Appendix A. Linear Regression for Quantitative Traits

Under models (1) and (2), $\boldsymbol{\theta} = (\boldsymbol{\beta}^{\mathrm{T}}, \gamma, \tau^2, \boldsymbol{\alpha}^{\mathrm{T}}, \sigma^2)^{\mathrm{T}}$. We set the initial values of $\boldsymbol{\beta}$ to $\mathbf{0}$, γ to 0, $\boldsymbol{\alpha}$ to $\mathbf{0}$, τ^2 to the sample variance of the Y_i, and σ^2 to the sample variance of the observed S_i.

The E-step involves calculation of $\widehat{E}(S_i)$ and $\widehat{E}(S_i^2)$ $(i = 1, \ldots, n)$. If $R_i = 1$, then $\widehat{E}(S_i) = S_i$ and $\widehat{E}(S_i^2) = S_i^2$. If $R_i = 0$ and $\mathcal{C}_i = (-\infty, \infty)$, then $\widehat{E}(S_i) = b_i$ and $\widehat{E}(S_i^2) = b_i^2 + a_i$, where $a_i = (\gamma^2/\tau^2 + 1/\sigma^2)^{-1}$, and $b_i = a_i \left\{\boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{X}_i/\sigma^2 + \gamma(Y_i - \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{Z}_i)/\tau^2\right\}$. If $R_i = 0$ and $\mathcal{C}_i = (-\infty, L_i)$, then

$$\widehat{E}(S_i) = b_i - \sqrt{a_i} \frac{\phi\left(\frac{L_i - b_i}{\sqrt{a_i}}\right)}{\Phi\left(\frac{L_i - b_i}{\sqrt{a_i}}\right)},$$

and

$$\widehat{E}(S_i^2) = \{\widehat{E}(S_i)\}^2 + a_i \left[1 - \frac{L_i - b_i}{\sqrt{a_i}} \frac{\phi\left(\frac{L_i - b_i}{\sqrt{a_i}}\right)}{\Phi\left(\frac{L_i - b_i}{\sqrt{a_i}}\right)} - \left\{\frac{\phi\left(\frac{L_i - b_i}{\sqrt{a_i}}\right)}{\Phi\left(\frac{L_i - b_i}{\sqrt{a_i}}\right)}\right\}^2\right],$$

where $\phi(x)$ and $\Phi(x)$ are, respectively, the density and distribution functions of the standard normal random variable. If $R_i = 0$ and $\mathcal{C}_i = (U_i, \infty)$, then

$$\widehat{E}(S_i) = b_i + \sqrt{a_i} \frac{\phi\left(\frac{-U_i + b_i}{\sqrt{a_i}}\right)}{\Phi\left(\frac{-U_i + b_i}{\sqrt{a_i}}\right)},$$

and

$$\widehat{E}(S_i^2) = \{\widehat{E}(S_i)\}^2 + a_i \left[1 - \frac{-U_i + b_i}{\sqrt{a_i}} \frac{\phi\left(\frac{-U_i + b_i}{\sqrt{a_i}}\right)}{\Phi\left(\frac{-U_i + b_i}{\sqrt{a_i}}\right)} - \left\{\frac{\phi\left(\frac{-U_i + b_i}{\sqrt{a_i}}\right)}{\Phi\left(\frac{-U_i + b_i}{\sqrt{a_i}}\right)}\right\}^2\right].$$

In the M-step, all parameter updates have explicit forms

$$\begin{bmatrix} \boldsymbol{\beta} \\ \gamma \end{bmatrix} = \left(\sum_{i=1}^n \begin{bmatrix} \boldsymbol{Z}_i \boldsymbol{Z}_i^{\mathrm{T}} & \widehat{E}(S_i)\boldsymbol{Z}_i \\ \widehat{E}(S_i)\boldsymbol{Z}_i^{\mathrm{T}} & \widehat{E}(S_i^2) \end{bmatrix}\right)^{-1} \sum_{i=1}^n \begin{bmatrix} Y_i \boldsymbol{Z}_i \\ Y_i \widehat{E}(S_i) \end{bmatrix},$$

$$\tau^2 = n^{-1} \sum_{i=1}^n \left\{\left(Y_i - \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{Z}_i\right)^2 + \gamma^2 \widehat{E}(S_i^2) - 2\left(Y_i - \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{Z}_i\right) \gamma \widehat{E}(S_i)\right\},$$

$$\boldsymbol{\alpha} = \left(\sum_{i=1}^n \boldsymbol{X}_i \boldsymbol{X}_i^{\mathrm{T}}\right)^{-1} \sum_{i=1}^n \widehat{E}(S_i) \boldsymbol{X}_i,$$

and

$$\sigma^2 = n^{-1} \sum_{i=1}^{n} \left\{ \widehat{E}(S_i^2) + (\boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{X}_i)^2 - 2(\boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{X}_i) \widehat{E}(S_i) \right\}.$$

