Biostatistics (2019), **0**, 0, *pp.* 1–39
doi:10.1093/biostatistics/Appendix

Supplementary Material

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1. Additional Figure

Figure 1 below presents the net survival curves obtained for the whole population of men lung cancer patients obtained with models M1–M3 as well as with the non-parametric Pohar-Perme estimator (Perme *and others*, 2012). In contrast, Figure 3 in the paper presents the net survival curves associated to specific patient characteristics (age, deprivation, comorbidities, and stage). Additional details on the models and the data set can be found in Section "Application: Lung Cancer data" of the paper.

2. Technical Details

Calculation of (2.6)

The overall survival function is obtained by integrating out γ

$$\tilde{S}_o(t;\mathbf{x}) = \int_0^\infty S_o(t \mid \gamma;\mathbf{x})dG(\gamma),$$

where $S_o(t \mid \gamma;\mathbf{x}) = \exp\left\{-\int_0^t h_o(s \mid \gamma;\mathbf{x})ds\right\}$. By using the decomposition (2.5), we can rewrite the conditional cumulative hazard as follows:

$$\tilde{H}_o(t \mid \gamma;\mathbf{x}) = H_E(t;\mathbf{x}) + \gamma\left[H_P(A+t;\mathbf{z}) - H_P(A;\mathbf{z})\right].$$

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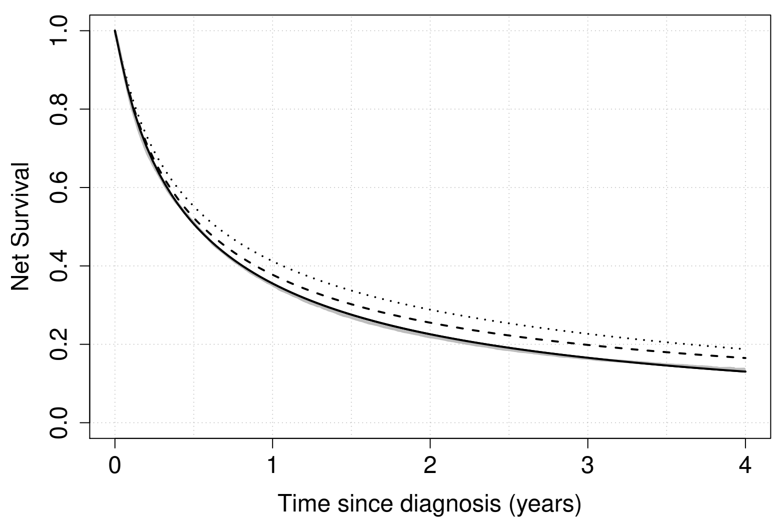

Fig. 1. Illustration for men lung cancer patients: comparison of net survival curves obtained with models M1 (continuous black line), M2 (dashed line), M3 (dotted line), as well as the non-parametric Pohar-Perme estimator (continuous grey line) for the whole population.

Consequently,

$$\begin{aligned}\tilde{S}_o(t;\mathbf{x}) &= \int_0^{\infty} \exp\left\{-\gamma\left[H_P(A+t;\mathbf{z}) - H_P(A;\mathbf{z})\right] - H_E(t;\mathbf{x})\right\} dG(\gamma) \\ &= \exp\left\{-H_E(t;\mathbf{x})\right\} \int_0^{\infty} \exp\left\{-\gamma\left[H_P(A+t;\mathbf{z}) - H_P(A;\mathbf{z})\right]\right\} dG(\gamma),\end{aligned}$$

This equation thus indicate that the overall survival can be written as the product of the net survival, $\exp\{-H_E(t;\mathbf{x})\}$, multiplied by the Laplace transform of the differences of the population cumulative hazard.

3. True values of the parameters and corresponding net survival

Simulation design I

Table 1 shows the net survival (NS) associated to the different covariate values in the simulation design I.

Simulation design II

For the Simulation design II, we consider an alternative baseline hazard function given by the Weibull distribution with scale parameter $\theta = 12$ and shape parameter $\kappa = 0.8$. This baseline hazard is decreasing. Moreover, when the baseline hazard is Weibull, it is known that the AFT, AH, and PH structures coincide, thus the additional parameters in the GH structure become redundant (Rubio *and others*, 2018). Thus, for

sex	comorb.	age	NS(1)	NS(5)
0	0	60	0.90	0.51
0	1	60	0.87	0.42
1	0	60	0.88	0.44
1	1	60	0.85	0.35
0	0	70	0.81	0.34
0	1	70	0.76	0.25
1	0	70	0.77	0.27
1	1	70	0.72	0.19
0	0	80	0.69	0.22
0	1	80	0.63	0.15
1	0	80	0.64	0.16
1	1	80	0.56	0.10

Table 1. Simulation design I: $(\kappa, \theta, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (age, sex, comorbidity) = (0.1, 0.1, 0.1)$, and $\beta_2 = (age, sex, comorbidity) = (0.05, 0.2, 0.25)$. Net Survival (NS) associated to the different covariate configurations at 1 and 5 years.

this scenario we only fit the PH model with a Weibull baseline hazard (we emphasise that this is not a restriction as the PH, the AH, and the AFT model coincide when we assume a Weibull baseline hazard). The fitted model now contains two parameters for the baseline hazard (θ, κ) and three regression parameters $(\beta_1, \beta_2, \beta_3)$. The simulation of the covariates is done in the same way as described in the Simulation Studies section in the main paper. Table 2 shows the net survival (NS) associated to the different covariate values in the simulation design I. Table 18 shows the proportion of selected hazard structures using AIC for both designs.

sex	comorb.	age	NS(1)	NS(5)
0	0	60	0.90	0.69
0	1	60	0.89	0.67
1	0	60	0.89	0.67
1	1	60	0.88	0.64
0	0	70	0.87	0.61
0	1	70	0.86	0.58
1	0	70	0.86	0.58
1	1	70	0.85	0.55
0	0	80	0.83	0.51
0	1	80	0.82	0.48
1	0	80	0.82	0.48
1	1	80	0.80	0.44

Table 2. Simulation design II: $(\theta, \kappa) = (12, 0.8)$, and $\beta = (age, sex, comorbidity) = (0.03, 0.1, 0.1)$. Net Survival (NS) associated to the different covariate configurations at 1 and 5 years.

4. SIMULATION STUDY

4.1 $n = 5000$: *Simulation design I*

The results associated to $n = 5000$ and design I are presented in Tables 3–5. The corresponding fitted hazards are presented in Figures 2–4.

Design I: $\gamma = 1$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	1.786	1.806	0.414	0.421	0.416	0.948
	κ (0.6)	0.606	0.607	0.065	0.066	0.065	0.946
	α (2.5)	2.529	2.440	0.469	0.458	0.470	0.954
	β_{11} (0.1)	0.102	0.102	0.015	0.014	0.015	0.957
	β_{12} (0.1)	0.083	0.089	0.256	0.239	0.256	0.948
	β_{13} (0.1)	0.082	0.085	0.239	0.238	0.240	0.957
	β_{21} (0.05)	0.050	0.050	0.003	0.003	0.003	0.949
	β_{22} (0.2)	0.200	0.198	0.042	0.042	0.042	0.955
	β_{23} (0.25)	0.250	0.249	0.045	0.042	0.045	0.934
M2	σ (1.75)	1.649	1.675	0.512	0.526	0.521	0.948
	κ (0.6)	0.589	0.587	0.099	0.102	0.100	0.916
	α (2.5)	2.802	2.595	0.964	0.898	1.010	0.926
	β_{11} (0.1)	0.101	0.101	0.014	0.014	0.014	0.951
	β_{12} (0.1)	0.107	0.116	0.244	0.234	0.244	0.945
	β_{13} (0.1)	0.085	0.089	0.235	0.241	0.235	0.964
	β_{21} (0.05)	0.049	0.049	0.005	0.005	0.005	0.935
	β_{22} (0.2)	0.195	0.200	0.057	0.058	0.057	0.950
	β_{23} (0.25)	0.255	0.251	0.055	0.053	0.055	0.935
	γ (1)	1.027	1.025	0.664	0.661	0.664	0.876
M3	σ (1.75)	1.601	1.601	0.558	0.607	0.577	0.970
	κ (0.6)	0.566	0.562	0.101	0.111	0.107	0.951
	α (2.5)	3.029	2.774	1.118	1.110	1.236	0.966
	β_{11} (0.1)	0.101	0.100	0.015	0.015	0.015	0.958
	β_{12} (0.1)	0.080	0.091	0.251	0.252	0.252	0.953
	β_{13} (0.1)	0.086	0.094	0.248	0.253	0.248	0.958
	β_{21} (0.05)	0.047	0.047	0.006	0.006	0.007	0.955
	β_{22} (0.2)	0.178	0.183	0.063	0.067	0.067	0.972
	β_{23} (0.25)	0.272	0.269	0.061	0.062	0.064	0.952
	b (0)	0.632	0.100	1.004	4.925	1.186	–
	μ (1)	1.482	1.448	0.791	0.874	0.926	0.788
M4	σ (1.75)	1.655	1.711	0.497	0.459	0.506	0.961
	κ (0.6)	0.584	0.591	0.088	0.075	0.089	0.922
	α (2.5)	2.831	2.587	0.978	0.717	1.032	0.942
	β_{11} (0.1)	0.100	0.100	0.016	0.014	0.016	0.935
	β_{12} (0.1)	0.086	0.093	0.253	0.235	0.253	0.941
	β_{13} (0.1)	0.091	0.090	0.240	0.235	0.240	0.948
	β_{21} (0.05)	0.049	0.049	0.005	0.004	0.005	0.921
	β_{22} (0.2)	0.193	0.197	0.055	0.048	0.055	0.944
	β_{23} (0.25)	0.258	0.253	0.054	0.047	0.055	0.914
	c (1)	1.234	1.000	0.739	–	0.775	–

Table 3. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 5000$, and no mismatch $\gamma = 1$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design I: $\gamma = 1$

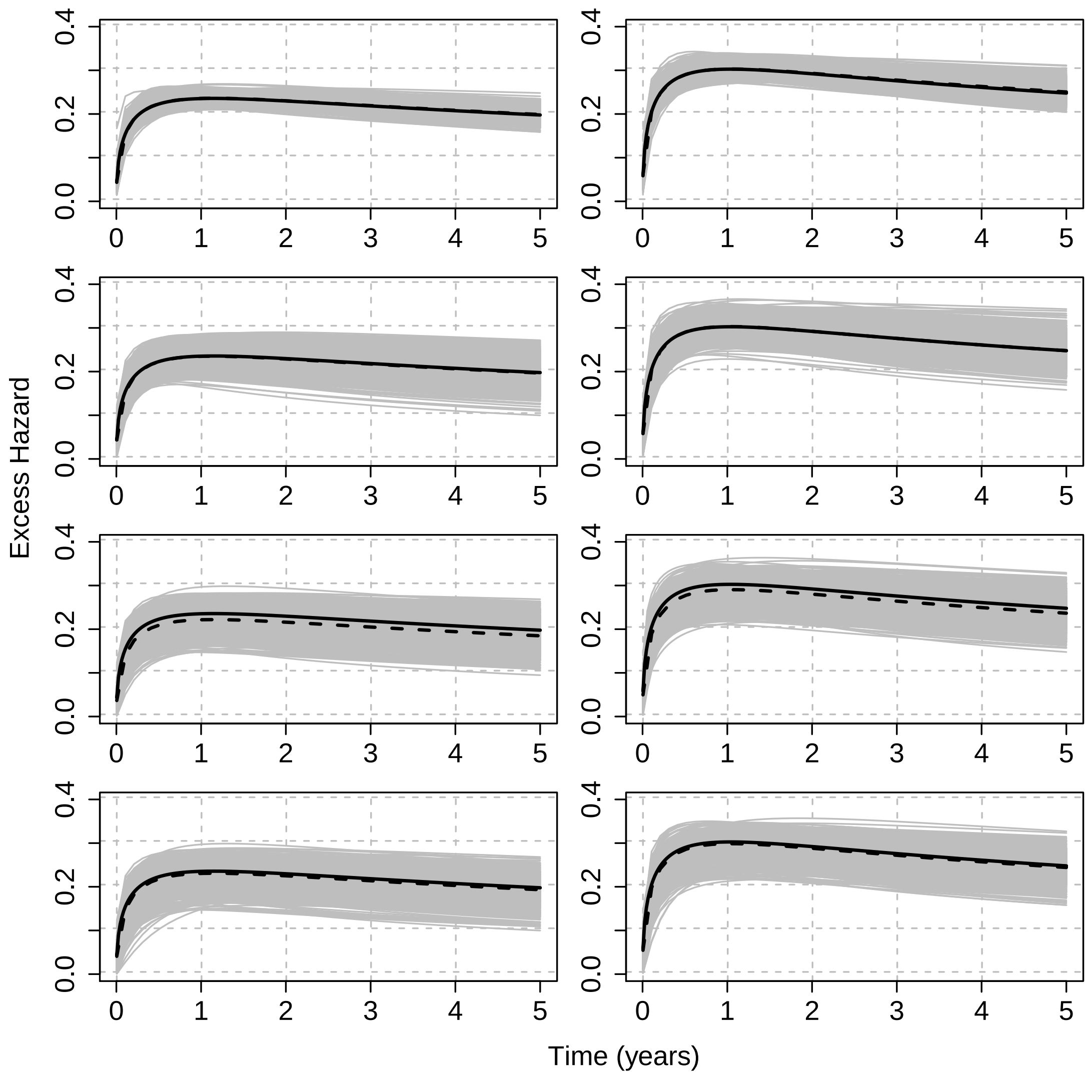

Fig. 2. Scenario with no mismatch $\gamma = 1$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(1.2, 0.02)$

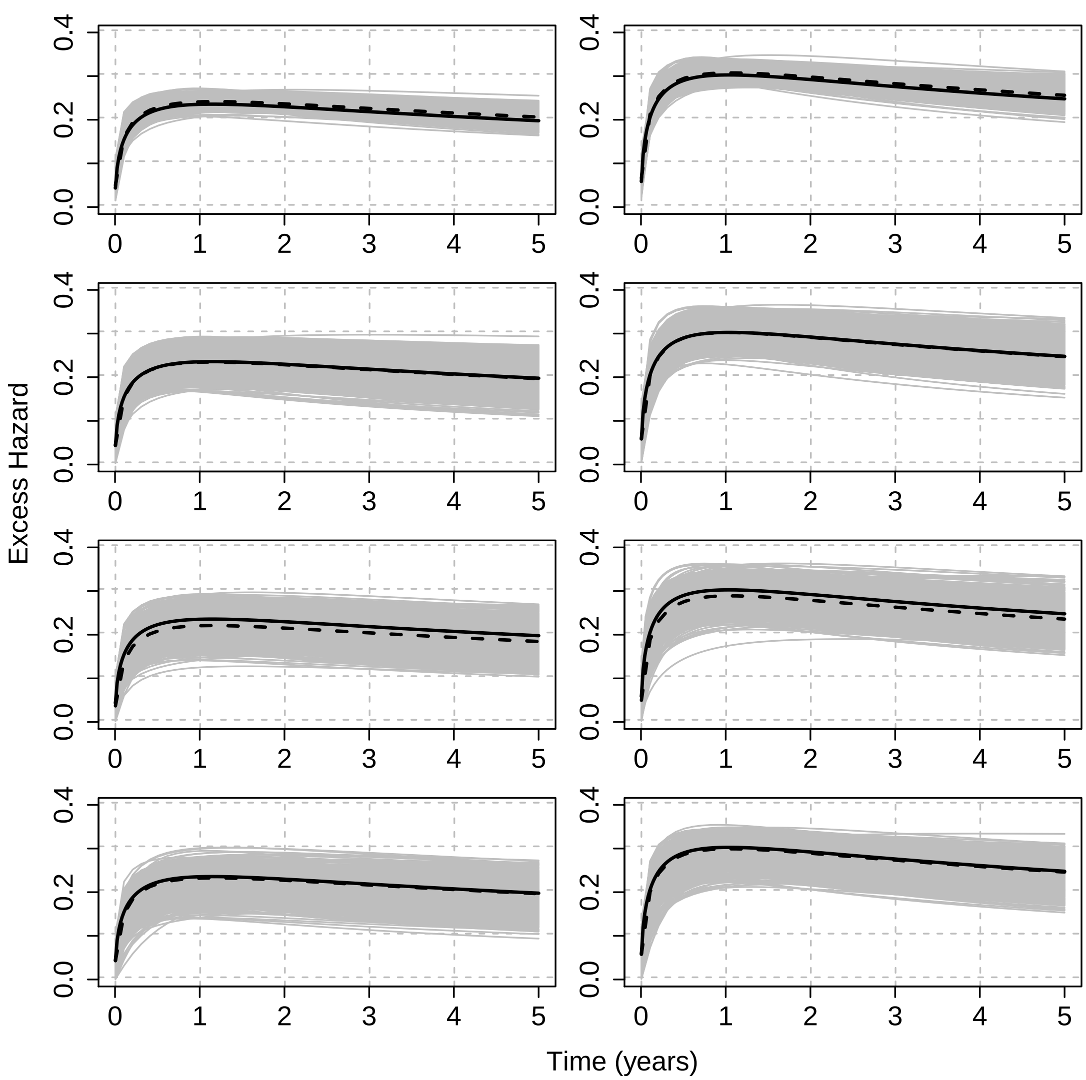

Fig. 3. Scenario with moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(1.875, 0.075)$

