Supporting Materials for Persistent spectral graph

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S1 Proprieties of the spectra of Laplacian

Property S1.1. *Let $G(V, E)$ be a simple graph of order N, then the largest eigenvalue*

$$\lambda_N \leq \max\{deg(u) + deg(v)|(u, v) \in E\}, \tag{S1.1}$$

where $deg(u)$ is the degree of vertex $u \in V$.

Property S1.2. *Let $G(V, E)$ be a simple graph of order N rather than a complete graph with vertex connectivity $\kappa(G)$ and edge connectivity $\kappa'(G)$. Then*

$$2\kappa'(G)(1 - \cos(\pi/N)) \leq \lambda_2(G) \leq \kappa(G) \leq \kappa'(G). \tag{S1.2}$$

The vertex connectivity $\kappa(G)$ is the minimum number of nodes whose deletion disconnects G and edge connectivity $\kappa'(G)$ to be the minimum number of edges whose deletion from a graph G disconnects G.

The charts in the top row of Figure 1 show 5 different types of regular convex polyhedrons, which are called platonic solids. The charts in the bottom row are platonic graphs that intuitively describe the vertices and edges as points and line segments in the Euclidean plane. In the three-dimensional (3D) space, objects (vertices) and the relationship (edges) between objects can be expressed by the Laplacian matrix and vice versa. Taking tetrahedron as an example, we denote the top vertex as v_1, while the other 3 vertices on the plane are denoted as v_2, v_3, and v_4. The Laplacian matrix of the tetrahedron can be expressed as:

$$\mathcal{L}_{\text{Tetra}} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

with eigenvalues being $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = 4$, and $\lambda_4 = 4$. Topological information can be extracted from this Laplacian matrix. First, the multiplicity of 0 eigenvalue is 1, which means there is only one connected component. Secondly, all of the entries in $\mathcal{L}_{\text{Tetra}}$ are non-zero and all the entries in the diagonal equal to 3, which means the corresponding graph is a complete 3-regular graph. Moreover, as stated by Properties S1.1 and S1.2, the largest eigenvalue $\lambda_4 \leq \max\{\deg(u) + \deg(v)|(u, v) \in E\} = 6$. The second smallest eigenvalue λ_2 is controlled by $\kappa(G) = 4$ and greater than $2\kappa'(G)(1 - \cos(\pi/N)) = 2 \times 3 \times (1 - \cos(\pi/4)) \approx 1.75735931288$. A similar analysis can be applied to other 4 platonic solids. Table 1 shows some characteristics of platonic solids.

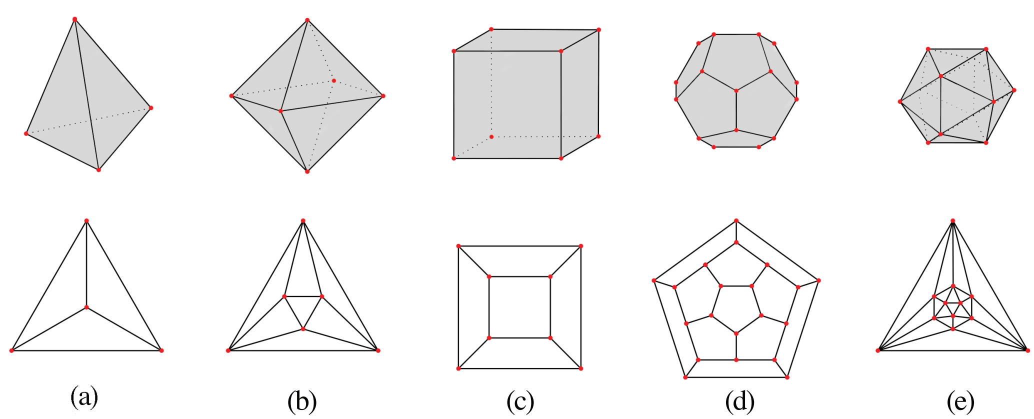


Figure 1: Platonic solids (top row) and its platonic graphs (bottom row). (a) Tetrahedron and tetrahedral graph. (b) Octahedron and octahedral graph. (c) Cube and cubical graph. (d) Dodecahedron and dodecahedral graph. (e) Icosahedron and icosahedral graph.

Table 1: Characteristics of platonic solids in Figure 1. $V, E, \beta_0, \kappa, \kappa', d$, and $\tilde{\lambda}_2$ stand for the number of vertices, the number of edges, the number of zero eigenvalues, vertex connectivity, edge connectivity, and the smallest non-zero eigenvalue, respectively. Here, all the platonic graphs are connected simple graphs, so $\lambda_2 = \tilde{\lambda}_2$.