Denote the final estimate of $\boldsymbol{\theta}$ as $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\beta}}^{\mathrm{T}}, \widehat{\gamma}, \widehat{\tau}^2, \widehat{\boldsymbol{\alpha}}^{\mathrm{T}}, \widehat{\sigma}^2)^{\mathrm{T}}$.

It is straightforward to show that

$$\boldsymbol{Q} = \begin{bmatrix} \sum_{i=1}^n \boldsymbol{Z}_i \boldsymbol{Z}_i^{\mathrm{T}} / \widehat{\tau}^2 & \sum_{i=1}^n \widehat{E}(S_i) \boldsymbol{Z}_i / \widehat{\tau}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \sum_{i=1}^n \widehat{E}(S_i) \boldsymbol{Z}_i^{\mathrm{T}} / \widehat{\tau}^2 & \sum_{i=1}^n \widehat{E}(S_i^2) / \widehat{\tau}^2 & 0 & \mathbf{0}^{\mathrm{T}} & 0 \\ \mathbf{0}^{\mathrm{T}} & 0 & n/(2\widehat{\tau}^4) & \mathbf{0}^{\mathrm{T}} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \sum_{i=1}^n \boldsymbol{X}_i \boldsymbol{X}_i^{\mathrm{T}} / \widehat{\sigma}^2 & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 & 0 & \mathbf{0}^{\mathrm{T}} & n/(2\widehat{\sigma}^4) \end{bmatrix}.$$

If S_i is missing, we compute

$$\mu_{1i} = b_i, \quad \mu_{2i} = b_i^2 + a_i, \quad \mu_{3i} = 3a_i b_i + b_i^3, \quad \mu_{4i} = 3a_i^2 + 6a_i b_i^2 + b_i^4.$$

If S_i is subject to a lower detection limit, we start with

$$f_{0i} = 1, \quad f_{1i} = -\frac{\phi((L_i - b_i)/\sqrt{a_i})}{\Phi((L_i - b_i)/\sqrt{a_i})}.$$

We recursively compute

$$f_{ji} = -\frac{\left\{(L_i - b_i)/\sqrt{a_i}\right\}^{j-1} \phi((L_i - b_i)/\sqrt{a_i})}{\Phi((L_i - b_i)/\sqrt{a_i})} + (j-1) f_{j-2,i}, \quad j = 2, 3, 4.$$

We then compute

$$\mu_{ki} = \sum_{j=0}^{k} \binom{k}{j} a_i^{j/2} b_i^{k-j} f_{ji}$$

for $k = 1, 2, 3, 4$. If S_i is subject to an upper detection limit, we set

$$f_{0i} = 1, \quad f_{1i} = -\frac{\phi((-U_i + b_i)/\sqrt{a_i})}{\Phi((-U_i + b_i)/\sqrt{a_i})}.$$

We recursively compute

$$f_{ji} = -\frac{\left\{(-U_i + b_i)/\sqrt{a_i}\right\}^{j-1} \phi((-U_i + b_i)/\sqrt{a_i})}{\Phi((-U_i + b_i)/\sqrt{a_i})} + (j-1) f_{j-2,i}, \quad j = 2, 3, 4.$$

We then compute

$$\mu_{ki} = (-1)^k \sum_{j=0}^{k} \binom{k}{j} a_i^{j/2} (-b_i)^{k-j} f_{ji}$$

for $k = 1, 2, 3, 4$.

It can be shown that

$$\boldsymbol{U}_i(S_i) = \boldsymbol{V}_i \begin{bmatrix} 1 \\ S_i \\ S_i^2 \end{bmatrix},$$

where

$$\boldsymbol{V}_i = \begin{bmatrix} (Y_i - \widehat{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{Z}_i)\boldsymbol{Z}_i/\widehat{\tau}^2 & -\widehat{\gamma}\boldsymbol{Z}_i/\widehat{\tau}^2 & \mathbf{0} \\ 0 & (Y_i - \widehat{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{Z}_i)/\widehat{\tau}^2 & -\widehat{\gamma}/\widehat{\tau}^2 \\ -1/(2\widehat{\tau}^2) + (Y_i - \widehat{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{Z}_i)^2/(2\widehat{\tau}^4) & -\widehat{\gamma}(Y_i - \widehat{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{Z}_i)/\widehat{\tau}^4 & \widehat{\gamma}^2/(2\widehat{\tau}^4) \\ -(\widehat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i)\boldsymbol{X}_i/\widehat{\sigma}^2 & \boldsymbol{X}_i/\widehat{\sigma}^2 & \mathbf{0} \\ -1/(2\widehat{\sigma}^2) + (\widehat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i)^2/(2\widehat{\sigma}^4) & -\widehat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i/\widehat{\sigma}^4 & 1/(2\widehat{\sigma}^4) \end{bmatrix}.$$