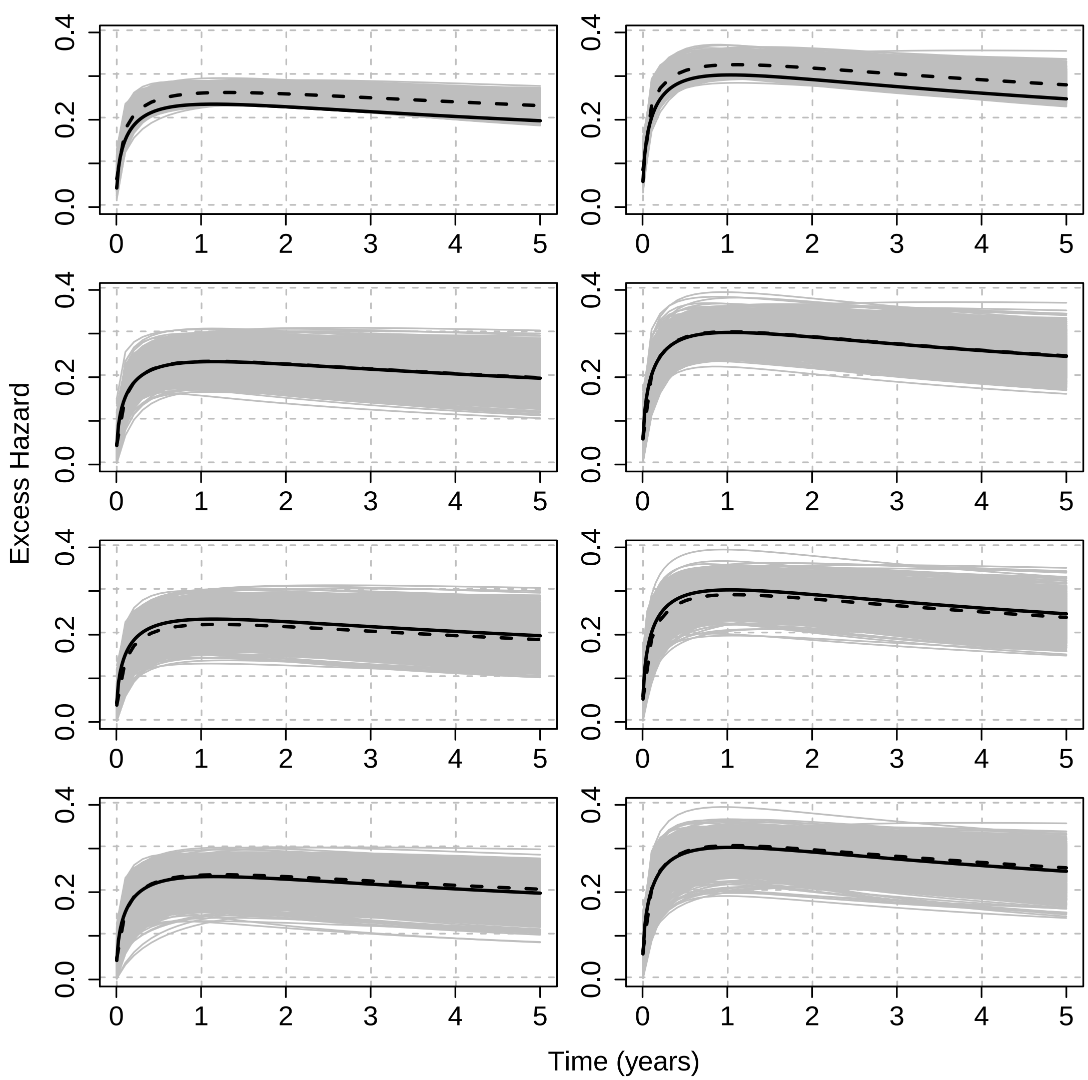

Fig. 4. Scenario with severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(1.2, 0.02)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	1.860	1.860	0.406	0.409	0.421	0.930
	κ (0.6)	0.626	0.626	0.066	0.066	0.071	0.925
	α (2.5)	2.399	2.341	0.433	0.417	0.444	0.931
	β_{11} (0.1)	0.101	0.099	0.015	0.015	0.015	0.955
	β_{12} (0.1)	0.083	0.080	0.254	0.247	0.254	0.958
	β_{13} (0.1)	0.107	0.108	0.250	0.246	0.250	0.951
	β_{21} (0.05)	0.051	0.051	0.003	0.003	0.003	0.922
	β_{22} (0.2)	0.211	0.213	0.043	0.042	0.044	0.943
	β_{23} (0.25)	0.241	0.242	0.042	0.042	0.043	0.950
M2	σ (1.75)	1.656	1.704	0.541	0.535	0.549	0.934
	κ (0.6)	0.590	0.591	0.107	0.105	0.107	0.916
	α (2.5)	2.818	2.578	1.034	0.940	1.081	0.928
	β_{11} (0.1)	0.101	0.100	0.015	0.014	0.015	0.956
	β_{12} (0.1)	0.108	0.112	0.239	0.241	0.239	0.960
	β_{13} (0.1)	0.097	0.099	0.250	0.249	0.250	0.948
	β_{21} (0.05)	0.049	0.050	0.005	0.005	0.005	0.924
	β_{22} (0.2)	0.196	0.201	0.058	0.060	0.058	0.954
	β_{23} (0.25)	0.254	0.250	0.055	0.055	0.056	0.942
	γ (1.2)	1.218	1.254	0.736	0.706	0.736	0.883
M3	σ (1.75)	1.609	1.637	0.591	0.628	0.607	0.957
	κ (0.6)	0.567	0.568	0.108	0.116	0.113	0.958
	α (2.5)	3.061	2.755	1.211	1.181	1.334	0.970
	β_{11} (0.1)	0.100	0.099	0.016	0.016	0.016	0.945
	β_{12} (0.1)	0.079	0.076	0.261	0.262	0.262	0.959
	β_{13} (0.1)	0.088	0.092	0.253	0.260	0.253	0.957
	β_{21} (0.05)	0.046	0.047	0.007	0.007	0.008	0.957
	β_{22} (0.2)	0.176	0.183	0.066	0.070	0.070	0.961
	β_{23} (0.25)	0.271	0.266	0.062	0.064	0.066	0.952
	b (0.02)	0.613	0.254	0.906	1.443	1.083	0.678
	μ (1.2)	1.711	1.718	0.856	0.930	0.997	0.774
M4	σ (1.75)	1.710	1.759	0.542	0.469	0.543	0.950
	κ (0.6)	0.596	0.609	0.098	0.077	0.098	0.921
	α (2.5)	2.785	2.458	1.125	0.749	1.160	0.939
	β_{11} (0.1)	0.100	0.099	0.017	0.015	0.017	0.943
	β_{12} (0.1)	0.089	0.085	0.255	0.246	0.255	0.953
	β_{13} (0.1)	0.113	0.106	0.260	0.247	0.260	0.951
	β_{21} (0.05)	0.049	0.050	0.006	0.004	0.006	0.902
	β_{22} (0.2)	0.197	0.203	0.059	0.049	0.059	0.936
	β_{23} (0.25)	0.253	0.248	0.058	0.048	0.058	0.937
	c (1.2)	1.348	1.000	0.817	–	0.830	–

Table 4. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 5000$, and moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

4.2 $n = 10000$*: Simulation design I*

The results associated to $n = 10000$ and design I are presented in Tables 6–9. The corresponding fitted hazards are presented in Figures 5–8.

4.3 $n = 5000$*: Simulation design II*

The results associated to $n = 5000$ and design II are presented in Tables 10–13. The corresponding fitted hazards are presented in Figures 9–12.

Design I: $\gamma = 1$

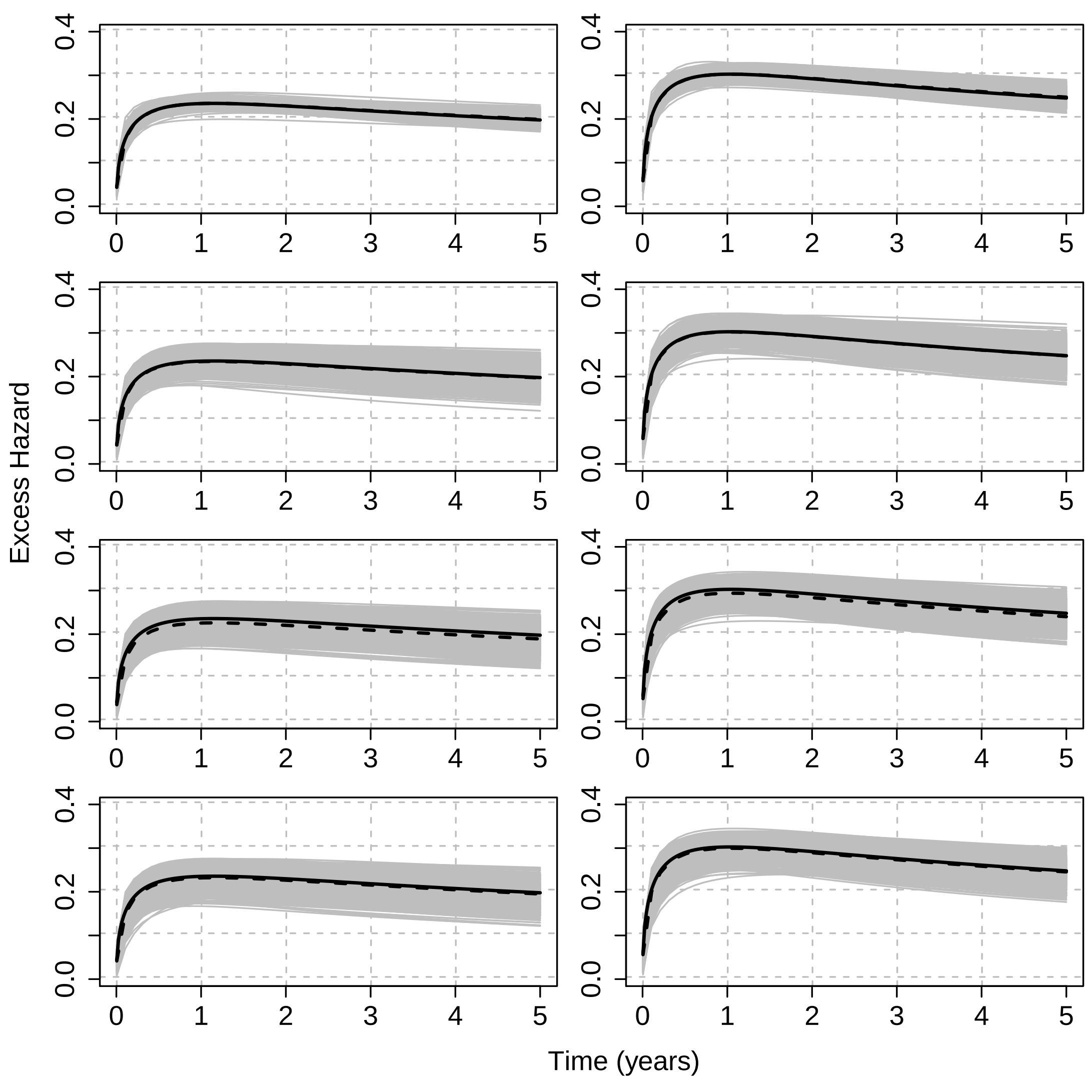

Fig. 5. Scenario with no mismatch $\gamma = 1$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(1.2, 0.02)$

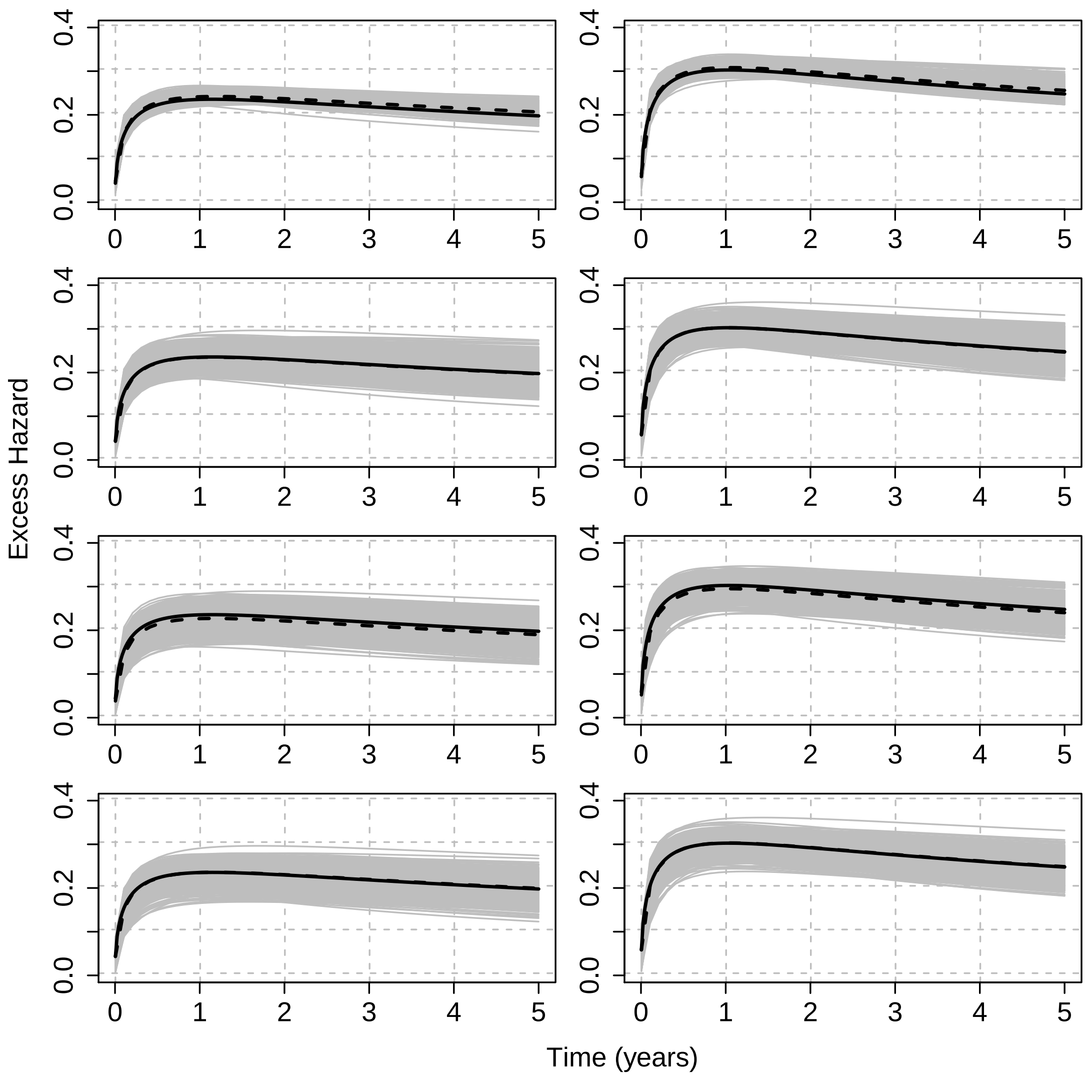

Fig. 6. Scenario with moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(1.875, 0.075)$

