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Supplementary Information for

Emergence of hierarchy in networked endorsement dynamics

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Supporting Information Text

1. Linear Stability

In this section, we prove a set of linear stability results that generalize Theorem 1 in the main text. Our generalizations account for (a) nonlinear features and (b) multiple updates per round.

Throughout this section, we consider a utility function of the form

$$u_{ij}(\mathbf{s}) = \sum_{\ell=1}^{k} \beta_\ell \phi^\ell_{ij}(\mathbf{s}) \;, \qquad \text{[S1]}$$

where each $\phi^\ell : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ is a smooth *feature map*; $\beta_\ell \in \mathbb{R}$ is a *preference parameter* indicating relative importance of the ℓth feature; and $\phi^\ell_{ij}(\mathbf{s})$ is the ijth entry of $\phi^\ell(\mathbf{s})$. We collect the parameters β in a vector $\boldsymbol{\beta} \in \mathbb{R}^k$. The utility function in Eq. (4) from the main text is a special case with linear feature map $\phi^1_{ij}(\mathbf{s}) = s_j$, and quadratic feature map, $\phi^2_{ij}(\mathbf{s}) = (s_i - s_j)^2$. We also define the *rate matrix* $\mathbf{G} = [n^{-1}p_{ij}]$, whose $(i,j)th$ entry gives the probability that, in a given time step, node i chosen uniformly at random endorses node j (see Eq. (5) in the main text for the definition of p_{ij}).

Since we aim to characterize the linear stability of egalitarian fixed points, we will consider the Jacobian of the rank vector $\boldsymbol{\gamma}$ evaluated at egalitarian fixed points. We will therefore evaluate the Jacobian at $\mathbf{s}_0 = \theta\mathbf{e}$, where $\theta \in \mathbb{R}$. By definition, $\boldsymbol{\gamma} = n^{-1}\mathbf{G}^T\mathbf{e} = n^{-1}\sum_i \boldsymbol{\gamma}_i$, where $\boldsymbol{\gamma}_i$ is the ith column of $\mathbf{G}$. Differentiating and applying the chain rule, we have

$$\frac{\partial \boldsymbol{\gamma}(\mathbf{s}_0)}{\partial \mathbf{s}} = \sum_i \left(\boldsymbol{\Gamma}_i - \boldsymbol{\gamma}_i\boldsymbol{\gamma}_i^T\right) \sum_{\ell=1}^{k} \beta_\ell \frac{\partial \phi^\ell_i}{\partial \mathbf{s}} \;,$$

where $\boldsymbol{\Gamma}_i = \mathrm{diag}\boldsymbol{\gamma}_i$ and $\phi^\ell_{i\cdot}(\mathbf{s}_0)$ is the ith row of the ℓth feature map evaluated at $\mathbf{s}_0$. At $\mathbf{s}_0 = \theta\mathbf{e}$, $\mathbf{G} = n^{-1}\mathbf{E}$. It follows that $\boldsymbol{\gamma}_i = n^{-1}\mathbf{e}$ and $\boldsymbol{\Gamma}_i = n^{-1}\mathbf{I}$. We thus have

$$\frac{\partial \boldsymbol{\gamma}(\mathbf{s}_0)}{\partial \mathbf{s}} = n^{-1}(\mathbf{I} - n^{-1}\mathbf{E}) \sum_{i=1}^{n} \sum_{\ell=1}^{k} \beta_\ell \frac{\partial \phi^\ell_{i\cdot}(\mathbf{s}_0)}{\partial \mathbf{s}} \triangleq \mathbf{M}(\mathbf{s}_0; \boldsymbol{\beta}) \;. \qquad \text{[S2]}$$

We will express our primary results in terms of this matrix.

When writing proofs involving dynamics, we will typically repress the time-argument of quantities like $\mathbf{s}$ and $\mathbf{A}$. When time step t is implied, we will use the somewhat informal notation $\delta\mathbf{s} = s(t+1) - \mathbf{s}(t)$ and $\delta\mathbf{A} = \mathbf{A}(t+1) - \mathbf{A}(t)$ to denote the increments of these and other quantities in the current time step.

A. Degree Scores.

Theorem S1 (Stable Egalitarianism with Degree Scores)**.** *When $\sigma(\mathbf{A}) = \mathbf{s} = \mathbf{A}^T\mathbf{e}$, the vector $\mathbf{s}_0 = d\mathbf{e}$ is a root of $\mathbf{f}$, where $d = \frac{m}{n}$, and is the only egalitarian root. Furthermore, $\mathbf{s}_0$ is linearly stable in the long-memory limit if and only if $\mathbf{M}(\mathbf{s}_0; \boldsymbol{\beta})$ has eigenvalues strictly smaller than $\frac{1}{m}$.*

