systems.'

Indeed these results have a much larger applicability than that in polarization optics. Having in view that Jones matrix formalism is a possible representation of the Lorentz group, the results obtained by the author, in this and in the previous papers, could be applied to a series of problems which shear the underlying mathematics of the Lorentz group (in specific problems of various fields of physics: polarization optics, multilayers, interferometry, laser cavity optics, geometrical optics, quantum optics, etc.)

- **Thanks for these observations and advices.**

A reach bibliography on this subject can be found, e. g. in: J.Opt. Soc. Amer., B 33 (5), 898 (2016) and a good introduction in this field in the older:

- **Thanks for bring me attention to this paper, I added to the reference list.**

This is one of the lines on which the author could enlarge his perspective in order to pass from incremental results based on a very specific mathematics to fundamental results in the field.

- **I really appreciate these comments of the reviewer.**

****

ADDITIONAL CHANGES

- Some minor orthographic typos has been corrected.
- Funding section has been revised.
- Acknowledgements section is not necessary in this paper.

Best regards

Dr. Julio C. Gutiérrez-Vega