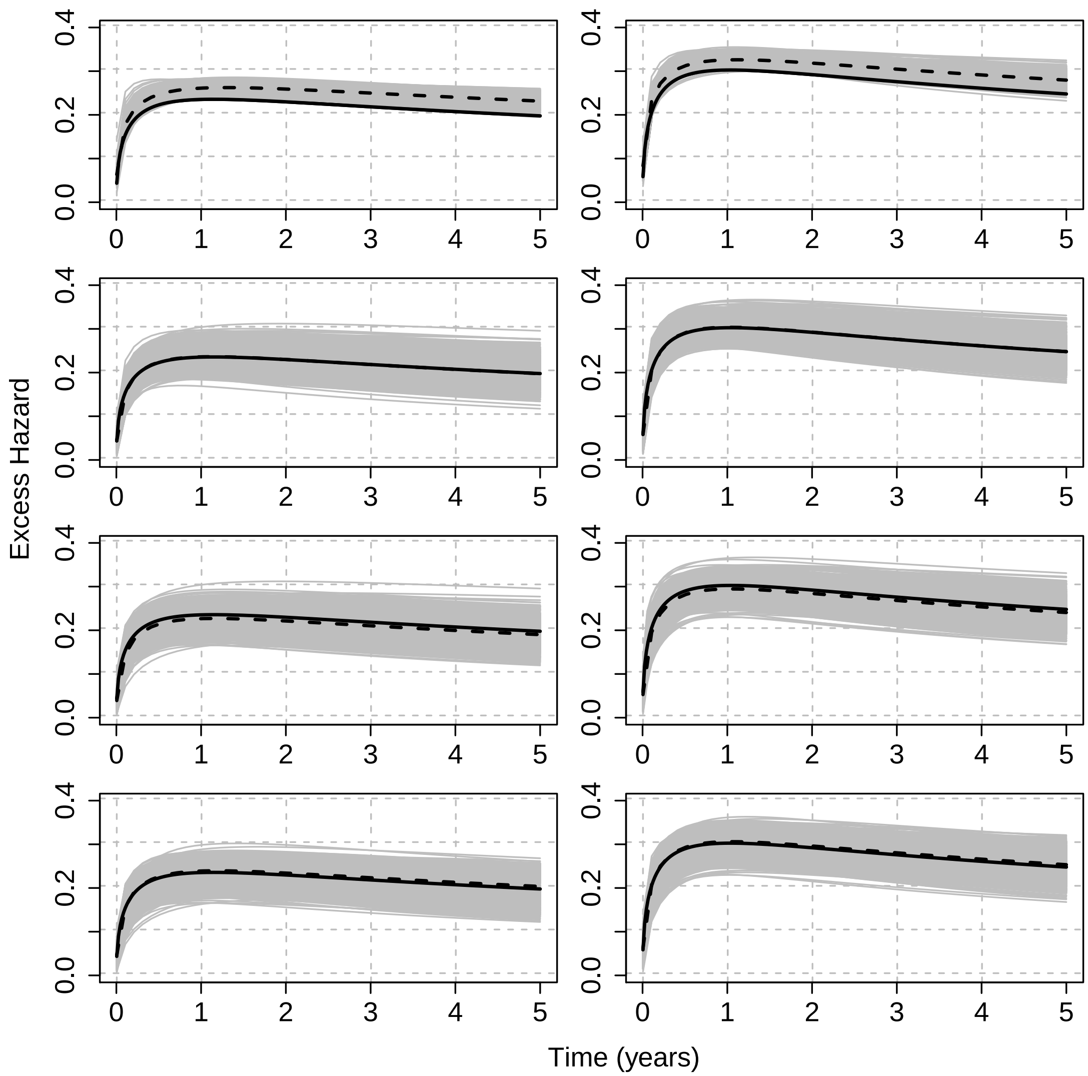

Fig. 7. Scenario with severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(6.5, 10)$

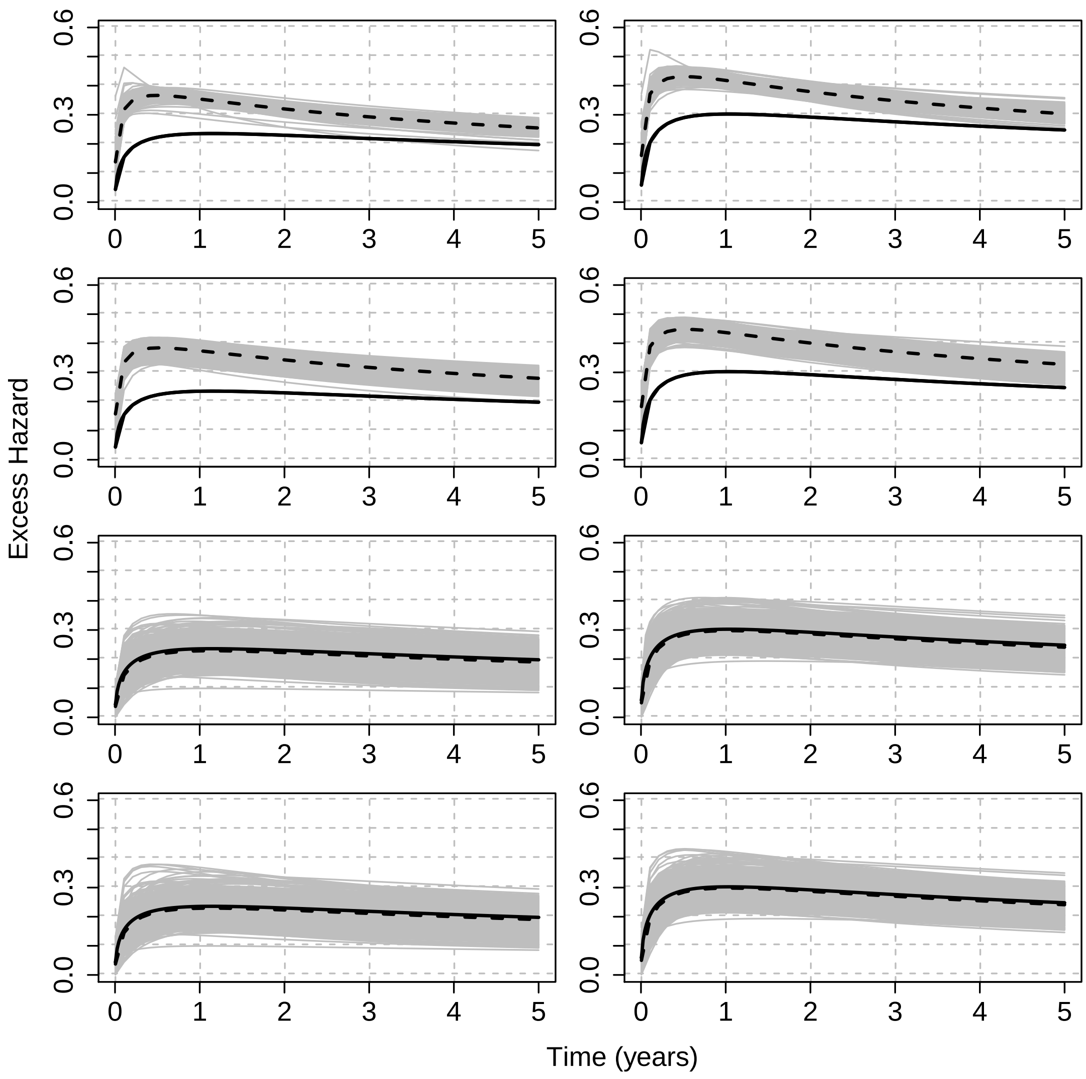

Fig. 8. Scenario with wide mismatch $\gamma \sim Ga(6.5, 10)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma = 1$

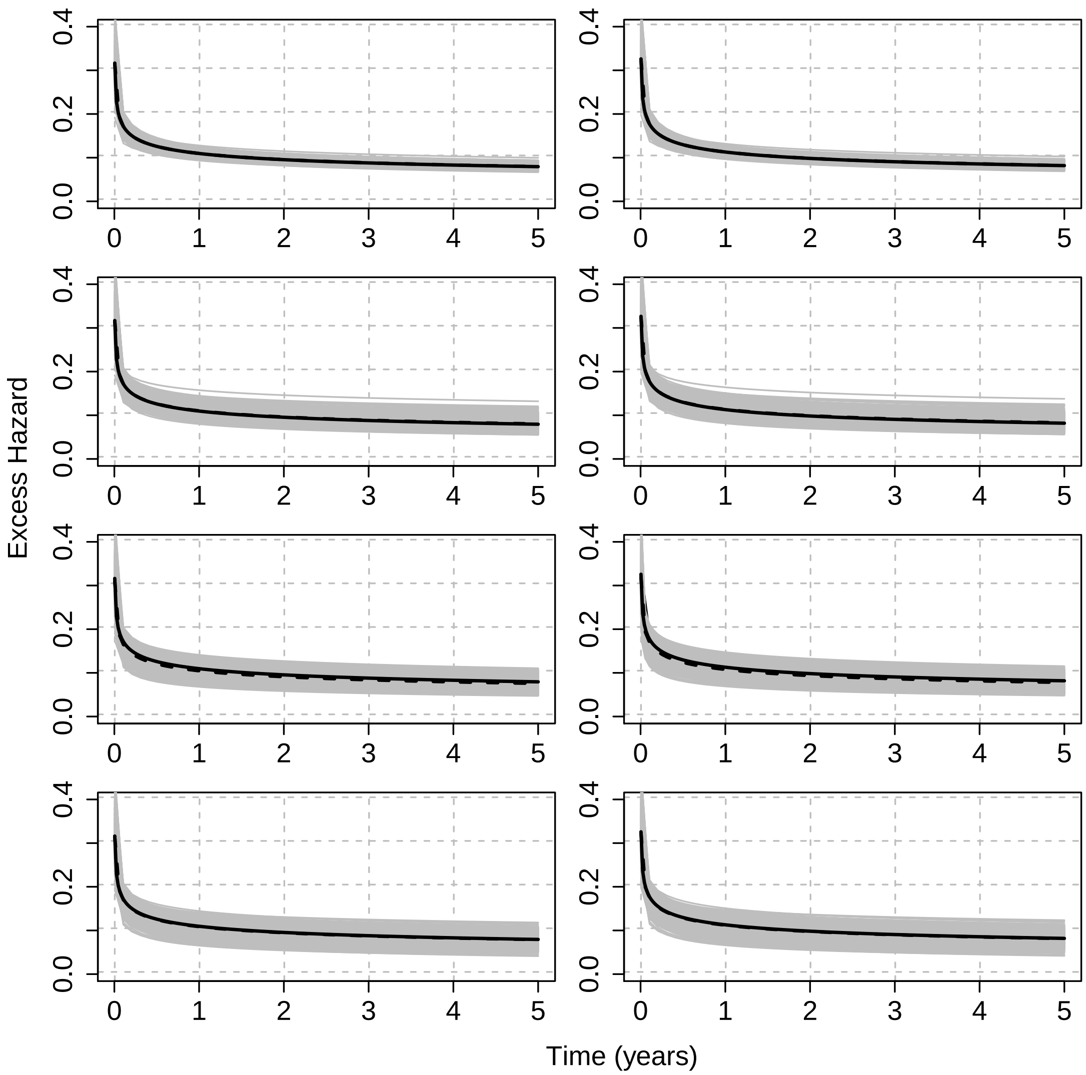

Fig. 9. Scenario with no mismatch $\gamma = 1$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma \sim Ga(1.2, 0.02)$

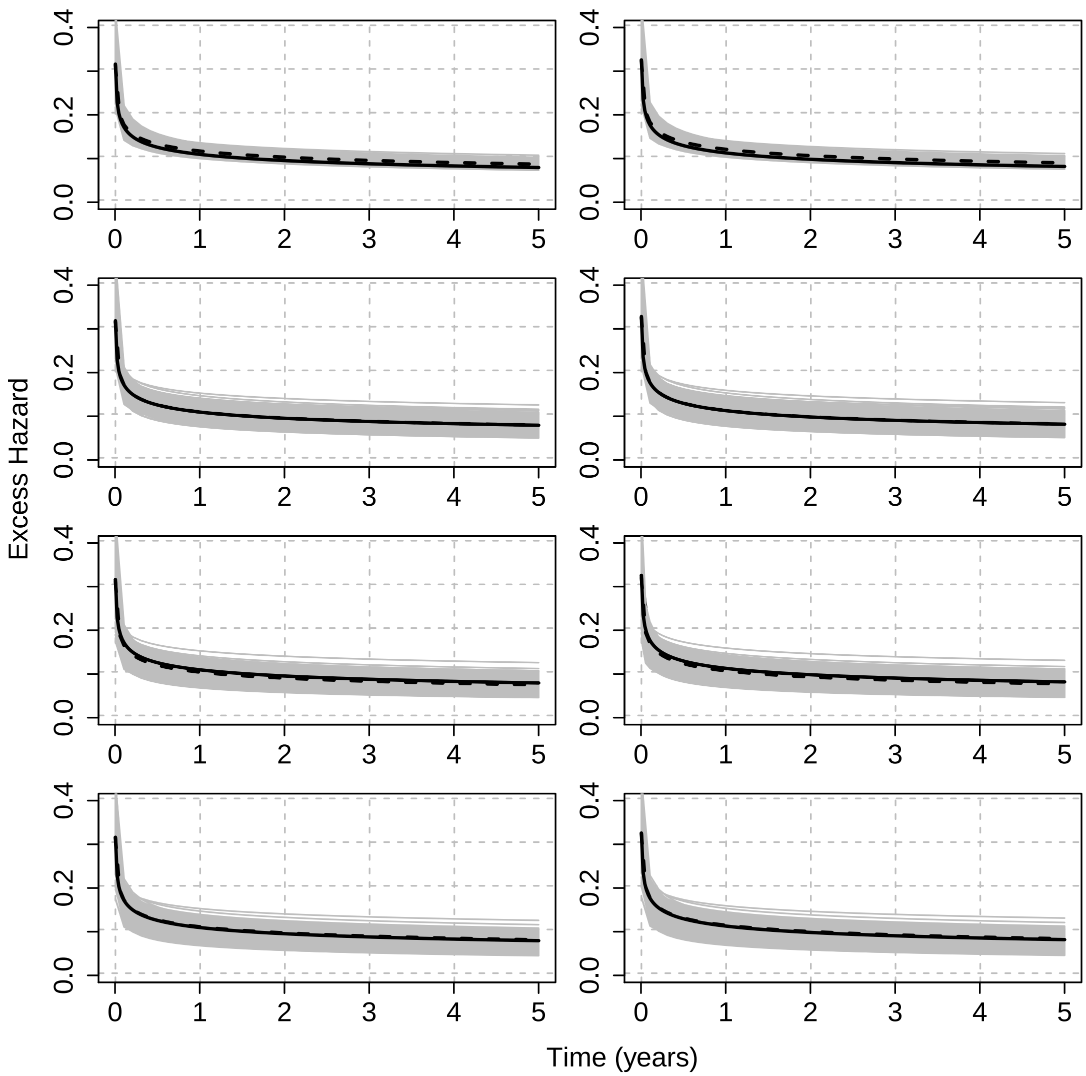

Fig. 10. Scenario with moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma \sim Ga(1.875, 0.075)$