Platonic Solid	V	E	κ	κ'	β_0	$\tilde{\lambda}_2$
Tetrahedron	4	6	4	3	1	4.00
Octahedron	6	12	4	4	1	4.00
Cube	8	12	3	3	1	2.00
Dodecahedron	20	30	3	3	1	0.7639
Icosahedron	12	30	5	5	1	2.76

S2 Examples of betti numbers of simplicial complexes

(A supplemental example for Section 2.2.) To illustrate the simplicial complex and its corresponding Betti number, we have designed two simple models as is shown in Figure 2. The Betti number of simplicial complexes are listed in Table 2. [1]

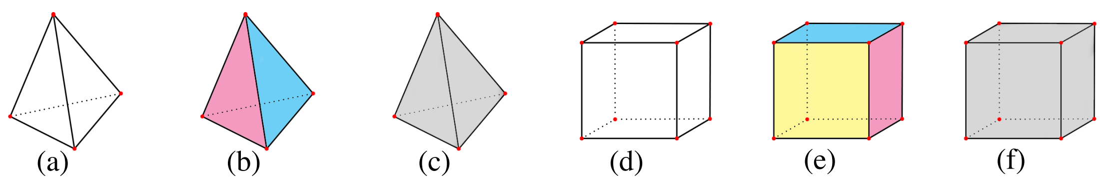


Figure 2: Illustrations of simplicial complexes

Table 2: The Betti number of simplicial complexes in Figure 2. Each color represents different faces. The tetrahedron-shaped simplicial complexes are demonstrated in (a)-(c), and the cube-shaped simplicial complexes are depicted in (d) - (f). (a) and (d) only has 0-simplices and 1-simplices, (b) has four 2-simplices, and (c) has one more 3-simplex. (e) and (f) do not have any 2-simplex.

Betti number	Fig. 3 (a)	Fig. 3 (b)	Fig. 3 (c)	Fig. 3 (d)	Fig. 3 (e)	Fig. 3 (f)
β_0	1	1	1	1	1	1
β_1	3	0	0	5	0	0
β_2	0	1	0	0	1	0

S3 The connection between Betti number and the dimension of the rank of the Laplacian

(A supplemental example for Section 2.3.2.) To illustrate the connection between Betti number and the dimension of the rank of q-combinatorial Laplacian matrix, we consider the tetrahedron-shaped structures in Figure 3. For the sake of brevity, we will use i to represents 0-simplex $[v_i]$, ij to represents 1-simplex $[v_i, v_j]$, and ijk to represents $[v_i, v_j, v_k]$. Then, 1- and 2-boundary operators map:

$$\partial_1(ij) = j - i,$$
$$\partial_2(ijk) = jk - ik + ij.$$

Since different orientations result in the same spectrum, there is no need to label the orientation in Figure 3. In the following, we analyze three tetrahedron-shaped simplicial complexes:

[1] These examples show in an intuitive way to count Betti numbers. However, In Section 2.3, it is impossible to generate structures (b), (e), and (f).

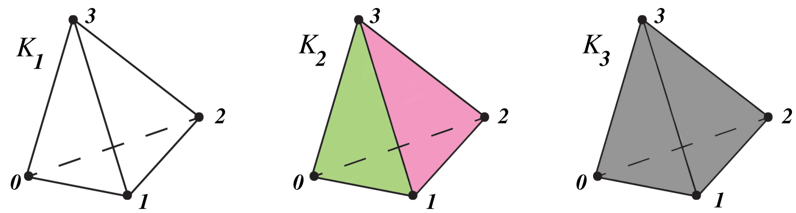


Figure 3: Illustration of three different tetrahedron-shaped simplicial complexes. There are four 0-simplices and six 1-simplices in K_1. Here, K_2 has four more 2-simplices than K_1 does, while K_3 owns one more 3-simplex than K_2 does.

$\boldsymbol{K_1}$. The left most chart in Figure 3 has four 0-simplices: $0, 1, 2$, and 3, and six 1-simplices: $01, 02, 03, 12, 13$, and 23. It is clear that $C_q(K_1)$ is an empty set and ∂_q is an zero map when $q \geq 2$. Then, its Laplacian operators are

$$\Delta_1 = \partial_1^* \partial_1,\ \Delta_0 = \partial_1 \partial_1^* + \partial_0^* \partial_0.$$

The combinatorial Laplacian matrices are:

$$\mathcal{L}_1 = \mathcal{B}_1^T \mathcal{B}_1,\ \mathcal{L}_0 = \mathcal{B}_1 \mathcal{B}_1^T + \mathcal{B}_0^T \mathcal{B}_0.$$

The matrix representation $\mathcal{B}_1$ for $\partial_1 : C_1(K_1) \longrightarrow C_0(K_1)$ is:

$$\mathcal{B}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 03 & 12 & 13 & 23 \end{array} \\ \left[\begin{array}{cccccc} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}. \tag{S3.1}$$

and B_0 is

$$B_0 = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Therefore, the associated combinatorial Laplacian matrices are

$$\mathcal{L}_1(K_1) = \left[\begin{array}{cccccc} 2 & 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 1 & 0 & -1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ -1 & 1 & 0 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 & 1 & 2 \end{array}\right],\ \mathcal{L}_0(K_1) = \left[\begin{array}{cccc} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{array}\right].$$

As shown in Table 3, we can calculate the spectra and ranks from combinatorial Laplacian matrices. We have $\beta_0 = 1$, $\beta_1 = 3$, which reveal that one connected component and three 1-cycles are exist in K_1.