Thus,

$$\widehat{E}\{\boldsymbol{U}_i(S_i)\} = \boldsymbol{V}_i \begin{bmatrix} 1 \\ \mu_{1i} \\ \mu_{2i} \end{bmatrix},$$

and

$$\widehat{E}\{\boldsymbol{U}_i(S_i)\boldsymbol{U}_i(S_i)^{\mathrm{T}}\} = \boldsymbol{V}_i \begin{bmatrix} 1 & \mu_{1i} & \mu_{2i} \\ \mu_{1i} & \mu_{2i} & \mu_{3i} \\ \mu_{2i} & \mu_{3i} & \mu_{4i} \end{bmatrix} \boldsymbol{V}_i^{\mathrm{T}}.$$

Appendix B. Logistic Regression for Binary Traits

Under models (1) and (3), $\boldsymbol{\theta} = (\boldsymbol{\beta}^{\mathrm{T}}, \gamma, \boldsymbol{\alpha}^{\mathrm{T}}, \sigma^2)^{\mathrm{T}}$. We set the initial values of $\boldsymbol{\beta}$ to $\mathbf{0}$, γ to 0, $\boldsymbol{\alpha}$ to $\mathbf{0}$, and σ^2 to the sample variance of the observed S_i.

In the E-step, we consider the following functions

$$g_1(S_i) = S_i, \quad g_2(S_i) = S_i^2,$$

$$g_3(S_i) = \frac{\exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)}{1 + \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)},$$

$$g_4(S_i) = \frac{S_i \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)}{1 + \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)},$$

$$g_5(S_i) = \frac{\exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)}{\{1 + \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)\}^2},$$

$$g_6(S_i) = \frac{S_i \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)}{\{1 + \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)\}^2},$$

and

$$g_7(S_i) = \frac{S_i^2 \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)}{\{1 + \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma S_i)\}^2}.$$

For $k = 1, \ldots, 7$, let $\widehat{E}\{g_k(S_i)\} = g_k(S_i)$ for $R_i = 1$ and calculate $\widehat{E}\{g_k(S_i)\}$ for $R_i = 0$ using the numerical integration given at the end of this appendix. In the M-step,

$$\begin{bmatrix} \boldsymbol{\beta}^{\text{new}} \\ \gamma^{\text{new}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}^{\text{old}} \\ \gamma^{\text{old}} \end{bmatrix} + \left(\sum_{i=1}^{n} \begin{bmatrix} \widehat{E}\{g_5(S_i)\}\boldsymbol{Z}_i\boldsymbol{Z}_i^{\text{T}} & \widehat{E}\{g_6(S_i)\}\boldsymbol{Z}_i \\ \widehat{E}\{g_6(S_i)\}\boldsymbol{Z}_i^{\text{T}} & \widehat{E}\{g_7(S_i)\} \end{bmatrix} \right)^{-1} \sum_{i=1}^{n} \begin{bmatrix} [Y_i - \widehat{E}\{g_3(S_i)\}]\boldsymbol{Z}_i \\ Y_i\widehat{E}(S_i) - \widehat{E}\{g_4(S_i)\} \end{bmatrix},$$

$$\boldsymbol{\alpha} = \left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{X}_i^{\text{T}} \right)^{-1} \sum_{i=1}^{n} \widehat{E}(S_i)\boldsymbol{X}_i,$$

and

$$\sigma^2 = n^{-1} \sum_{i=1}^{n} \left\{ \widehat{E}(S_i^2) + (\boldsymbol{\alpha}^{\text{T}}\boldsymbol{X}_i)^2 - 2(\boldsymbol{\alpha}^{\text{T}}\boldsymbol{X}_i)\widehat{E}(S_i) \right\}.$$

Denote the final estimate of $\boldsymbol{\theta}$ as $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\beta}}^{\text{T}}, \widehat{\gamma}, \widehat{\boldsymbol{\alpha}}^{\text{T}}, \widehat{\sigma}^2)^{\text{T}}$.