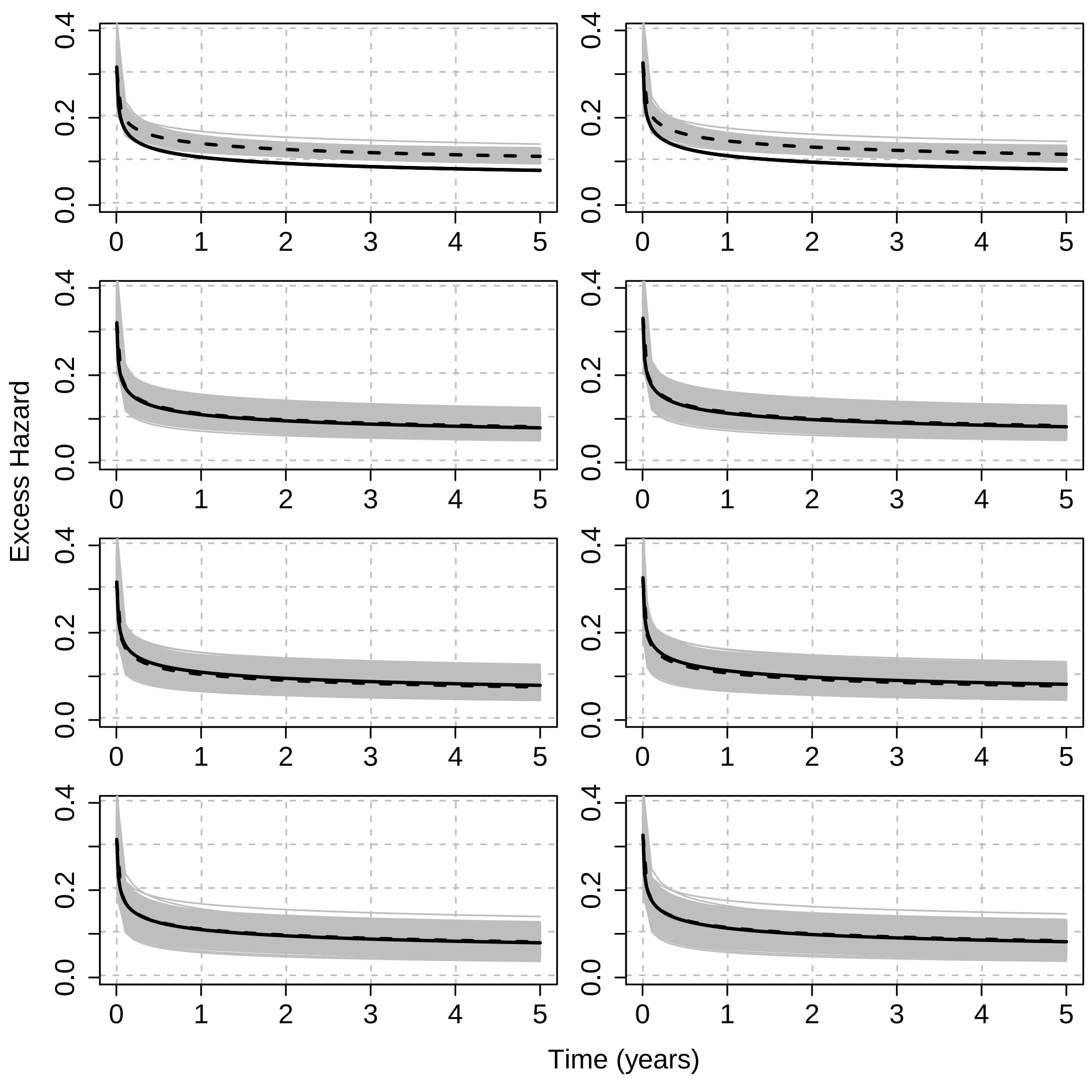

Fig. 11. Scenario with severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma \sim Ga(6.5, 10)$

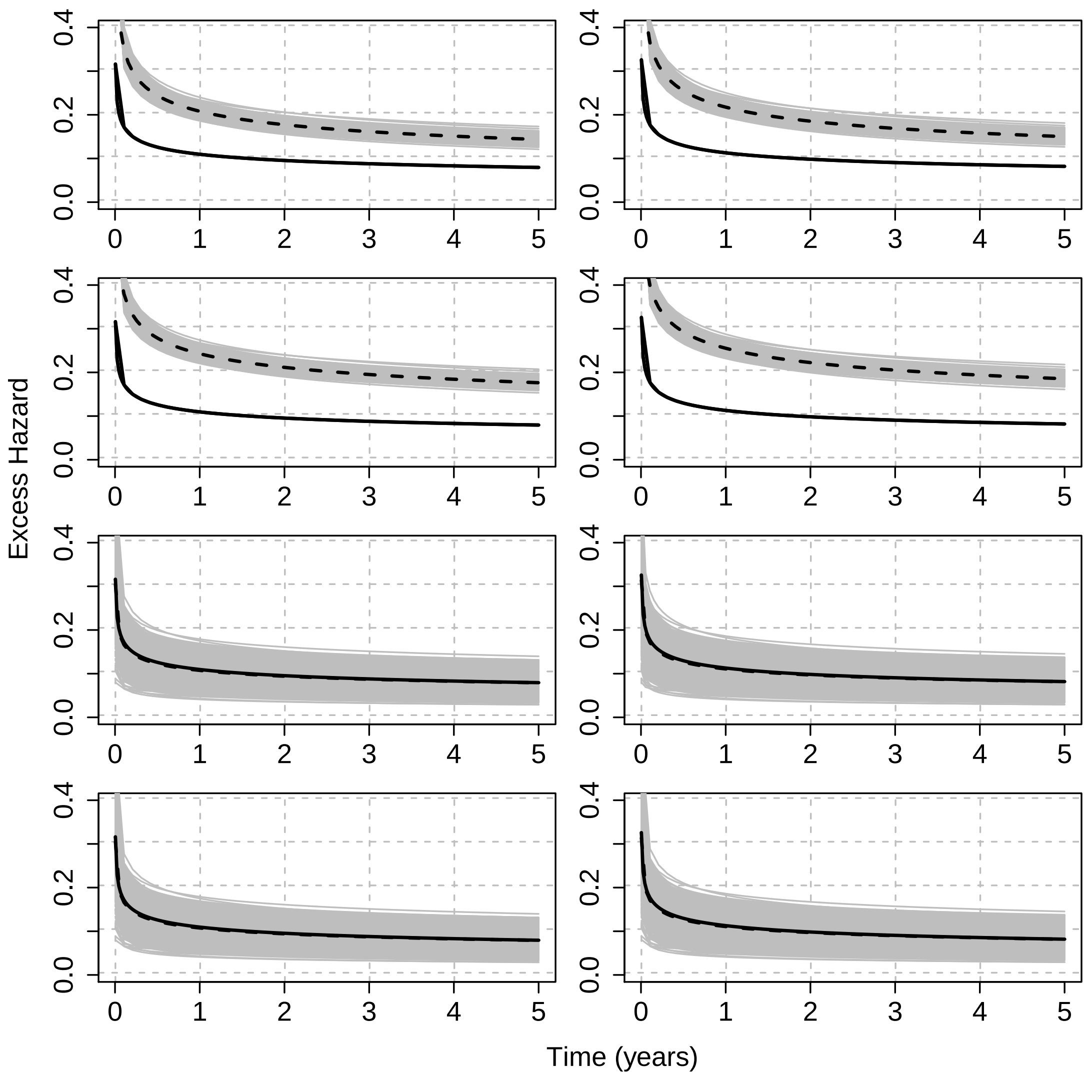

Fig. 12. Scenario with wide mismatch $\gamma \sim Ga(6.5, 10)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 5000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(1.875, 0.075)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	2.039	2.045	0.368	0.370	0.468	0.825
	κ (0.6)	0.688	0.686	0.067	0.067	0.111	0.687
	α (2.5)	2.064	2.026	0.320	0.318	0.541	0.697
	β_{11} (0.1)	0.102	0.101	0.016	0.016	0.016	0.962
	β_{12} (0.1)	0.079	0.092	0.278	0.275	0.279	0.952
	β_{13} (0.1)	0.170	0.174	0.281	0.275	0.290	0.958
	β_{21} (0.05)	0.054	0.054	0.003	0.003	0.005	0.701
	β_{22} (0.2)	0.241	0.240	0.040	0.040	0.057	0.832
	β_{23} (0.25)	0.216	0.216	0.040	0.040	0.053	0.859
M2	σ (1.75)	1.657	1.674	0.564	0.567	0.571	0.915
	κ (0.6)	0.594	0.589	0.113	0.115	0.113	0.938
	α (2.5)	2.810	2.578	1.008	1.014	1.054	0.940
	β_{11} (0.1)	0.102	0.101	0.016	0.016	0.016	0.954
	β_{12} (0.1)	0.115	0.124	0.263	0.263	0.264	0.957
	β_{13} (0.1)	0.110	0.114	0.276	0.272	0.276	0.959
	β_{21} (0.05)	0.049	0.050	0.006	0.006	0.006	0.946
	β_{22} (0.2)	0.195	0.201	0.067	0.066	0.067	0.933
	β_{23} (0.25)	0.253	0.253	0.058	0.060	0.058	0.938
	γ (1.875)	1.845	1.935	0.845	0.854	0.845	0.910
M3	σ (1.75)	1.659	1.664	0.622	0.653	0.628	0.935
	κ (0.6)	0.580	0.575	0.117	0.124	0.119	0.957
	α (2.5)	2.976	2.690	1.157	1.198	1.251	0.965
	β_{11} (0.1)	0.100	0.099	0.017	0.017	0.017	0.944
	β_{12} (0.1)	0.079	0.101	0.291	0.288	0.292	0.964
	β_{13} (0.1)	0.112	0.111	0.284	0.287	0.284	0.957
	β_{21} (0.05)	0.047	0.048	0.007	0.007	0.008	0.958
	β_{22} (0.2)	0.182	0.189	0.072	0.075	0.075	0.955
	β_{23} (0.25)	0.268	0.265	0.066	0.068	0.068	0.949
	b (0.075)	0.543	0.206	0.814	7.701	0.939	0.783
	μ (1.875)	2.304	2.327	0.951	1.023	1.043	0.837
M4	σ (1.75)	1.770	1.860	0.607	0.497	0.607	0.889
	κ (0.6)	0.619	0.643	0.121	0.087	0.123	0.807
	α (2.5)	2.729	2.269	1.359	0.872	1.377	0.813
	β_{11} (0.1)	0.100	0.099	0.018	0.016	0.018	0.936
	β_{12} (0.1)	0.079	0.105	0.306	0.276	0.307	0.951
	β_{13} (0.1)	0.142	0.130	0.294	0.274	0.297	0.941
	β_{21} (0.05)	0.050	0.052	0.007	0.005	0.007	0.793
	β_{22} (0.2)	0.208	0.220	0.077	0.056	0.077	0.872
	β_{23} (0.25)	0.247	0.239	0.064	0.052	0.065	0.899
	c (1.875)	1.786	1.000	1.101	–	1.104	–

Table 5. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 5000$, and severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

4.4 $n = 10000$: *Simulation design II*

The results associated to $n = 10000$ and design II are presented in Tables 14–17. The corresponding fitted hazards are presented in Figures 13–16.

5. Results for national life tables

Table 19 shows the results for the fitted excess hazard models using life tables without the deprivation variable (*i.e.* national life tables). See the Section "Application: Lung Cancer data" of the paper for more

Design II: $\gamma = 1$

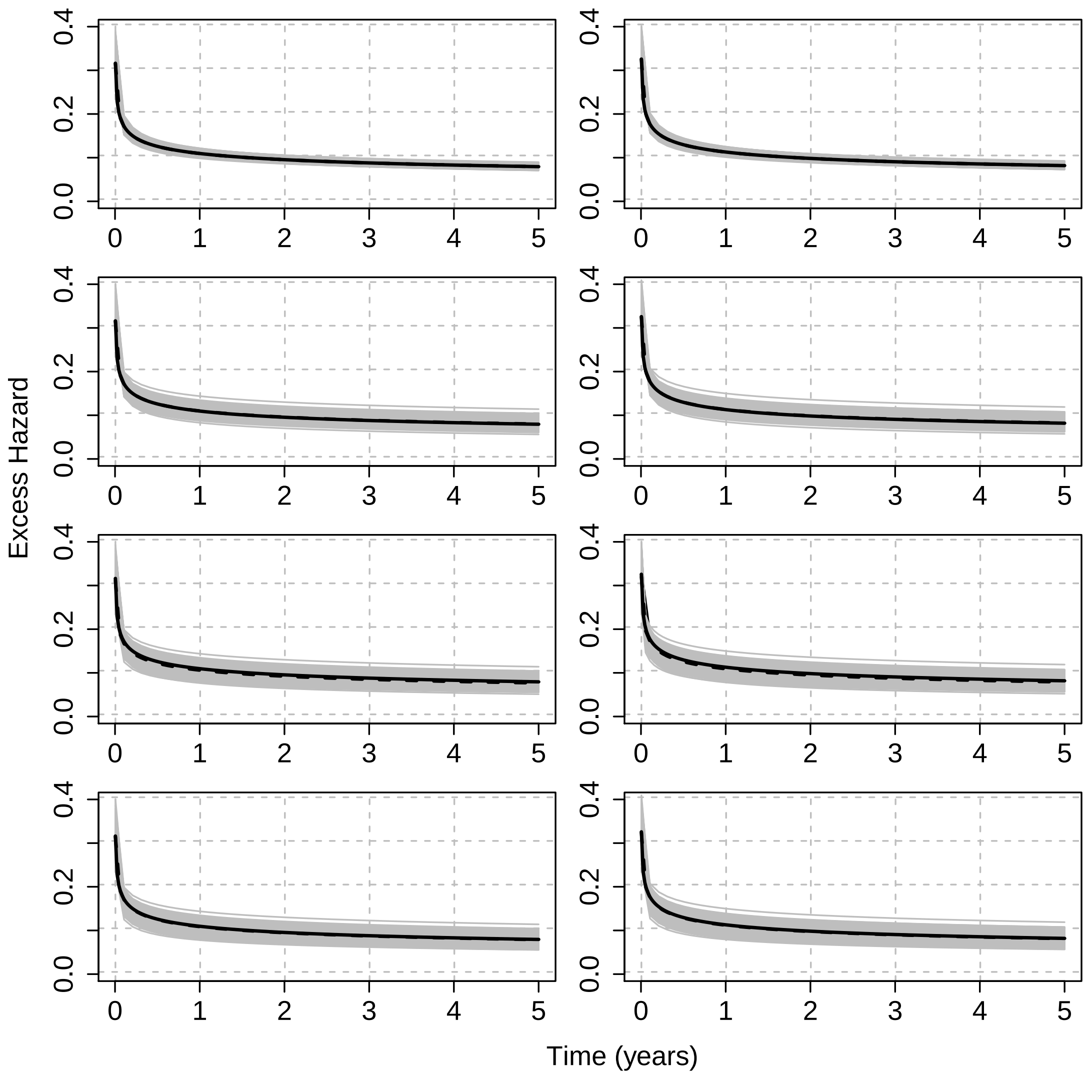

Fig. 13. Scenario with no mismatch $\gamma = 1$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma \sim Ga(1.2, 0.02)$