Table 3: Table of dimensions, ranks, nullity, spectra and Betti numbers of combinatorial Laplacian matrices $\mathcal{L}_0$, and $\mathcal{L}_1$ for simplicial complex K_1.

	$\mathcal{L}_1(K_1)$	$\mathcal{L}_0(K_1)$
Betti number	$\beta_1 = 3$	$\beta_0 = 1$
dim	6	4
rank	3	3
nullity	3	1
Spectra	$\{0, 0, 0, 4, 4, 4\}$	$\{0, 4, 4, 4\}$

$\boldsymbol{K_2}$. We analyze the middle chart in Figure 3 in a similar manner. As one can see, K_2 has four 0-simplices: $0, 1, 2$, and 3, six 1-simplices: $01, 02, 03, 12, 13$, and 23, and four 2-simplices: $012, 013, 023$, and 123. The associated Laplacian operators are

$$\Delta_2 = \partial_2^* \partial_2,\ \Delta_1 = \partial_2 \partial_2^* + \partial_1^* \partial_1,\ \Delta_0 = \partial_1 \partial_1^* + \partial_0^* \partial_0.$$

The resulting combinatorial Laplacian matrices are

$$\mathcal{L}_2 = \mathcal{B}_2^T \mathcal{B}_2,\ \mathcal{L}_1 = \mathcal{B}_2 \mathcal{B}_2^T + \mathcal{B}_1^T \mathcal{B}_1,\ \mathcal{L}_0 = \mathcal{B}_1 \mathcal{B}_1^T + \mathcal{B}_0^T \mathcal{B}_0.$$

The corresponding matrix representations for $\mathcal{B}_2$ and $\mathcal{B}_1$ are respectively

$$\mathcal{B}_2 = \begin{array}{c} \\ 01 \\ 02 \\ 03 \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{cccc} 012 & 013 & 023 & 123 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}, \quad \mathcal{B}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 03 & 12 & 13 & 23 \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}. \tag{S3.2}$$

Then, associated combinatorial Laplacian matrices are

$$\mathcal{L}_2(K_2) = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}, \ \mathcal{L}_1(K_2) = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

and $\mathcal{L}_0(K_2) = \mathcal{L}_0(K_1)$. Similarly, from Table 4, we see that there are one connected component and one 2-cycle (void) in K_2.

Table 4: Table of dimensions, ranks, nullity, spectra and Betti numbers of combinatorial Laplacian matrices $\mathcal{L}_0, \mathcal{L}_1$, and $\mathcal{L}_2$ for simplicial complex K_2.

	$\mathcal{L}_2(K_2)$	$\mathcal{L}_1(K_2)$	$\mathcal{L}_0(K_2)$
Betti number	$\beta_2 = 1$	$\beta_1 = 0$	$\beta_0 = 1$
dim	4	6	4
rank	3	6	3
nullity	1	0	1
Spectra	$\{0, 4, 4, 4\}$	$\{4, 4, 4, 4, 4, 4\}$	$\{0, 4, 4, 4\}$

S4 Persistence Homology

Persistence Homology is an algebraic topology-based method for the multiscale analysis of the topological invariants of functions and datasets. It has been widely applied in the field of topological data analysis. We provide a brief introduction to persistent homology and the interested readers are referred to the literature [1,2] for more detail.

S4.1 Homology

For a topological space X, a sequences of complexes $C_0(X), C_1(X), \cdots$ describes different dimensional information of the topological space X, which are connected by homomorphisms (or boundary operators)

$\partial_k : C_k \longrightarrow C_{k-1}$ such that $\operatorname{im}\partial_k \subseteq \ker\partial_{k-1}$, i.e., $\partial_{k-1}\partial_k = 0$. With a k-simplex $\sigma_k = [v_0, \cdots, v_k]$ where v_i are all the vertices of σ_k, $\partial_k\sigma_k$ can be given by a formal sum with coefficients in the $\mathbb{Z}_2$ field

$$\partial_k\sigma_k = \sum_{i=0}^{k} \sigma_{k-1}^{i}, \tag{S4.1}$$

where σ_i^{k-1} is the $(k-1)$-simplex with its ith vertex v_i being omitted. The algebraic construction to connect a sequence of complexes by boundary maps is called a chain complex

$$\cdots \xrightarrow{\partial_{i+1}} C_i(X) \xrightarrow{\partial_i} C_{i-1}(X) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

and the kth homology group is the quotient group defined by

$$H_k = \ker\partial_k / \operatorname{im}\partial_{k+1}. \tag{S4.2}$$

By studying homology groups, one can derive homological properties of the space K. The Betti numbers are defined by the ranks of kth homology group H_k which counts $k-$dimensional holes, especially, $\operatorname{rank}(H_0)$ reflects the number of connected components, $\operatorname{rank}(H_1)$ reflects the number of loops, and $\operatorname{rank}(H_2)$ reveals the number of voids or cavities. However, $\operatorname{rank}(H_k)$ only allows us to express the topological information for a specific setup. Persistent homology is devised to track the multiscale topological information over different scales along a filtration.