It can be shown that

$$\boldsymbol{Q} = \begin{bmatrix} \sum_{i=1}^{n} \widehat{E}\{g_5(S_i)\}\boldsymbol{Z}_i\boldsymbol{Z}_i^{\text{T}} & \sum_{i=1}^{n} \widehat{E}\{g_6(S_i)\}\boldsymbol{Z}_i & \mathbf{0} & \mathbf{0} \\ \sum_{i=1}^{n} \widehat{E}\{g_6(S_i)\}\boldsymbol{Z}_i^{\text{T}} & \sum_{i=1}^{n} \widehat{E}\{g_7(S_i)\} & \mathbf{0}^{\text{T}} & 0 \\ \mathbf{0} & \mathbf{0} & \sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{X}_i^{\text{T}}/\widehat{\sigma}^2 & \mathbf{0} \\ \mathbf{0}^{\text{T}} & 0 & \mathbf{0}^{\text{T}} & n/(2\widehat{\sigma}^4) \end{bmatrix}.$$

In addition,

$$\boldsymbol{U}_i(S_i) = \boldsymbol{V}_i \begin{bmatrix} 1 \\ S_i \\ S_i^2 \\ \exp(\widehat{\boldsymbol{\beta}}^{\text{T}}\boldsymbol{Z}_i + \widehat{\gamma}S_i)/\{1 + \exp(\widehat{\boldsymbol{\beta}}^{\text{T}}\boldsymbol{Z}_i + \widehat{\gamma}S_i)\} \\ S_i\exp(\widehat{\boldsymbol{\beta}}^{\text{T}}\boldsymbol{Z}_i + \widehat{\gamma}S_i)/\{1 + \exp(\widehat{\boldsymbol{\beta}}^{\text{T}}\boldsymbol{Z}_i + \widehat{\gamma}S_i)\} \end{bmatrix},$$

where

$$\boldsymbol{V}_i = \begin{bmatrix} Y_i\boldsymbol{Z}_i & \mathbf{0} & \mathbf{0} & -\boldsymbol{Z}_i & \mathbf{0} \\ 0 & Y_i & 0 & 0 & -1 \\ -(\widehat{\boldsymbol{\alpha}}^{\text{T}}\boldsymbol{X}_i)\boldsymbol{X}_i/\widehat{\sigma}^2 & \boldsymbol{X}_i/\widehat{\sigma}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -1/(2\widehat{\sigma}^2) + (\widehat{\boldsymbol{\alpha}}^{\text{T}}\boldsymbol{X}_i)^2/(2\widehat{\sigma}^4) & -\widehat{\boldsymbol{\alpha}}^{\text{T}}\boldsymbol{X}_i/\widehat{\sigma}^4 & 1/(2\widehat{\sigma}^4) & 0 & 0 \end{bmatrix}.$$

Thus,

$$\widehat{E}\{\boldsymbol{U}_i(S_i)\} = \boldsymbol{V}_i \begin{bmatrix} 1 \\ \widehat{E}(S_i) \\ \widehat{E}(S_i^2) \\ \widehat{E}\{g_3(S_i)\} \\ \widehat{E}\{g_4(S_i)\} \end{bmatrix},$$

and

$$\widehat{E}\{\boldsymbol{U}_i(S_i)\boldsymbol{U}_i(S_i)^{\mathrm{T}}\} = \boldsymbol{V}_i \begin{bmatrix} 1 & \widehat{E}(S_i) & \widehat{E}(S_i^2) & \widehat{E}\{g_3(S_i)\} & \widehat{E}\{g_4(S_i)\} \\ \widehat{E}(S_i) & \widehat{E}(S_i^2) & \widehat{E}(S_i^3) & \widehat{E}\{g_4(S_i)\} & \widehat{E}\{S_i^2 g_3(S_i)\} \\ \widehat{E}(S_i^2) & \widehat{E}(S_i^3) & \widehat{E}(S_i^4) & \widehat{E}\{S_i^2 g_3(S_i)\} & \widehat{E}\{S_i^3 g_3(S_i)\} \\ \widehat{E}\{g_3(S_i)\} & \widehat{E}\{g_4(S_i)\} & \widehat{E}\{S_i^2 g_3(S_i)\} & \widehat{E}\{g_3^2(S_i)\} & \widehat{E}\{S_i g_3^2(S_i)\} \\ \widehat{E}\{g_4(S_i)\} & \widehat{E}\{S_i^2 g_3(S_i)\} & \widehat{E}\{S_i^3 g_3(S_i)\} & \widehat{E}\{S_i g_3^2(S_i)\} & \widehat{E}\{S_i^2 g_3^2(S_i)\} \end{bmatrix} \boldsymbol{V}_i^{\mathrm{T}}.$$