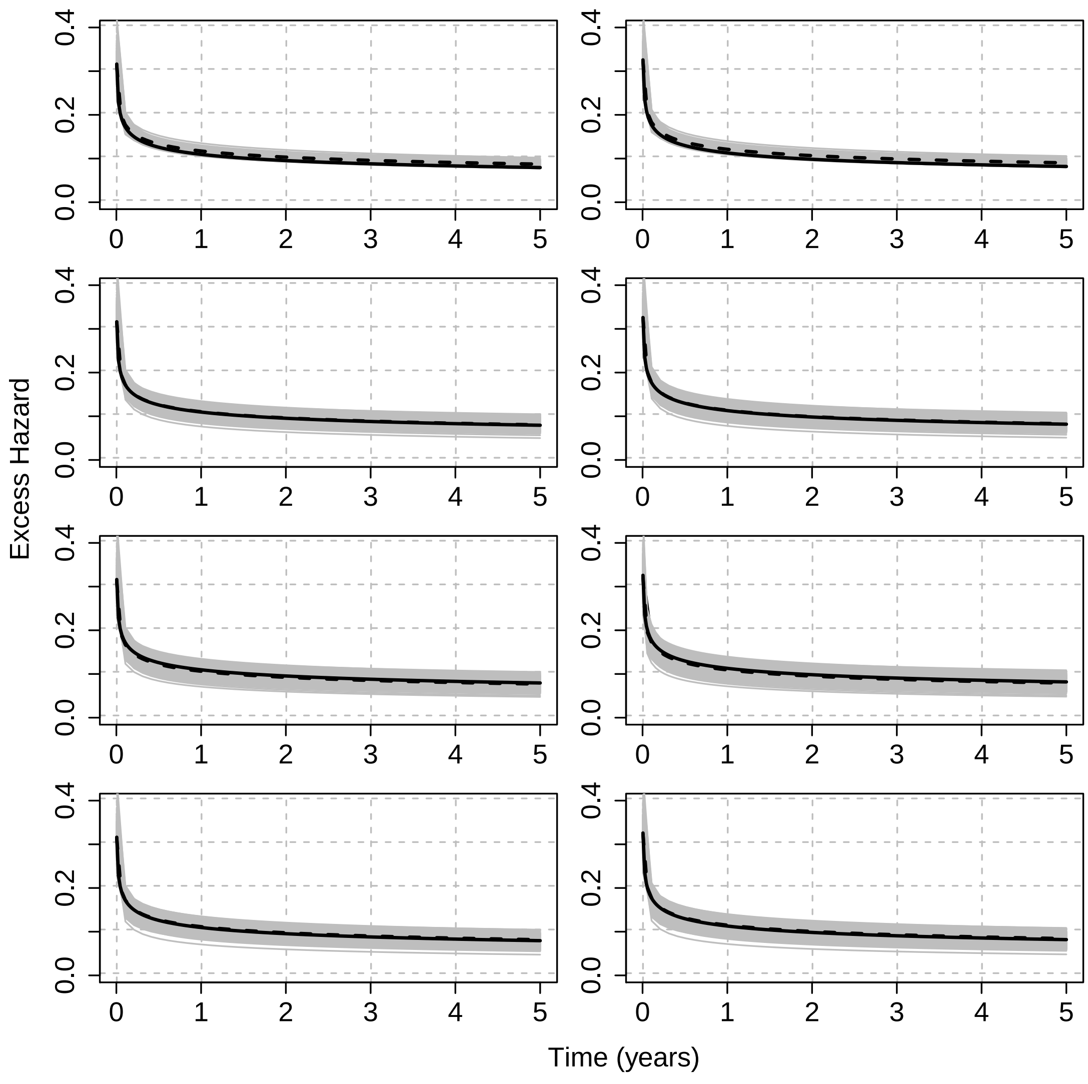

Fig. 14. Scenario with moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma \sim Ga(1.875, 0.075)$

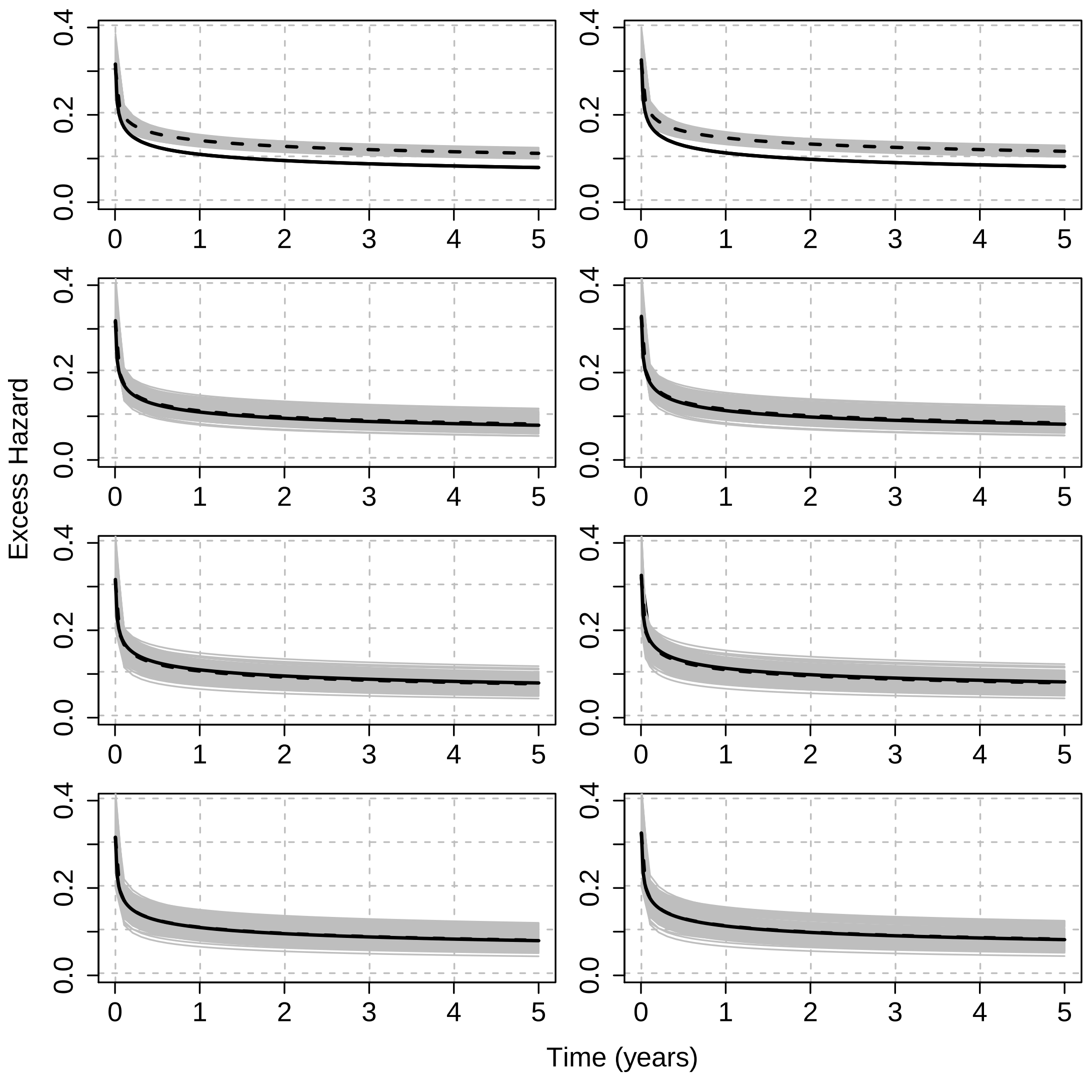

Fig. 15. Scenario with severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design II: $\gamma \sim Ga(6.5, 10)$

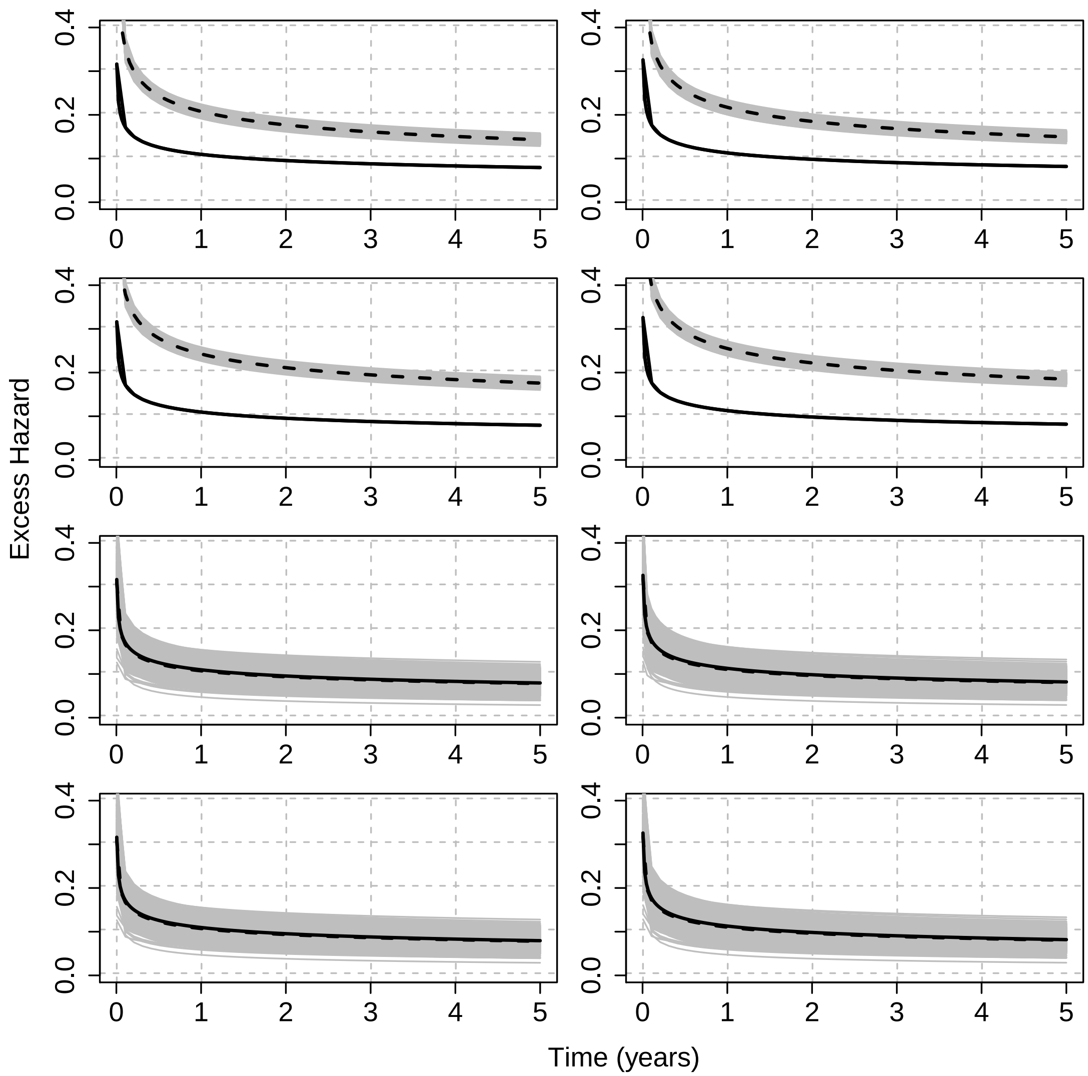

Fig. 16. Scenario with wide mismatch $\gamma \sim Ga(6.5, 10)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 1000 sample-specific fitted hazards (grey lines) for $n = 10000$ and 45% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma = 1$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	1.797	1.772	0.322	0.298	0.325	0.912
	κ (0.6)	0.608	0.605	0.050	0.046	0.050	0.920
	α (2.5)	2.488	2.483	0.332	0.313	0.332	0.922
	β_{11} (0.1)	0.102	0.101	0.010	0.010	0.011	0.956
	β_{12} (0.1)	0.092	0.098	0.179	0.166	0.179	0.936
	β_{13} (0.1)	0.095	0.094	0.177	0.166	0.177	0.940
	β_{21} (0.05)	0.050	0.050	0.002	0.002	0.002	0.942
	β_{22} (0.2)	0.200	0.200	0.030	0.030	0.030	0.942
	β_{23} (0.25)	0.250	0.250	0.030	0.030	0.030	0.943
M2	σ (1.75)	1.701	1.718	0.397	0.384	0.399	0.941
	κ (0.6)	0.595	0.596	0.077	0.075	0.077	0.931
	α (2.5)	2.650	2.532	0.622	0.572	0.639	0.932
	β_{11} (0.1)	0.101	0.101	0.010	0.010	0.010	0.958
	β_{12} (0.1)	0.105	0.105	0.168	0.164	0.168	0.953
	β_{13} (0.1)	0.098	0.091	0.174	0.170	0.174	0.944
	β_{21} (0.05)	0.049	0.050	0.004	0.004	0.004	0.929
	β_{22} (0.2)	0.196	0.197	0.041	0.042	0.041	0.936
	β_{23} (0.25)	0.253	0.253	0.037	0.038	0.037	0.953
	γ (1)	1.036	1.049	0.519	0.511	0.520	0.887
M3	σ (1.75)	1.670	1.684	0.423	0.425	0.430	0.956
	κ (0.6)	0.580	0.577	0.078	0.080	0.081	0.950
	α (2.5)	2.781	2.667	0.695	0.659	0.750	0.961
	β_{11} (0.1)	0.100	0.099	0.010	0.010	0.010	0.960
	β_{12} (0.1)	0.083	0.086	0.176	0.174	0.177	0.959
	β_{13} (0.1)	0.097	0.092	0.179	0.175	0.179	0.953
	β_{21} (0.05)	0.048	0.048	0.004	0.004	0.005	0.947
	β_{22} (0.2)	0.184	0.187	0.044	0.046	0.047	0.957
	β_{23} (0.25)	0.264	0.262	0.041	0.042	0.044	0.963
	b (0)	0.489	0.128	0.772	1.154	0.914	–
	μ (1)	1.341	1.360	0.604	0.645	0.694	0.810
M4	σ (1.75)	1.708	1.720	0.378	0.326	0.380	0.935
	κ (0.6)	0.593	0.596	0.067	0.053	0.068	0.909
	α (2.5)	2.656	2.550	0.593	0.426	0.613	0.922
	β_{11} (0.1)	0.100	0.100	0.010	0.010	0.010	0.949
	β_{12} (0.1)	0.098	0.101	0.174	0.165	0.174	0.945
	β_{13} (0.1)	0.100	0.100	0.173	0.165	0.173	0.946
	β_{21} (0.05)	0.049	0.050	0.004	0.003	0.004	0.907
	β_{22} (0.2)	0.194	0.197	0.039	0.033	0.039	0.920
	β_{23} (0.25)	0.255	0.253	0.036	0.032	0.037	0.945
	c (1)	1.143	1.000	0.503	–	0.523	–

Table 6. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 10000$, and no mismatch $\gamma = 1$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

details.