S4.2 Persistent homology

A filtration of a topology space K is a sequence of sub-spaces $(K_t)_{t=0}^m$ of K such that

$$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K. \tag{S4.3}$$

A sequence of chain complexes induced by the filtration is defined as

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_3} & C_2^1 & \xrightarrow{\partial_2} & C_1^1 & \xrightarrow{\partial_1} & C_0^1 & \xrightarrow{\partial_0} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial_3} & C_2^2 & \xrightarrow{\partial_2} & C_1^2 & \xrightarrow{\partial_1} & C_0^2 & \xrightarrow{\partial_0} & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \xrightarrow{\partial_3} & C_2^m & \xrightarrow{\partial_2} & C_1^m & \xrightarrow{\partial_1} & C_0^m & \xrightarrow{\partial_0} & 0
\end{array} \tag{S4.4}$$

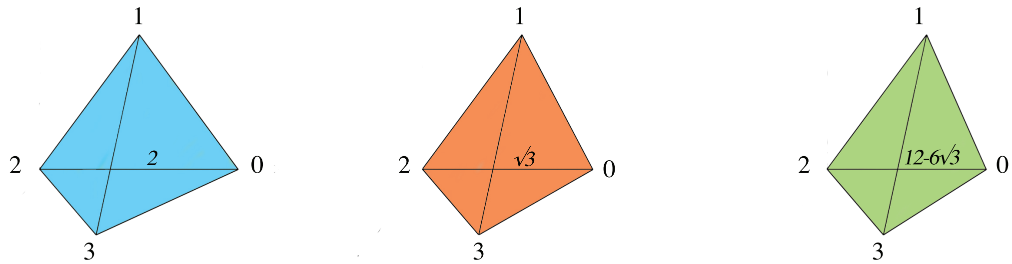
with $C_k^t := C_k(K^t)$ and $\downarrow$ denotes the inclusion [3]. The p-persistent kth homology group of K^t is defined as

$$H_k^p(K^t) = \ker\partial_k(K^t)/(\operatorname{im}\partial_{k+1}(K^{t+p}) \cap \ker\partial_k(K^t)), \tag{S4.5}$$

Intuitively, this homology group records the homology classes of K^t that are persistent at least until K^{t+p}. When $k=0$, the rank of $H_k^p(K^t)$ reveals the number of connected components in K^t.

S5 Implement PST to distinguish different topological structures

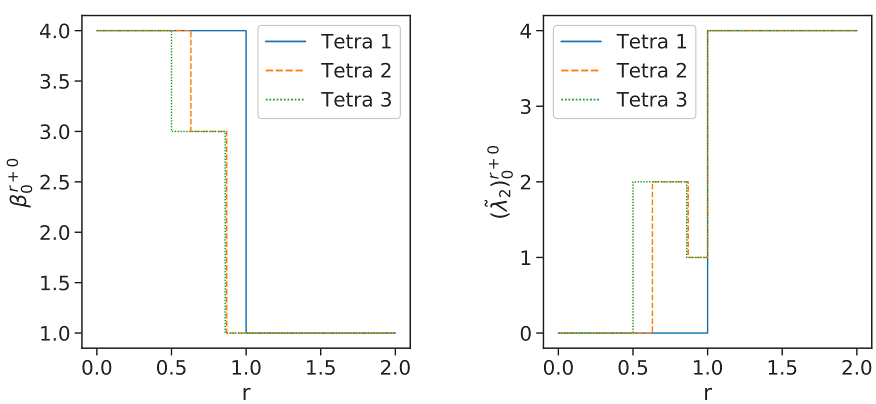
(A supplementary example to distinguish different topological structures by implementing PST in the Section 2.3.3.)



(a) Tetra 1 (b) Tetra 2 (c) Tetra 3

Figure 4: Three tetrahedrons with different topological shape in $\mathbb{R}^3$. (a) Regular tetrahedron with edge length 2. (b) Move v_0 along the edge $[v_0, v_1]$ and construct a new tetrahedron with the length of $[v_0, v_1]$ to be $\sqrt{3}$. (c) Move v_0 along the edge $[v_0, v_1]$ and construct a new tetrahedron with the length of $[v_0, v_1]$ being $12 - 6\sqrt{3}$.