To calculate $\widehat{E}\{g(S_i)\}$, we use the Gaussian-Hermite approximation

$$\int_{-\infty}^{\infty} h(x)e^{-x^2}dx \approx \sum_{k=1}^{K} w_k h(x_k),$$

where (w_k, x_k) $(k = 1, \ldots, K)$ are the weights and abacissi. Note that the posterior density of S_i given the observed data is proportional to $\exp\{-(s-b_i)^2/(2a_i)\}d_i(s)I(s \in \mathcal{C}_i)$, where $a_i = \sigma^2$, $b_i = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}_i$, and $d_i(s) = \exp\{Y_i(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma s)\}/\{1 + \exp(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}_i + \gamma s)\}$. Thus, in the case of a lower detection limit,

$$\widehat{E}\{g(S_i)\} = \frac{\int_{-\infty}^{L_i} g(s)d_i(s)\exp\{-(s-b_i)^2/(2a_i)\}ds}{\int_{-\infty}^{L_i} d_i(s)\exp\{-(s-b_i)^2/(2a_i)\}ds}$$

$$= \frac{\int_{-\infty}^{c_i} g(\sqrt{2a_i}s + b_i)d_i(\sqrt{a_i}s + b_i)e^{-s^2}ds}{\int_{-\infty}^{c_i} d_i(\sqrt{a_i}s + b_i)e^{-s^2}ds},$$

where $c_i = (L_i - b_i)/\sqrt{2a_i}$. Clearly,

$$\begin{aligned} & \int_{-\infty}^{c_i} g(\sqrt{2a_i}s + b_i)d_i(\sqrt{2a_i}s + b_i)e^{-s^2}ds \\ = & \int_{-\infty}^{0} g(\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)d_i(\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)e^{-2c_i s - c_i^2}e^{-s^2}ds \\ = & \int_{-\infty}^{\infty} \widetilde{g}_i(s)e^{-s^2}ds, \end{aligned}$$

where

$$\widetilde{g}_i(s) = \Big\{ g(\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)d_i(\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)e^{-2c_i s - c_i^2}I(s \le 0)$$

$$+ g(-\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)d_i(-\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)e^{2c_i s - c_i^2}I(s > 0)\Big\}/2.$$

Then

$$\widehat{E}\{g(S_i)\} \approx \frac{\sum_{k=1}^{K} \widetilde{g}_i(x_k)w_k}{\sum_{k=1}^{K} \widetilde{d}_i(x_k)w_k},$$

where

$$\widetilde{d}_i(s) = \left\{ d_i(\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)e^{-2c_i s - c_i^2} I(s \le 0) + d_i(-\sqrt{2a_i}s + \sqrt{2a_i}c_i + b_i)e^{2c_i s - c_i^2} I(s > 0) \right\} /2.$$

If S_i is subject to an upper limit, then we set $c_i = (-U_i + b_i)/\sqrt{2a_i}$,

$$\widetilde{g}_i(s) = \left\{ g(-\sqrt{2a_i}s + \sqrt{2a_i}c_i - b_i)d_i(-\sqrt{2a_i}s + \sqrt{2a_i}c_i - b_i)e^{-2c_i s - c_i^2} I(s \le 0) \right.$$
$$\left. + g(\sqrt{2a_i}s + \sqrt{2a_i}c_i - b_i)d_i(\sqrt{2a_i}s + \sqrt{2a_i}c_i - b_i)e^{2c_i s - c_i^2} I(s > 0) \right\} /2,$$

and

$$\widetilde{d}_i(s) = \left\{ d_i(-\sqrt{2a_i}s + \sqrt{2a_i}c_i - b_i)e^{-2c_i s - c_i^2} I(s \le 0) + d_i(\sqrt{2a_i}s + \sqrt{2a_i}c_i - b_i)e^{2c_i s - c_i^2} I(s > 0) \right\} /2.$$

If S_i is missing, then we set

$$\widehat{E}\{g(S_i)\} = \frac{\sum_{k=1}^{K} g(\sqrt{2a_i}x_k + b_i)d_i(\sqrt{2a_i}x_k + b_i)w_k}{\sum_{k=1}^{K} d_i(\sqrt{2a_i}x_k + b_i)w_k}.$$