Design I: $\gamma \sim Ga(1.2, 0.02)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	1.846	1.846	0.305	0.291	0.319	0.906
	κ (0.6)	0.625	0.623	0.049	0.047	0.055	0.890
	α (2.5)	2.384	2.360	0.308	0.291	0.329	0.898
	β_{11} (0.1)	0.101	0.100	0.010	0.010	0.010	0.949
	β_{12} (0.1)	0.092	0.098	0.185	0.170	0.185	0.934
	β_{13} (0.1)	0.119	0.119	0.177	0.171	0.178	0.952
	β_{21} (0.05)	0.051	0.051	0.002	0.002	0.002	0.910
	β_{22} (0.2)	0.209	0.209	0.030	0.029	0.031	0.936
	β_{23} (0.25)	0.241	0.241	0.030	0.029	0.031	0.933
M2	σ (1.75)	1.671	1.697	0.409	0.392	0.417	0.940
	κ (0.6)	0.591	0.590	0.080	0.077	0.081	0.929
	α (2.5)	2.692	2.571	0.655	0.603	0.682	0.936
	β_{11} (0.1)	0.100	0.100	0.010	0.010	0.010	0.953
	β_{12} (0.1)	0.110	0.112	0.174	0.167	0.174	0.943
	β_{13} (0.1)	0.105	0.099	0.173	0.172	0.173	0.954
	β_{21} (0.05)	0.049	0.050	0.004	0.004	0.004	0.921
	β_{22} (0.2)	0.195	0.196	0.044	0.043	0.045	0.943
	β_{23} (0.25)	0.253	0.251	0.039	0.039	0.039	0.944
	γ (1.2)	1.224	1.270	0.556	0.536	0.556	0.900
M3	σ (1.75)	1.648	1.662	0.430	0.430	0.442	0.961
	κ (0.6)	0.579	0.578	0.080	0.081	0.083	0.953
	α (2.5)	2.804	2.679	0.708	0.679	0.771	0.959
	β_{11} (0.1)	0.099	0.099	0.010	0.010	0.010	0.952
	β_{12} (0.1)	0.090	0.092	0.183	0.176	0.183	0.947
	β_{13} (0.1)	0.107	0.098	0.177	0.177	0.177	0.957
	β_{21} (0.05)	0.048	0.048	0.004	0.004	0.005	0.939
	β_{22} (0.2)	0.184	0.186	0.047	0.047	0.049	0.947
	β_{23} (0.25)	0.263	0.260	0.042	0.043	0.044	0.955
	b (0.02)	0.401	0.081	0.663	0.670	0.764	0.717
	μ (1.2)	1.506	1.543	0.605	0.655	0.678	0.834
M4	σ (1.75)	1.728	1.761	0.391	0.328	0.391	0.924
	κ (0.6)	0.601	0.607	0.072	0.054	0.072	0.886
	α (2.5)	2.612	2.473	0.625	0.428	0.634	0.901
	β_{11} (0.1)	0.100	0.099	0.011	0.010	0.011	0.948
	β_{12} (0.1)	0.100	0.098	0.177	0.169	0.177	0.944
	β_{13} (0.1)	0.116	0.107	0.174	0.169	0.175	0.951
	β_{21} (0.05)	0.050	0.051	0.004	0.003	0.004	0.872
	β_{22} (0.2)	0.198	0.202	0.042	0.034	0.042	0.910
	β_{23} (0.25)	0.251	0.249	0.038	0.033	0.038	0.920
	c (1.2)	1.219	1.000	0.530	–	0.530	–

Table 7. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 10000$, and moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

6. Misspecification of the frailty distribution

In this section, we present a simulation study where we assess the effect of misspecifiying the distribution of the correction parameter γ in model (2.5). We simulate γ from a lognormal distribution with log-scale parameter $\sigma = 0.2$ and log-location parameter $\mu = 0.63$. This frailty distribution has mean 1.91 and variance 0.5. The Kullback-Liebler divergence between this distribution and the Ga$(1.875, 0.075)$, employed in the severe mismatch scenario, is 0.01. Thus, this represents a scenario where the correction is not a gamma variate, but the shape of its distribution can be closely reproduced with a gamma distribution. Results are presented in Tables 20–21 and Figures 17–18, which very similar to those obtained in the

Design I: $\gamma \sim Ga(1.875, 0.075)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	2.024	2.026	0.261	0.264	0.378	0.757
	κ (0.6)	0.684	0.684	0.046	0.047	0.096	0.522
	α (2.5)	2.059	2.040	0.223	0.224	0.494	0.528
	β_{11} (0.1)	0.101	0.101	0.011	0.011	0.011	0.952
	β_{12} (0.1)	0.088	0.085	0.201	0.189	0.202	0.949
	β_{13} (0.1)	0.151	0.151	0.192	0.189	0.199	0.948
	β_{21} (0.05)	0.054	0.054	0.002	0.002	0.005	0.507
	β_{22} (0.2)	0.242	0.242	0.027	0.028	0.050	0.681
	β_{23} (0.25)	0.217	0.218	0.028	0.028	0.043	0.788
M2	σ (1.75)	1.661	1.693	0.424	0.413	0.433	0.945
	κ (0.6)	0.591	0.590	0.084	0.083	0.085	0.955
	α (2.5)	2.703	2.560	0.685	0.654	0.714	0.953
	β_{11} (0.1)	0.101	0.101	0.011	0.011	0.011	0.945
	β_{12} (0.1)	0.114	0.106	0.184	0.180	0.184	0.954
	β_{13} (0.1)	0.092	0.096	0.182	0.186	0.182	0.957
	β_{21} (0.05)	0.050	0.050	0.004	0.004	0.004	0.941
	β_{22} (0.2)	0.198	0.202	0.048	0.046	0.048	0.945
	β_{23} (0.25)	0.252	0.251	0.042	0.042	0.042	0.942
	γ (1.875)	1.854	1.899	0.639	0.618	0.639	0.910
M3	σ (1.75)	1.654	1.679	0.458	0.456	0.468	0.956
	κ (0.6)	0.580	0.578	0.087	0.087	0.089	0.958
	α (2.5)	2.813	2.654	0.762	0.734	0.824	0.967
	β_{11} (0.1)	0.100	0.100	0.012	0.011	0.011	0.940
	β_{12} (0.1)	0.084	0.088	0.193	0.193	0.194	0.958
	β_{13} (0.1)	0.097	0.098	0.189	0.193	0.189	0.961
	β_{21} (0.05)	0.048	0.048	0.005	0.005	0.005	0.956
	β_{22} (0.2)	0.188	0.192	0.050	0.051	0.052	0.947
	β_{23} (0.25)	0.262	0.259	0.045	0.046	0.046	0.959
	b (0.075)	0.355	0.134	0.543	0.538	0.610	0.812
	μ (1.875)	2.178	2.207	0.706	0.732	0.768	0.843
M4	σ (1.75)	1.733	1.792	0.467	0.372	0.467	0.870
	κ (0.6)	0.609	0.622	0.097	0.065	0.097	0.726
	α (2.5)	2.620	2.395	0.766	0.546	0.775	0.744
	β_{11} (0.1)	0.100	0.100	0.012	0.011	0.012	0.937
	β_{12} (0.1)	0.097	0.094	0.194	0.186	0.194	0.951
	β_{13} (0.1)	0.112	0.109	0.193	0.186	0.193	0.947
	β_{21} (0.05)	0.050	0.051	0.005	0.003	0.005	0.723
	β_{22} (0.2)	0.205	0.213	0.053	0.039	0.053	0.802
	β_{23} (0.25)	0.247	0.241	0.046	0.037	0.046	0.872
	c (1.875)	1.780	1.813	0.840	–	0.845	–

Table 8. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 10000$, and severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

gamma frailty scenario. This indicates that model M3 can capture mismatches in the life tables even in cases where the mismatch is not generated by a gamma distribution, but as long as the gamma distribution can approximate the shape of the true generating correction for some values of the parameters.

REFERENCES

PERME, M.P., STARE, J. AND ESTÈVE, J. (2012). On estimation in relative survival. *Biometrics* **68**, 113–120.

Misspecified Frailty

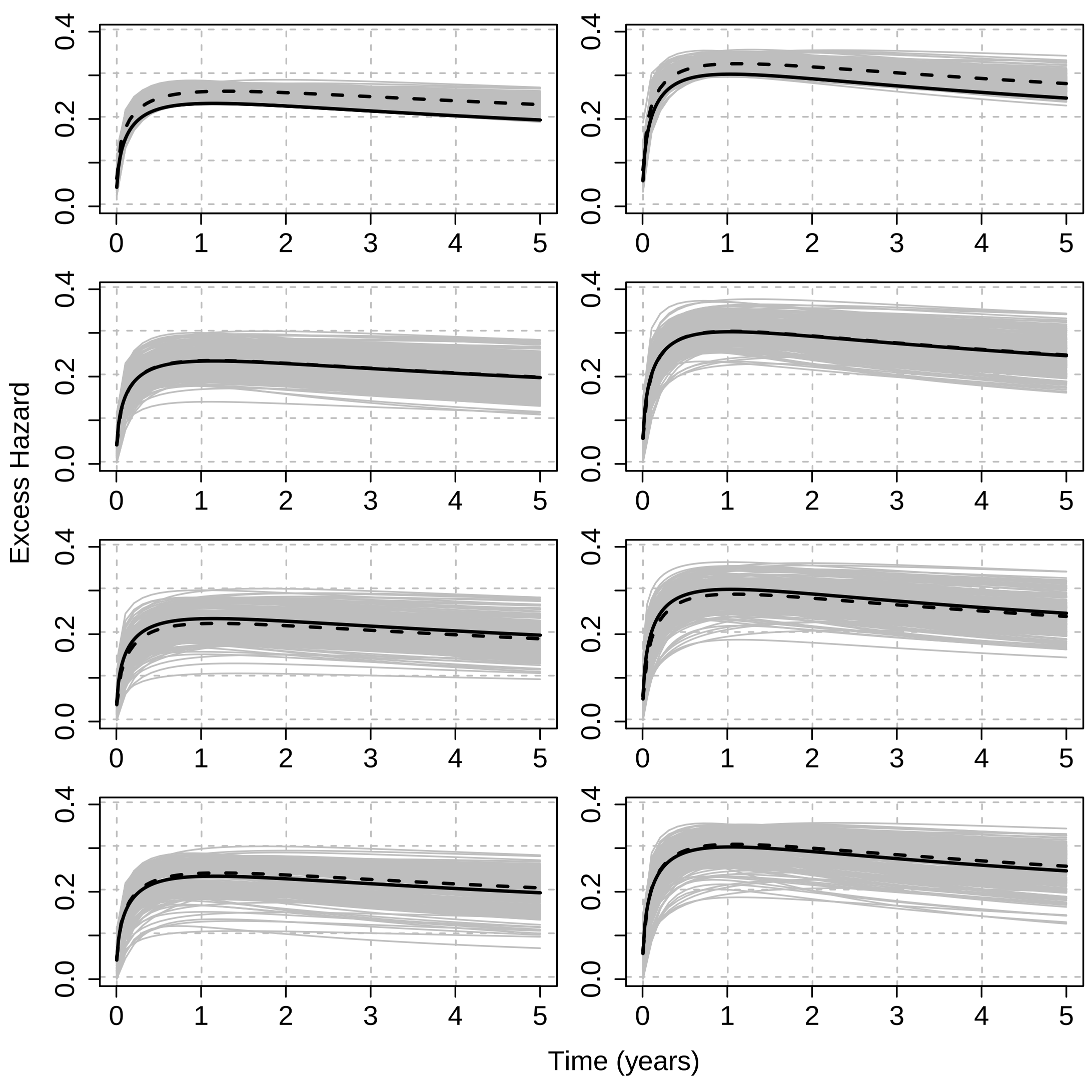

Fig. 17. Scenario with misspecified mismatch $\gamma \sim \text{lognormal}(0.63, 0.2)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 250 sample-specific fitted hazards (grey lines) for $n = 5000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Misspecified Frailty

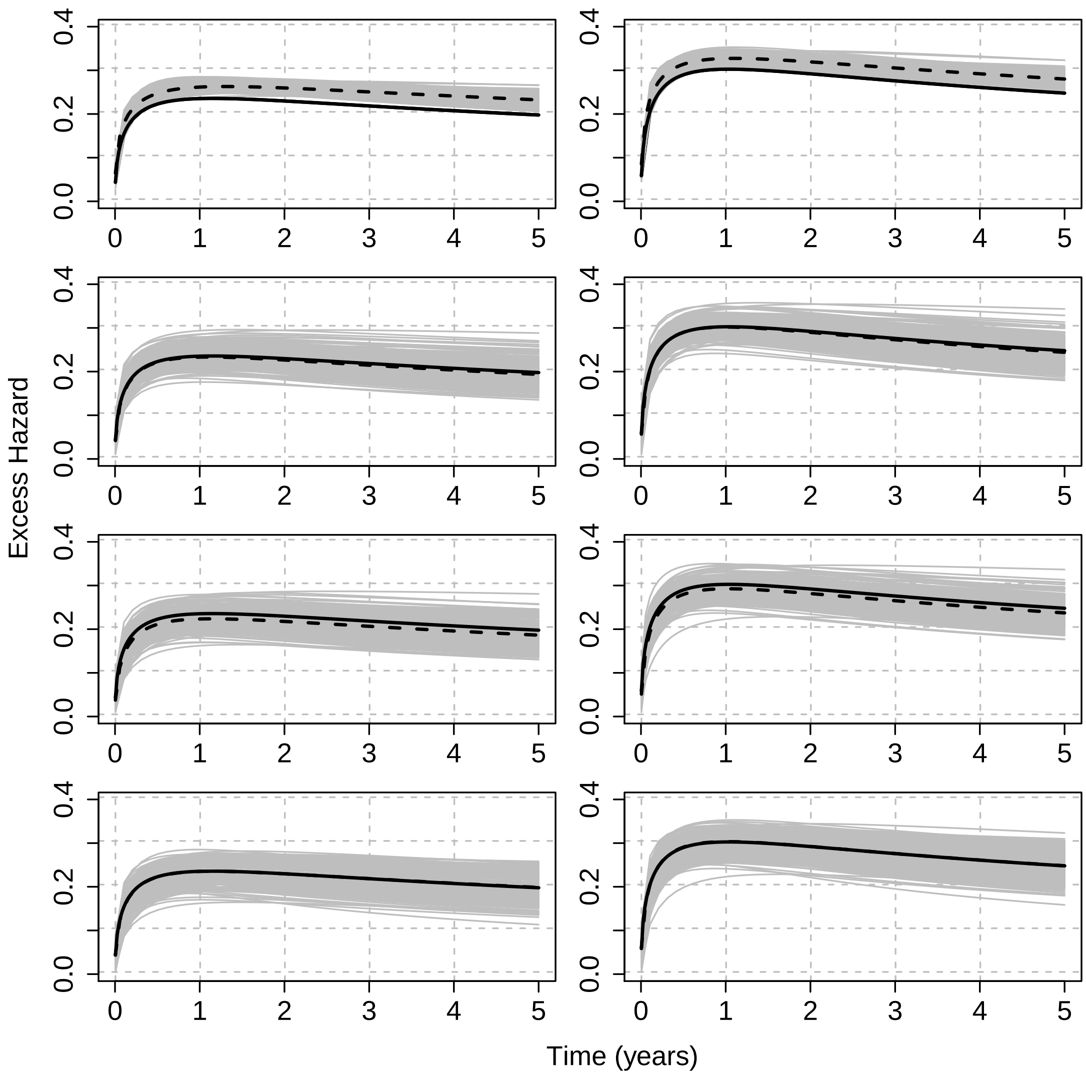

Fig. 18. Scenario with misspecified mismatch $\gamma \sim \text{lognormal}(0.63, 0.2)$. Models M1–M4 from top to bottom. Mean of the fitted hazards (dashed lines), compared to the true generating hazard (continuous lines), and 250 sample-specific fitted hazards (grey lines) for $n = 10000$ and 30% censoring. Panels from left to right correspond to covariate values (age, sex, comorbidity)=$(70, 0, 0)$, $(70, 0, 1)$, respectively.