Figure 4 and Figure 5 exemplify the capacity of persistent spectral theory to discriminate between different structures in $\mathbb{R}^3$. In Figure 5(a), we employ the persistent spectral analysis based on the β_0^{r+0} tendency along the filtration to distinguish three tetrahedrons. As r grows, isolated points (0-simplices) will gradually grow into solid 2-spheres, and a new isolated component will be created once two spheres corresponding to two isolated points overlap with each other. Since β_0^{r+0} represents the number of isolated components, the value of β_0^{r+0} will finally decrease to 1. Take Tetra 2 as an example. It is seen that at the initial setup ($r = 0.0$), the number of isolated components is 4, which represents the number of isolated points. When r is around 0.63, two spheres centered at v_0 and v_2 with radius 0.63 overlapped with each other. Therefore, β_0^{r+0} reduces to 3 at this point. With r keeping growing, the sphere centered at v_0 overlaps with spheres centered at v_1, v_2, and v_3, which results $\beta_0^{r+0} = 1$ after $r = 0.87$. Similarly, the smallest non-zero eigenvalue $(\tilde{\lambda}_2)_0^{r+0}$ changes at radius 0.63 and 0.87 in Figure 5(b), which also affirms that the solid spheres get overlapped at these specific filtration parameters. It is clear that Tetrahedron 1, 2, and 3 have different β_0^{r+0} and $(\tilde{\lambda}_2)_0^{r+0}$ values. Since 1-cycle and 2-cycle are not formed along with the filtration, analysis of β_1^{r+0} and β_2^{r+0} will not be mentioned in this case.



(a) (b)

Figure 5: (a) Plot of β_0^{r+0} with radius filtration r among 3 different tetrahedrons. (b) Plot of $(\tilde{\lambda}_2)_0^{r+0}$ with radius filtration r among 3 different tetrahedrons.

S6 Additional Laplacian matrices and their properties

In this section, we give a further description of additional boundary and Laplacian matrices and their properties involved in the filtration process in Figure 6. Detailed information are listed in the Table 5 - Table 23.

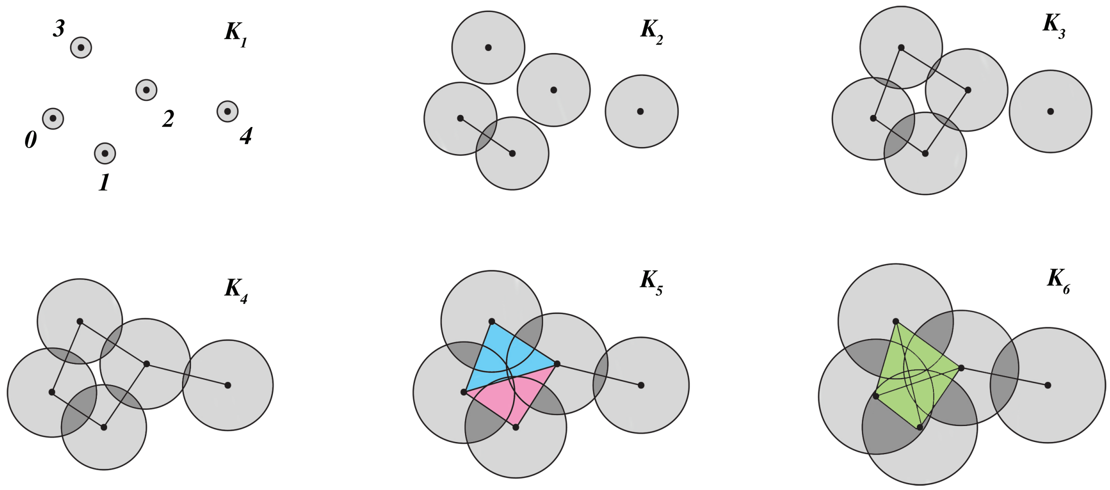


Figure 6: Illustration of filtration. We use $0, 1, 2, 3$, and 4 to stand for 0-simplices, $01, 12, 23, 03, 24, 02$, and 13 for 1-simplices, $012, 023, 013$, and 123 for 2-simplices, and 0123 for the 3-simplex. Here, K_1 has five 0-cycles, K_2 has four 0-cycles, K_3 has two 0-cycles and a 1-cycle, K_4 has a 0-cycle and a 1-cycle, K_5 has one 0-cycle, and K_6 has a 0-cycle.

Table 5: $K_1 \to K_1$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{1+0}$	/	/	/
$\mathcal{B}_q^1$	$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \end{array}$ $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	/	/
$\mathcal{L}_q^{1+0}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	/	/
β_q^{1+0}	5	/	/
$\dim(\mathcal{L}_q^{1+0})$	5	/	/
$\mathrm{rank}(\mathcal{L}_q^{1+0})$	0	/	/
$\mathrm{nullity}(\mathcal{L}_q^{1+0})$	5	/	/
$\mathrm{Spectra}(\mathcal{L}_q^{1+0})$	$\{0, 0, 0, 0, 0\}$	/	/

Table 6: $K_2 \to K_2$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{2+0}$	$\begin{array}{c c} & 01 \\ \begin{array}{c}0\\1\\2\\3\\4\end{array} & \left[\begin{array}{c}-1\\1\\0\\0\\0\end{array}\right]\end{array}$	/	/
$\mathcal{B}_q^2$	$\begin{array}{c}\begin{array}{ccccc}0&1&2&3&4\end{array}\\ \left[\begin{array}{ccccc}0&0&0&0&0\end{array}\right]\end{array}$	$\begin{array}{c c} & 01 \\ \begin{array}{c}0\\1\\2\\3\\4\end{array} & \left[\begin{array}{c}-1\\1\\0\\0\\0\end{array}\right]\end{array}$	/
$\mathcal{L}_q^{2+0}$	$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[2]	/
β_q^{2+0}	4	0	/
$\dim(\mathcal{L}_q^{2+0})$	5	1	/
$\text{rank}(\mathcal{L}_q^{2+0})$	1	1	/
$\text{nullity}(\mathcal{L}_q^{2+0})$	4	0	/
$\text{Spectra}(\mathcal{L}_q^{2+0})$	$\{0,0,0,0,2\}$	2	/

Table 7: $K_3 \to K_3$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{3+0}$	$\begin{array}{cc} & \begin{matrix} 01 & 12 & 23 & 03 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_{q}^{3}$	$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{cc} & \begin{matrix} 01 & 12 & 23 & 03 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{L}_{q}^{3+0}$	$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$	/
β_{q}^{3+0}	2	1	/
$\dim(\mathcal{L}_{q}^{3+0})$	5	4	/
$\text{rank}(\mathcal{L}_{q}^{3+0})$	3	3	/
$\text{nullity}(\mathcal{L}_{q}^{3+0})$	2	1	/
$\text{Spectra}(\mathcal{L}_{q}^{3+0})$	$\{0,0,2,2,4\}$	$\{0,2,2,4\}$	/

Table 8: $K_5 \to K_5$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{5+0}$	$\begin{matrix} & 01 & 12 & 23 & 03 & 24 & 02 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & -1 & 1 \\ 3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$	$\begin{matrix} & 012 & 023 \\ 01 & 1 & 0 \\ 12 & 1 & 0 \\ 23 & 0 & 1 \\ 03 & 0 & -1 \\ 24 & 0 & 0 \\ 02 & -1 & 1 \end{matrix}$	$\begin{matrix} & 0123 \\ 012 & -1 \\ 023 & 1 \end{matrix}$
$\mathcal{B}_q^{5}$	$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} & 01 & 12 & 23 & 03 & 24 & 02 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & -1 & 1 \\ 3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$	$\begin{matrix} & 012 & 023 \\ 01 & 1 & 0 \\ 12 & 1 & 0 \\ 23 & 0 & 1 \\ 03 & 0 & -1 \\ 24 & 0 & 0 \\ 02 & -1 & 1 \end{matrix}$
$\mathcal{L}_q^{5+0}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
β_q^{5+0}	1	0	0
$\dim(\mathcal{L}_q^{5+0})$	5	6	2
$\mathrm{rank}(\mathcal{L}_q^{5+0})$	4	6	2
$\mathrm{nullity}(\mathcal{L}_q^{5+0})$	1	0	0
$\mathrm{Spectra}(\mathcal{L}_q^{5+0})$	$\{0, 1, 2, 4, 5\}$	$\{1, 2, 2, 4, 4, 5\}$	$\{4, 4\}$

Table 9: $K_1 \to K_2$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{1+1}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 01 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^1$	$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ [\,0 & 0 & 0 & 0 & 0\,] \end{array}$	/	/
$\mathcal{L}_q^{1+1}$	$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	/	/
β_q^{1+1}	4	/	/
$\dim(\mathcal{L}_q^{1+1})$	5	/	/
$\text{rank}(\mathcal{L}_q^{1+1})$	1	/	/
$\text{nullity}(\mathcal{L}_q^{1+1})$	4	/	/
$\text{Spectra}(\mathcal{L}_q^{1+1})$	$\{0,0,0,0,2\}$	/	/

Table 10: $K_1 \to K_3$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{1+2}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 01 & 12 & 23 & 03 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^1$	$\begin{array}{c} \\ / \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{L}_q^{1+2}$	$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	/	/
β_q^{1+2}	2	/	/
$\dim(\mathcal{L}_q^{1+2})$	5	/	/
$\text{rank}(\mathcal{L}_q^{1+2})$	3	/	/
$\text{nullity}(\mathcal{L}_q^{1+2})$	2	/	/
$\text{Spectrum}(\mathcal{L}_q^{1+2})$	$\{0, 0, 2, 2, 4\}$	/	/

Table 11: $K_1 \to K_4$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{1+3}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 01 & 12 & 23 & 03 & 24 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^1$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{L}_q^{1+3}$	$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	/	/
β_q^{1+3}	1	/	/
$\dim(\mathcal{L}_q^{1+3})$	5	/	/
$\text{rank}(\mathcal{L}_q^{1+3})$	4	/	/
$\text{nullity}(\mathcal{L}_q^{1+3})$	1	/	/
$\text{Spectra}(\mathcal{L}_q^{1+3})$	$\{0, 0.8299, 2, 2.6889, 4.4812\}$	/	/