Design I: $\gamma \sim Ga(6.5, 10)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	1.216	1.215	0.161	0.154	0.558	0.123
	κ (0.6)	0.592	0.593	0.033	0.032	0.034	0.933
	α (2.5)	2.337	2.313	0.221	0.211	0.275	0.850
	β_{11} (0.1)	0.120	0.119	0.009	0.009	0.021	0.384
	β_{12} (0.1)	0.404	0.409	0.174	0.154	0.351	0.473
	β_{13} (0.1)	-0.045	-0.046	0.165	0.155	0.219	0.857
	β_{21} (0.05)	0.065	0.065	0.002	0.002	0.015	0.000
	β_{22} (0.2)	0.293	0.294	0.036	0.032	0.100	0.167
	β_{23} (0.25)	0.162	0.163	0.033	0.032	0.094	0.197
M2	σ (1.75)	1.318	1.324	0.151	0.166	0.458	0.338
	κ (0.6)	0.635	0.639	0.040	0.046	0.054	0.747
	α (2.5)	2.108	2.066	0.233	0.252	0.456	0.544
	β_{11} (0.1)	0.120	0.119	0.009	0.009	0.021	0.390
	β_{12} (0.1)	0.408	0.409	0.157	0.157	0.346	0.497
	β_{13} (0.1)	-0.017	-0.018	0.152	0.158	0.192	0.900
	β_{21} (0.05)	0.067	0.067	0.002	0.002	0.017	0.000
	β_{22} (0.2)	0.308	0.310	0.031	0.032	0.113	0.097
	β_{23} (0.25)	0.153	0.153	0.029	0.031	0.101	0.123
	γ (6.5)	0.323	0.037	0.450	0.412	6.193	0.805
M3	σ (1.75)	1.542	1.554	0.538	0.530	0.576	0.975
	κ (0.6)	0.564	0.573	0.094	0.089	0.101	0.945
	α (2.5)	3.052	2.755	1.144	0.930	1.270	0.970
	β_{11} (0.1)	0.100	0.100	0.020	0.018	0.020	0.908
	β_{12} (0.1)	0.108	0.101	0.247	0.239	0.247	0.942
	β_{13} (0.1)	0.084	0.083	0.240	0.231	0.241	0.947
	β_{21} (0.05)	0.048	0.049	0.006	0.006	0.007	0.960
	β_{22} (0.2)	0.185	0.192	0.065	0.062	0.067	0.936
	β_{23} (0.25)	0.264	0.258	0.060	0.057	0.062	0.954
	b (10)	12.014	9.648	9.291	6.876	9.503	0.853
	μ (6.5)	6.951	6.985	1.230	1.244	1.309	0.877
M4	σ (1.75)	1.539	1.554	0.540	0.528	0.579	0.966
	κ (0.6)	0.564	0.573	0.094	0.089	0.101	0.943
	α (2.5)	3.053	2.755	1.144	0.927	1.271	0.967
	β_{11} (0.1)	0.100	0.100	0.020	0.018	0.020	0.905
	β_{12} (0.1)	0.109	0.102	0.247	0.239	0.247	0.940
	β_{13} (0.1)	0.084	0.083	0.240	0.231	0.240	0.947
	β_{21} (0.05)	0.048	0.049	0.007	0.006	0.007	0.954
	β_{22} (0.2)	0.186	0.193	0.066	0.062	0.067	0.933
	β_{23} (0.25)	0.263	0.258	0.061	0.057	0.062	0.949
	c (6.5)	6.948	6.985	1.271	–	1.347	–

Table 9. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 10000$, and wide mismatch $\gamma \sim Ga(6.5, 10)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

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[*Received August 1, 2010; revised October 1, 2010; accepted for publication November 1, 2010*]

Design II: $\gamma = 1$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	12.020	11.978	0.839	0.856	0.839	0.946
	κ (0.8)	0.801	0.800	0.020	0.020	0.020	0.954
	β_1 (0.03)	0.030	0.030	0.002	0.002	0.002	0.954
	β_2 (0.1)	0.100	0.101	0.057	0.058	0.057	0.952
	β_3 (0.1)	0.099	0.099	0.058	0.058	0.058	0.948
M2	σ (12)	12.163	11.963	1.886	1.899	1.892	0.952
	κ (0.8)	0.801	0.801	0.029	0.028	0.029	0.949
	β_1 (0.03)	0.030	0.030	0.005	0.005	0.005	0.953
	β_2 (0.1)	0.099	0.103	0.071	0.070	0.071	0.954
	β_3 (0.1)	0.099	0.099	0.059	0.059	0.059	0.952
	γ (1)	0.984	1.007	0.242	0.239	0.243	0.936
M3	σ (12)	13.179	12.841	2.555	2.730	2.813	0.967
	κ (0.8)	0.796	0.796	0.030	0.030	0.030	0.960
	β_1 (0.03)	0.028	0.028	0.005	0.006	0.005	0.956
	β_2 (0.1)	0.083	0.087	0.079	0.079	0.081	0.958
	β_3 (0.1)	0.105	0.104	0.063	0.063	0.064	0.958
	b (0)	0.260	0.022	0.386	0.328	0.465	–
	μ (1)	1.158	1.136	0.331	0.357	0.367	0.892
M4	σ (12)	12.365	12.066	2.080	1.195	2.110	0.903
	κ (0.8)	0.800	0.800	0.027	0.022	0.027	0.935
	β_1 (0.03)	0.030	0.030	0.004	0.003	0.004	0.910
	β_2 (0.1)	0.096	0.099	0.070	0.061	0.070	0.932
	β_3 (0.1)	0.101	0.100	0.060	0.059	0.060	0.949
	c (1)	1.028	1.000	0.250	–	0.252	–

Table 10. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 5000$, and no mismatch $\gamma = 1$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma \sim Ga(1.2, 0.02)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	10.827	10.778	0.709	0.718	1.370	0.639
	κ (0.8)	0.816	0.815	0.019	0.020	0.025	0.879
	β_1 (0.03)	0.033	0.033	0.002	0.002	0.004	0.741
	β_2 (0.1)	0.128	0.127	0.055	0.055	0.062	0.913
	β_3 (0.1)	0.092	0.092	0.054	0.055	0.055	0.955
M2	σ (12)	12.257	11.966	2.033	1.990	2.048	0.953
	κ (0.8)	0.799	0.800	0.028	0.029	0.028	0.953
	β_1 (0.03)	0.030	0.030	0.005	0.005	0.005	0.959
	β_2 (0.1)	0.096	0.098	0.072	0.072	0.072	0.949
	β_3 (0.1)	0.100	0.098	0.060	0.061	0.060	0.954
	γ (1.2)	1.188	1.206	0.248	0.249	0.248	0.948
M3	σ (12)	13.322	12.807	2.673	2.855	2.980	0.967
	κ (0.8)	0.794	0.795	0.030	0.031	0.030	0.969
	β_1 (0.03)	0.028	0.028	0.005	0.006	0.006	0.955
	β_2 (0.1)	0.078	0.081	0.080	0.082	0.083	0.960
	β_3 (0.1)	0.105	0.104	0.064	0.066	0.064	0.958
	b (0.02)	0.251	0.069	0.356	0.306	0.425	0.719
	μ (1.2)	1.374	1.351	0.341	0.370	0.382	0.898
M4	σ (12)	12.061	11.172	2.482	1.371	2.482	0.705
	κ (0.8)	0.804	0.807	0.028	0.023	0.029	0.904
	β_1 (0.03)	0.030	0.032	0.005	0.003	0.005	0.772
	β_2 (0.1)	0.103	0.107	0.073	0.063	0.073	0.924
	β_3 (0.1)	0.098	0.098	0.059	0.059	0.059	0.951
	c (1.2)	1.162	1.000	0.306	–	0.308	–

Table 11. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 5000$, and moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma \sim Ga(1.875, 0.075)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	8.262	8.247	0.453	0.457	3.765	0.000
	κ (0.8)	0.854	0.854	0.018	0.018	0.057	0.140
	β_1 (0.03)	0.041	0.041	0.002	0.002	0.012	0.001
	β_2 (0.1)	0.203	0.202	0.049	0.050	0.115	0.465
	β_3 (0.1)	0.077	0.075	0.050	0.050	0.055	0.928
M2	σ (12)	12.020	11.728	2.269	2.179	2.268	0.939
	κ (0.8)	0.802	0.799	0.030	0.031	0.030	0.945
	β_1 (0.03)	0.031	0.031	0.005	0.005	0.005	0.950
	β_2 (0.1)	0.106	0.110	0.078	0.078	0.078	0.941
	β_3 (0.1)	0.103	0.102	0.066	0.066	0.066	0.948
	γ (1.875)	1.783	1.801	0.305	0.298	0.318	0.970
M3	σ (12)	13.365	12.727	2.994	3.219	3.289	0.971
	κ (0.8)	0.795	0.794	0.031	0.033	0.032	0.961
	β_1 (0.03)	0.028	0.028	0.006	0.006	0.006	0.949
	β_2 (0.1)	0.083	0.087	0.086	0.089	0.088	0.956
	β_3 (0.1)	0.109	0.109	0.069	0.072	0.070	0.960
	b (0.075)	0.244	0.138	0.300	0.282	0.344	0.794
	μ (1.875)	2.033	2.017	0.402	0.438	0.432	0.917
M4	σ (12)	12.448	11.984	3.286	2.329	3.315	0.807
	κ (0.8)	0.802	0.798	0.035	0.029	0.035	0.836
	β_1 (0.03)	0.030	0.030	0.007	0.005	0.007	0.795
	β_2 (0.1)	0.102	0.107	0.089	0.078	0.089	0.876
	β_3 (0.1)	0.104	0.102	0.068	0.067	0.068	0.947
	c (1.875)	1.827	1.839	0.491	–	0.493	–

Table 12. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 5000$, and severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma \sim Ga(6.5, 10)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	5.485	5.480	0.273	0.274	6.520	0.000
	κ (0.8)	0.769	0.768	0.014	0.014	0.034	0.397
	β_1 (0.03)	0.045	0.045	0.002	0.002	0.015	0.000
	β_2 (0.1)	0.203	0.202	0.044	0.042	0.111	0.337
	β_3 (0.1)	0.052	0.052	0.042	0.042	0.064	0.800
M2	σ (12)	4.445	4.446	0.177	0.179	7.557	0.000
	κ (0.8)	0.801	0.801	0.012	0.012	0.013	0.944
	β_1 (0.03)	0.050	0.050	0.002	0.002	0.020	0.000
	β_2 (0.1)	0.248	0.247	0.036	0.036	0.152	0.011
	β_3 (0.1)	0.043	0.043	0.036	0.036	0.067	0.629
	γ (6.5)	10^{-8}	0.000	0.000	0.000	6.500	0.000
M3	σ (12)	13.441	12.309	5.018	4.593	5.219	0.935
	κ (0.8)	0.805	0.804	0.044	0.042	0.044	0.948
	β_1 (0.03)	0.029	0.029	0.008	0.008	0.008	0.930
	β_2 (0.1)	0.088	0.095	0.109	0.108	0.110	0.944
	β_3 (0.1)	0.107	0.104	0.087	0.090	0.087	0.973
	b (10)	10.832	10.209	2.896	2.627	3.012	0.939
	μ (6.5)	6.692	6.696	0.917	0.934	0.936	0.923
M4	σ (12)	13.441	12.309	5.018	4.593	5.219	0.935
	κ (0.8)	0.805	0.804	0.044	0.042	0.044	0.948
	β_1 (0.03)	0.029	0.029	0.008	0.008	0.008	0.930
	β_2 (0.1)	0.088	0.095	0.109	0.108	0.110	0.944
	β_3 (0.1)	0.107	0.104	0.087	0.090	0.087	0.973
	c (6.5)	6.692	6.696	0.917	–	0.936	–

Table 13. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 5000$, and wide mismatch $\gamma \sim Ga(6.5, 10)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma = 1$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	12.027	11.999	0.601	0.603	0.602	0.946
	κ (0.8)	0.801	0.800	0.013	0.014	0.013	0.964
	β_1 (0.03)	0.030	0.030	0.002	0.002	0.002	0.952
	β_2 (0.1)	0.100	0.101	0.041	0.041	0.041	0.952
	β_3 (0.1)	0.102	0.102	0.040	0.041	0.040	0.947
M2	σ (12)	12.071	11.958	1.340	1.313	1.342	0.942
	κ (0.8)	0.801	0.801	0.019	0.020	0.019	0.953
	β_1 (0.03)	0.030	0.030	0.003	0.003	0.003	0.947
	β_2 (0.1)	0.100	0.103	0.050	0.049	0.050	0.940
	β_3 (0.1)	0.102	0.102	0.040	0.041	0.040	0.960
	γ (1)	0.988	0.990	0.173	0.167	0.174	0.946
M3	σ (12)	12.742	12.471	1.731	1.792	1.882	0.953
	κ (0.8)	0.797	0.797	0.020	0.021	0.020	0.954
	β_1 (0.03)	0.029	0.029	0.004	0.004	0.004	0.951
	β_2 (0.1)	0.089	0.090	0.055	0.054	0.056	0.949
	β_3 (0.1)	0.106	0.105	0.042	0.044	0.042	0.962
	b (0)	0.178	0.026	0.257	0.225	0.312	–
	μ (1)	1.108	1.093	0.237	0.248	0.260	0.906
M4	σ (12)	12.244	12.025	1.379	0.819	1.400	0.901
	κ (0.8)	0.800	0.800	0.018	0.015	0.018	0.940
	β_1 (0.03)	0.030	0.030	0.003	0.002	0.003	0.913
	β_2 (0.1)	0.097	0.100	0.049	0.043	0.049	0.936
	β_3 (0.1)	0.103	0.102	0.041	0.041	0.041	0.950
	c (1)	1.022	1.000	0.183	–	0.184	–

Table 14. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 10000$, and no mismatch $\gamma = 1$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma \sim Ga(1.2, 0.02)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	10.797	10.790	0.491	0.503	1.299	0.372
	κ (0.8)	0.816	0.816	0.014	0.014	0.021	0.794
	β_1 (0.03)	0.033	0.033	0.002	0.002	0.003	0.541
	β_2 (0.1)	0.128	0.128	0.039	0.039	0.048	0.886
	β_3 (0.1)	0.094	0.094	0.038	0.039	0.039	0.948
M2	σ (12)	12.006	11.888	1.385	1.352	1.384	0.951
	κ (0.8)	0.802	0.801	0.020	0.020	0.020	0.961
	β_1 (0.03)	0.030	0.030	0.003	0.003	0.003	0.947
	β_2 (0.1)	0.100	0.106	0.051	0.050	0.051	0.951
	β_3 (0.1)	0.101	0.100	0.042	0.043	0.042	0.963
	γ (1.2)	1.176	1.184	0.175	0.176	0.177	0.960
M3	σ (12)	12.735	12.540	1.796	1.865	1.940	0.963
	κ (0.8)	0.798	0.798	0.021	0.022	0.021	0.965
	β_1 (0.03)	0.029	0.029	0.004	0.004	0.004	0.945
	β_2 (0.1)	0.087	0.091	0.055	0.056	0.057	0.953
	β_3 (0.1)	0.105	0.104	0.044	0.045	0.044	0.963
	b (0.02)	0.181	0.054	0.243	0.214	0.291	0.727
	μ (1.2)	1.312	1.297	0.246	0.263	0.270	0.925
M4	σ (12)	11.862	11.164	1.780	0.980	1.784	0.569
	κ (0.8)	0.805	0.807	0.021	0.017	0.022	0.847
	β_1 (0.03)	0.031	0.032	0.004	0.002	0.004	0.664
	β_2 (0.1)	0.105	0.113	0.056	0.045	0.056	0.904
	β_3 (0.1)	0.100	0.099	0.042	0.042	0.042	0.952
	c (1.2)	1.155	1.000	0.244	–	0.248	–