Table 12: $K_1 \to K_5$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{1+4}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 12 & 23 & 03 & 24 & 02 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^1$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{L}_q^{1+4}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	/	/
β_q^{1+4}	1	/	/
$\dim(\mathcal{L}_q^{1+4})$	5	/	/
$\mathrm{rank}(\mathcal{L}_q^{1+4})$	4	/	/
$\mathrm{nullity}(\mathcal{L}_q^{1+4})$	1	/	/
$\mathrm{Spectra}(\mathcal{L}_q^{1+4})$	$\{0,1,2,4,5\}$	/	/

Table 13: $K_1 \to K_6$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{1+5}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccccc} 01 & 12 & 23 & 03 & 24 & 02 & 13 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^1$	$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \end{array}$ $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	/	/
$\mathcal{L}_q^{1+5}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	/	/
β_q^{1+5}	1	/	/
$\dim(\mathcal{L}_q^{1+5})$	5	/	/
$\text{rank}(\mathcal{L}_q^{1+5})$	4	/	/
$\text{nullity}(\mathcal{L}_q^{1+5})$	1	/	/
$\text{Spectra}(\mathcal{L}_q^{1+5})$	$\{0, 1, 4, 4, 5\}$	/	/

Table 14: $K_2 \to K_3$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{2+1}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 01 & 12 & 23 & 03 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^2$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 01 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{2+1}$	$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[2]	/
β_q^{2+1}	2	0	/
$\dim(\mathcal{L}_q^{2+1})$	5	1	/
$\text{rank}(\mathcal{L}_q^{2+1})$	3	1	/
$\text{nullity}(\mathcal{L}_q^{2+1})$	2	0	/
$\text{Spectra}(\mathcal{L}_q^{2+1})$	$\{0, 0, 2, 2, 4\}$	2	/

Table 15: $K_2 \to K_4$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{2+2}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 01 & 12 & 23 & 03 & 24 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^2$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 01 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{2+2}$	$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	[2]	/
β_q^{2+2}	1	0	/
$\dim(\mathcal{L}_q^{2+2})$	5	1	/
$\mathrm{rank}(\mathcal{L}_q^{2+2})$	4	1	/
$\mathrm{nullity}(\mathcal{L}_q^{2+2})$	1	0	/
$\mathrm{Spectra}(\mathcal{L}_q^{2+2})$	$\{0, 0.8299, 2, 2.6889, 4.4812\}$	2	/

Table 16: $K_2 \to K_5$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{2+3}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 12 & 23 & 03 & 24 & 02 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$	$\begin{array}{cc} & \begin{array}{cc} 012 & 023 \end{array} \\ 01 & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{B}_q^2$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 01 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{2+3}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	[3]	/
β_q^{2+3}	1	0	/
$\dim(\mathcal{L}_q^{2+3})$	5	1	/
$\mathrm{rank}(\mathcal{L}_q^{2+3})$	4	1	/
$\mathrm{nullity}(\mathcal{L}_q^{2+3})$	1	0	/
$\mathrm{Spectra}(\mathcal{L}_q^{2+3})$	$\{0, 1, 2, 4, 5\}$	3	/

Table 17: $K_2 \to K_6$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{2+4}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 01 & 12 & 23 & 03 & 24 & 02 & 13 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 01 \end{array} \begin{array}{c} \begin{array}{cccc} 012 & 023 & 013 & 123 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{B}_q^2$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 01 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{2+4}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	[4]	/
β_q^{2+4}	1	0	/
$\dim(\mathcal{L}_q^{2+4})$	5	1	/
$\mathrm{rank}(\mathcal{L}_q^{2+4})$	4	1	/
$\mathrm{nullity}(\mathcal{L}_q^{2+4})$	1	0	/
$\mathrm{Spectra}(\mathcal{L}_q^{2+4})$	$\{0,1,4,4,5\}$	4	/

Table 18: $K_3 \to K_5$

q	$q = 0$	$q = 1$	$q = 2$
$\mathcal{B}_{q+1}^{3+2}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 12 & 23 & 03 & 24 & 02 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 01 \\ 12 \\ 23 \\ 03 \end{array} \begin{array}{c} \begin{array}{cc} 012 & 023 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \end{array}$	/
$\mathcal{B}_q^3$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 01 & 12 & 23 & 03 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{3+2}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$	/
β_q^{3+2}	1	0	/
$\dim(\mathcal{L}_q^{3+2})$	5	4	/
$\mathrm{rank}(\mathcal{L}_q^{3+2})$	4	4	/
$\mathrm{nullity}(\mathcal{L}_q^{3+2})$	1	0	/
$\mathrm{Spectra}(\mathcal{L}_q^{3+2})$	$\{0, 1, 2, 4, 5\}$	$\{2, 2, 4, 4\}$	/

Table 19: $K_3 \to K_6$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{3+3}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 01 & 12 & 23 & 03 & 24 & 02 & 13 \end{array} \\ \left[\begin{array}{ccccccc} -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$	$\begin{array}{c} \\ 01 \\ 12 \\ 23 \\ 03 \end{array} \begin{array}{c} \begin{array}{cccc} 012 & 023 & 013 & 123 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right] \end{array}$	/
$\mathcal{B}_q^{3}$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 01 & 12 & 23 & 03 \end{array} \\ \left[\begin{array}{cccc} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$	/
$\mathcal{L}_q^{3+3}$	$\left[\begin{array}{ccccc} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{cccc} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right]$	/
β_q^{3+3}	1	0	/
$\dim(\mathcal{L}_q^{3+3})$	5	4	/
$\mathrm{rank}(\mathcal{L}_q^{3+3})$	4	4	/
$\mathrm{nullity}(\mathcal{L}_q^{3+3})$	1	0	/
$\mathrm{Spectra}(\mathcal{L}_q^{3+3})$	$\{0, 1, 4, 4, 5\}$	$\{4, 4, 4, 4\}$	/

Table 20: $K_4 \to K_4$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{4+0}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 01 & 12 & 23 & 03 & 24 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	/	/
$\mathcal{B}_q^4$	/ $\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 01 & 12 & 23 & 03 & 24 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{4+0}$	$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{bmatrix}$	/
β_q^{4+0}	1	1	/
$\dim(\mathcal{L}_q^{4+0})$	5	5	/
$\mathrm{rank}(\mathcal{L}_q^{4+0})$	4	4	/
$\mathrm{nullity}(\mathcal{L}_q^{4+0})$	1	1	/
$\mathrm{Spectra}(\mathcal{L}_q^{4+0})$	$\{0, 0.8299, 2, 2.6889, 4.4812\}$	$\{0, 0.8299, 2, 2.6889, 4.4812\}$	/

Table 21: $K_4 \to K_5$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{4+1}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 12 & 23 & 03 & 24 & 02 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 01 \\ 12 \\ 23 \\ 03 \\ 24 \end{array} \begin{array}{c} \begin{array}{cc} 012 & 023 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{B}_q^4$	/ $\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 01 & 12 & 23 & 03 & 24 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{4+1}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{bmatrix}$	/
β_q^{4+1}	1	0	/
$\dim(\mathcal{L}_q^{4+1})$	5	5	/
$\text{rank}(\mathcal{L}_q^{4+1})$	4	5	/
$\text{nullity}(\mathcal{L}_q^{4+1})$	1	0	/
$\text{Spectra}(\mathcal{L}_q^{4+1})$	$\{0, 1, 2, 4, 5\}$	$\{1.2677, 2, 2, 4, 4.7321\}$	/

Table 22: $K_4 \to K_6$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{4+2}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 01 & 12 & 23 & 03 & 24 & 02 & 13 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 01 \\ 12 \\ 23 \\ 03 \\ 24 \end{array} \begin{array}{c} \begin{array}{cccc} 012 & 023 & 013 & 123 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{B}_q^4$	$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 01 & 12 & 23 & 03 & 24 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	/
$\mathcal{L}_q^{4+2}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{bmatrix}$	/
β_q^{4+2}	1	0	/
$\dim(\mathcal{L}_q^{4+2})$	5	5	/
$\text{rank}(\mathcal{L}_q^{4+2})$	4	5	/
$\text{nullity}(\mathcal{L}_q^{4+2})$	1	0	/
$\text{Spectra}(\mathcal{L}_q^{4+2})$	$\{0, 1, 4, 4, 5\}$	$\{1.2679, 4, 4, 4, 4.7321\}$	/

Table 23: $K_5 \to K_6$

q	$q=0$	$q=1$	$q=2$
$\mathcal{B}_{q+1}^{5+1}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 12 & 23 & 03 & 24 & 02 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 01 \\ 12 \\ 23 \\ 03 \\ 24 \\ 02 \end{array} \begin{array}{c} \begin{array}{cccc} 012 & 023 & 013 & 123 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$	/
$\mathcal{B}_{q}^{5}$	$/ \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 12 & 23 & 03 & 24 & 02 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$	$\begin{array}{c} \\ 01 \\ 12 \\ 23 \\ 03 \\ 24 \\ 02 \end{array} \begin{array}{c} \begin{array}{cc} 012 & 023 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \end{array}$
$\mathcal{L}_{q}^{5+1}$	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & -1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$
β_{q}^{5+1}	1	0	0
$\dim(\mathcal{L}_{q}^{5+1})$	5	6	2
$\mathrm{rank}(\mathcal{L}_{q}^{5+1})$	4	6	2
$\mathrm{nullity}(\mathcal{L}_{q}^{5+1})$	1	0	0
$\mathrm{Spectra}(\mathcal{L}_{q}^{5+1})$	$\{0,1,2,4,5\}$	$\{1,4,4,4,4,5\}$	$\{2,4\}$

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