Table 15. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 10000$, and moderate mismatch $\gamma \sim Ga(1.2, 0.02)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma \sim Ga(1.875, 0.075)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	8.221	8.209	0.311	0.319	3.792	0.000
	κ (0.8)	0.855	0.855	0.013	0.013	0.056	0.010
	β_1 (0.03)	0.041	0.041	0.002	0.002	0.012	0.000
	β_2 (0.1)	0.202	0.202	0.035	0.035	0.108	0.178
	β_3 (0.1)	0.075	0.074	0.035	0.035	0.043	0.885
M2	σ (12)	11.747	11.550	1.530	1.473	1.550	0.940
	κ (0.8)	0.803	0.803	0.021	0.022	0.021	0.950
	β_1 (0.03)	0.031	0.031	0.004	0.004	0.004	0.955
	β_2 (0.1)	0.109	0.111	0.053	0.054	0.053	0.947
	β_3 (0.1)	0.098	0.097	0.047	0.046	0.047	0.935
	γ (1.875)	1.783	1.793	0.208	0.209	0.228	0.973
M3	σ (12)	12.719	12.348	2.040	2.101	2.162	0.963
	κ (0.8)	0.798	0.798	0.022	0.023	0.022	0.959
	β_1 (0.03)	0.029	0.029	0.004	0.004	0.004	0.960
	β_2 (0.1)	0.091	0.095	0.059	0.061	0.060	0.962
	β_3 (0.1)	0.103	0.102	0.050	0.049	0.050	0.941
	b (0.075)	0.184	0.113	0.217	0.201	0.243	0.817
	μ (1.875)	1.975	1.956	0.294	0.312	0.310	0.926
M4	σ (12)	12.171	11.822	2.076	1.646	2.082	0.928
	κ (0.8)	0.801	0.801	0.023	0.022	0.023	0.934
	β_1 (0.03)	0.030	0.030	0.004	0.004	0.004	0.932
	β_2 (0.1)	0.101	0.104	0.059	0.056	0.059	0.934
	β_3 (0.1)	0.100	0.098	0.049	0.047	0.049	0.935
	c (1.875)	1.857	1.825	0.321	–	0.321	–

Table 16. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 10000$, and severe mismatch $\gamma \sim Ga(1.875, 0.075)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Design II: $\gamma \sim Ga(6.5, 10)$

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (12)	5.491	5.493	0.191	0.193	6.512	0.000
	κ (0.8)	0.768	0.768	0.009	0.010	0.033	0.099
	β_1 (0.03)	0.045	0.044	0.001	0.001	0.015	0.000
	β_2 (0.1)	0.203	0.202	0.030	0.030	0.107	0.063
	β_3 (0.1)	0.052	0.053	0.030	0.030	0.056	0.642
M2	σ (12)	4.447	4.448	0.124	0.126	7.554	0.000
	κ (0.8)	0.800	0.800	0.008	0.009	0.008	0.959
	β_1 (0.03)	0.050	0.050	0.001	0.001	0.020	0.000
	β_2 (0.1)	0.247	0.246	0.025	0.025	0.149	0.000
	β_3 (0.1)	0.044	0.044	0.025	0.025	0.062	0.393
	γ (6.5)	10^{-8}	0.000	0.000	0.000	6.500	0.000
M3	σ (12)	12.989	12.394	3.463	3.040	3.600	0.936
	κ (0.8)	0.800	0.800	0.030	0.029	0.030	0.939
	β_1 (0.03)	0.029	0.029	0.006	0.006	0.006	0.927
	β_2 (0.1)	0.088	0.097	0.078	0.076	0.079	0.945
	β_3 (0.1)	0.107	0.104	0.063	0.063	0.063	0.956
	b (10)	10.382	10.011	1.780	1.662	1.820	0.947
	μ (6.5)	6.647	6.634	0.661	0.661	0.677	0.916
M4	σ (12)	12.989	12.394	3.463	3.040	3.600	0.936
	κ (0.8)	0.800	0.800	0.030	0.029	0.030	0.939
	β_1 (0.03)	0.029	0.029	0.006	0.006	0.006	0.927
	β_2 (0.1)	0.088	0.097	0.078	0.076	0.079	0.945
	β_3 (0.1)	0.107	0.104	0.063	0.063	0.063	0.956
	c (6.5)	6.647	6.634	0.661	–	0.677	–

Table 17. Simulation results for the scenario GH with $(\sigma, \kappa) = (12, 0.8)$, $\beta = (0.03, 0.1, 0.1)$, $n = 10000$, and wide mismatch $\gamma \sim Ga(6.5, 10)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage).

Scenario	M1	M2	M3
$n = 5000$ – Design I			
$\gamma = 1$	0.701	0.216	0.083
$\gamma \sim$ Ga$(1.2, 0.02)$	0.708	0.209	0.083
$\gamma \sim$ Ga$(1.875, 0.075)$	0.598	0.298	0.104
$\gamma \sim$ Ga$(6.5, 10)$	0.074	0.023	0.903
$n = 5000$ – Design II			
$\gamma = 1$	0.815	0.142	0.043
$\gamma \sim$ Ga$(1.2, 0.02)$	0.703	0.250	0.047
$\gamma \sim$ Ga$(1.875, 0.075)$	0.165	0.729	0.106
$\gamma \sim$ Ga$(6.5, 10)$	0.000	0.000	1.000
$n = 10000$ – Design I			
$\gamma = 1$	0.741	0.194	0.065
$\gamma \sim$ Ga$(1.2, 0.02)$	0.724	0.216	0.060
$\gamma \sim$ Ga$(1.875, 0.075)$	0.482	0.411	0.104
$\gamma \sim$ Ga$(6.5, 10)$	0.005	0.000	0.995
$n = 10000$ – Design II			
$\gamma = 1$	0.799	0.158	0.043
$\gamma \sim$ Ga$(1.2, 0.02)$	0.620	0.320	0.060
$\gamma \sim$ Ga$(1.875, 0.075)$	0.034	0.836	0.130
$\gamma \sim$ Ga$(6.5, 10)$	0.000	0.000	1.000

Table 18. Proportion of selected models using AIC.

	M1	M2	M3
b	–	–	12.49 (2.71)
$\gamma \mid \mu$	–	2.59 (0.21)	6.88 (0.71)
θ	0.05 (0.01)	0.03 (0.01)	0.03 (0.01)
κ	0.38 (0.01)	0.35 (0.01)	0.35 (0.01)
α	4.63 (0.33)	5.56 (0.47)	5.77 (0.56)
Age-t	0.3 (0.04)	0.29 (0.04)	0.16 (0.04)
Dep-t	0.11 (0.04)	0.11 (0.04)	0.13 (0.04)
Stage 1-t	-2.7 (0.25)	-2.29 (0.32)	-6.07 (1)
Stage 2-t	-2.21 (0.2)	-2.03 (0.22)	-2.87 (0.32)
Stage 3-t	-1.66 (0.11)	-1.58 (0.11)	-1.78 (0.12)
CV-t	0.31 (0.11)	0.32 (0.11)	0.42 (0.11)
COPD-t	0.14 (0.11)	0.09 (0.12)	0.41 (0.12)
Age	0.27 (0.01)	0.23 (0.02)	0.16 (0.02)
Dep	0.07 (0.01)	0.07 (0.01)	0.08 (0.01)
Stage 1	-2.85 (0.06)	-3.13 (0.09)	-3.26 (0.36)
Stage 2	-2.16 (0.06)	-2.31 (0.07)	-2.65 (0.08)
Stage 3	-1.23 (0.03)	-1.26 (0.04)	-1.36 (0.04)
CV	0.24 (0.04)	0.26 (0.04)	0.29 (0.04)
COPD	0.19 (0.04)	0.18 (0.04)	0.27 (0.04)
AIC	20310.12	20255.2	**20236.05**

Table 19. Regression parameter estimates (standard errors) using models M1–M3, with their corresponding AIC using national life tables (not deprivation-specific). Note: The time- dependent effects are indicated with -t. For model M2, γ is estimated, while μ is estimated for model M3. Age=Age at diagnosis (centred at 70, and divided by 10), Dep=Income Deprivation Score (centred at 0.1, and divided by 10), CV=CardioVascular comorbidity, COPD=Chronic Obstructive Pulmonary Disease, AIC=Akaike Information Criteria (best model indicated in bold font).

Misspecified Frailty

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	2.010	1.992	0.394	0.366	0.472	0.832
	κ (0.6)	0.686	0.682	0.073	0.066	0.113	0.712
	α (2.5)	2.091	2.042	0.360	0.325	0.544	0.692
	β_{11} (0.1)	0.101	0.100	0.017	0.016	0.017	0.936
	β_{12} (0.1)	0.092	0.076	0.290	0.274	0.290	0.952
	β_{13} (0.1)	0.149	0.143	0.285	0.275	0.289	0.972
	β_{21} (0.05)	0.054	0.054	0.003	0.003	0.005	0.676
	β_{22} (0.2)	0.243	0.245	0.041	0.040	0.060	0.804
	β_{23} (0.25)	0.214	0.210	0.041	0.040	0.054	0.872
M2	σ (1.75)	1.612	1.676	0.605	0.559	0.620	0.936
	κ (0.6)	0.586	0.595	0.122	0.115	0.123	0.916
	α (2.5)	3.109	2.536	2.751	1.371	2.812	0.940
	β_{11} (0.1)	0.101	0.100	0.018	0.015	0.018	0.916
	β_{12} (0.1)	0.131	0.119	0.277	0.261	0.279	0.952
	β_{13} (0.1)	0.087	0.096	0.264	0.267	0.264	0.972
	β_{21} (0.05)	0.049	0.049	0.006	0.006	0.006	0.944
	β_{22} (0.2)	0.195	0.202	0.070	0.066	0.070	0.920
	β_{23} (0.25)	0.253	0.245	0.061	0.060	0.060	0.944
	γ (1.91)	1.887	1.905	0.885	0.846	0.883	0.892
M3	σ (1.75)	1.658	1.716	0.685	0.637	0.690	0.932
	κ (0.6)	0.582	0.594	0.132	0.122	0.133	0.920
	α (2.5)	3.431	2.579	4.505	1.972	4.592	0.948
	β_{11} (0.1)	0.099	0.099	0.021	0.018	0.021	0.924
	β_{12} (0.1)	0.078	0.066	0.318	0.288	0.318	0.956
	β_{13} (0.1)	0.106	0.097	0.299	0.288	0.299	0.980
	β_{21} (0.05)	0.047	0.048	0.008	0.007	0.008	0.964
	β_{22} (0.2)	0.183	0.189	0.080	0.074	0.082	0.928
	β_{23} (0.25)	0.263	0.256	0.072	0.068	0.073	0.952
	b (–)	1.512	0.252	7.690	2.927	–	–
	μ (1.91)	2.349	2.314	1.005	1.041	1.095	0.84
M4	σ (1.75)	1.856	1.957	0.537	0.445	0.546	0.884
	κ (0.6)	0.645	0.663	0.110	0.086	0.119	0.804
	α (2.5)	2.603	2.147	2.642	0.892	2.638	0.808
	β_{11} (0.1)	0.101	0.100	0.018	0.016	0.018	0.916
	β_{12} (0.1)	0.101	0.084	0.294	0.270	0.293	0.948
	β_{13} (0.1)	0.123	0.120	0.272	0.271	0.272	0.980
	β_{21} (0.05)	0.052	0.053	0.006	0.004	0.006	0.776
	β_{22} (0.2)	0.223	0.235	0.062	0.051	0.066	0.840
	β_{23} (0.25)	0.230	0.220	0.058	0.049	0.061	0.888
	c (1.91)	1.424	1.000	0.826	–	–	–

Table 20. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 5000$, and $\gamma \sim \text{lognormal}(0.63, 0.2)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage). The RMSE and coverage of b are omitted as they require the true scale parameter, which is undefined in this case due to model misspecification.

Misspecified Frailty

Model	Parameter	MMLE	mMLE	ESD	Mean Std Error	RMSE	Coverage
M1	σ (1.75)	2.037	2.034	0.247	0.262	0.378	0.760
	κ (0.6)	0.687	0.685	0.045	0.047	0.098	0.524
	α (2.5)	2.042	2.031	0.209	0.220	0.503	0.512
	β_{11} (0.1)	0.100	0.099	0.011	0.011	0.011	0.948
	β_{12} (0.1)	0.092	0.096	0.196	0.189	0.196	0.956
	β_{13} (0.1)	0.172	0.183	0.188	0.190	0.200	0.948
	β_{21} (0.05)	0.055	0.055	0.002	0.002	0.005	0.400
	β_{22} (0.2)	0.241	0.243	0.029	0.028	0.050	0.676
	β_{23} (0.25)	0.219	0.218	0.027	0.028	0.042	0.800
M2	σ (1.75)	1.614	1.622	0.422	0.426	0.442	0.960
	κ (0.6)	0.579	0.575	0.086	0.083	0.088	0.960
	α (2.5)	2.810	2.645	0.917	0.717	0.966	0.960
	β_{11} (0.1)	0.100	0.100	0.010	0.010	0.010	0.944
	β_{12} (0.1)	0.111	0.123	0.183	0.180	0.183	0.964
	β_{13} (0.1)	0.100	0.100	0.183	0.186	0.183	0.968
	β_{21} (0.05)	0.050	0.050	0.004	0.004	0.004	0.960
	β_{22} (0.2)	0.191	0.194	0.047	0.047	0.048	0.940
	β_{23} (0.25)	0.258	0.255	0.040	0.043	0.041	0.960
	γ (1.91)	1.977	2.053	0.612	0.610	0.615	0.904
M3	σ (1.75)	1.633	1.644	0.472	0.473	0.486	0.952
	κ (0.6)	0.574	0.566	0.091	0.088	0.095	0.948
	α (2.5)	2.894	2.738	0.990	0.791	1.063	0.952
	β_{11} (0.1)	0.098	0.098	0.012	0.011	0.012	0.940
	β_{12} (0.1)	0.080	0.090	0.194	0.196	0.195	0.960
	β_{13} (0.1)	0.116	0.109	0.195	0.196	0.196	0.960
	β_{21} (0.05)	0.048	0.048	0.005	0.005	0.005	0.972
	β_{22} (0.2)	0.182	0.182	0.051	0.051	0.054	0.940
	β_{23} (0.25)	0.267	0.267	0.045	0.047	0.048	0.940
	b (–)	0.712	0.223	2.632	0.865	–	–
	μ (1.91)	2.320	2.358	0.685	0.736	0.797	0.844
M4	σ (1.75)	1.772	1.844	0.431	0.370	0.430	0.868
	κ (0.6)	0.618	0.632	0.092	0.069	0.093	0.776
	α (2.5)	2.548	2.321	0.926	0.546	0.926	0.772
	β_{11} (0.1)	0.100	0.099	0.011	0.011	0.011	0.948
	β_{12} (0.1)	0.098	0.100	0.191	0.186	0.191	0.964
	β_{13} (0.1)	0.129	0.115	0.192	0.189	0.194	0.944
	β_{21} (0.05)	0.051	0.052	0.004	0.003	0.005	0.724
	β_{22} (0.2)	0.208	0.217	0.051	0.040	0.051	0.820
	β_{23} (0.25)	0.244	0.240	0.041	0.037	0.041	0.884
	c (1.91)	1.668	1.524	0.749		–	–

Table 21. Simulation results for the scenario GH with $(\sigma, \kappa, \alpha) = (1.75, 0.6, 2.5)$, $\beta_1 = (0.1, 0.1, 0.1)$, $\beta_2 = (0.05, 0.2, 0.25)$, $n = 10000$, and $\gamma \sim \text{lognormal}(0.63, 0.2)$. Mean of the MLEs (MMLE), median of the MLEs (mMLE), empirical standard deviation (ESD), mean (estimated) standard error, root-mean-square error (RMSE), and coverage proportions (Coverage). The RMSE and coverage of b are omitted as they require the true scale parameter, which is undefined in this case due to model misspecification.