Proof. We first derive the functional form of $\mathbf{f}$. We can write

$$\begin{aligned}
\mathbb{E}[\mathbf{s}(t+1)|\mathbf{A}(t)] &= \mathbb{E}[\mathbf{A}(t+1)|\mathbf{A}(t)]^T\mathbf{e} \\
&= \lambda\mathbf{A}(t)\mathbf{e} + (1-\lambda)\mathbb{E}[\boldsymbol{\Delta}(t)]^T\mathbf{e} \\
&= \lambda\mathbf{A}(t)\mathbf{e} + (1-\lambda)mn^{-1}\mathbf{G}(t)^T\mathbf{e} \;.
\end{aligned}$$

Inserting this expression into Eq. (**??**), and recognizing $n^{-1}\mathbf{G}(t)\mathbf{e} = \boldsymbol{\gamma}(t)$, we have

$$\mathbf{f}(\mathbf{s}) = mn^{-1}\mathbb{E}[\mathbf{G}]\mathbf{e} - \mathbf{A}(t)\mathbf{e} = m\boldsymbol{\gamma} - \mathbf{s} \;.$$

We can now check that $\mathbf{s}_0$ is indeed the unique egalitarian root of $\mathbf{f}$. Suppose that $\mathbf{s} = s\mathbf{e}$ for some scalar s. Then,

$$\mathbf{f}(\mathbf{s}) = m\boldsymbol{\gamma}(\mathbf{s}) - \mathbf{s} = (mn^{-1} - s)\mathbf{e} \;,$$

which is only equal to zero when $s = \frac{m}{n}$, as needed.

Now computing derivatives, we have

$$\frac{\partial \mathbf{f}(\mathbf{s})}{\partial \mathbf{s}} = m\mathbf{M}(\mathbf{s}; \boldsymbol{\beta}) - \mathbf{I} \;.$$

This matrix has strictly negative eigenvalues provided that the eigenvalues of $\mathbf{M}(\mathbf{s_0}; \boldsymbol{\beta})$ are strictly smaller than $\frac{1}{m}$, completing the proof. □

Corrolary S1. *Using the Root-Degree score function, $\mathbf{s}_0 = \frac{m}{n}\mathbf{e}$ is a linearly stable fixed point of $\mathbf{f}$ if and only if $\beta < 2\sqrt{\frac{n}{m}}$.*

Proof. It is convenient to treat the operation of taking the square root as part of the feature map, rather than part of the score function. We therefore suppose that s_j is the in-degree of node j and that $\phi_j(\mathbf{s}) = \sqrt{s_j}$. Computing from Eq. (S2), we obtain

$$\mathbf{M}(\mathbf{s}_0;\boldsymbol{\beta}) = \frac{1}{2}\frac{n^{-1}}{\sqrt{d}}\beta(\mathbf{I} - n^{-1}\mathbf{E}) \; .$$

This matrix again has a zero eigenvalue associated with the direction $\mathbf{e}$. For any direction $\mathbf{v} \perp \mathbf{e}$, there is an eigenvalue $\frac{1}{2}\frac{n^{-1}}{\sqrt{d}}\beta$. From Theorem S1, $\mathbf{s}_0$ will be linearly stable provided that

$$\frac{1}{m} > \frac{1}{2}\frac{n^{-1}}{\sqrt{d}}\beta \; .$$

or

$$\beta < 2\sqrt{d}\frac{n}{m} = 2\sqrt{\frac{n}{m}} \; ,$$

as required. □

B. PageRank Scores. The PageRank score (1, 2) is the solution $\mathbf{s}$ of the linear system

$$\left[\alpha\mathbf{A}^T(\mathbf{D}^o)^{-1} + (1-\alpha)n^{-1}\mathbf{E}\right]\mathbf{s} = \mathbf{s} \; , \tag{S3}$$

where $\mathbf{D}^o = \mathrm{diag}(\mathbf{Ae})$. The Perron-Frobenius Theorem (3) ensures that $\mathbf{s}$ is strictly positive entrywise. We assume $\mathbf{s}$ to be normalized so that $\mathbf{s}^T\mathbf{e} = n$, which is contrary to the usual normalization $\mathbf{s}^T\mathbf{e} = 1$. This choice amounts to a rescaling of the parameters $\boldsymbol{\beta}$, and does not otherwise impact the analysis.

In the case of PageRank, it is difficult to derive a result for general features and we therefore work directly with the PageRank model with linear features.

Theorem S2. *The vector $\mathbf{s}_0 = \mathbf{e}$ is the unique egalitarian root of $\mathbf{f}$ under PageRank scores. In the PageRank-Linear model, the egalitarian root is linearly stable if and only if $\beta < \frac{1}{\alpha}$.*

Proof. Uniqueness is a direct consequence of normalization: if $\mathbf{s} = \theta\mathbf{e}$ and $\mathbf{e}^T\mathbf{s} = n$, then we must have $\theta = 1$.

We next obtain a necessary condition describing roots of $\mathbf{f}$. We start with a useful simplification. At any fixed point of $\mathbf{f}$, we must have $\mathbf{D}^o = m\mathbf{I}$. This is because, at any such fixed point, we must have $\mathbf{A} = m\mathbf{G}$, and $n\mathbf{G}$ is row-stochastic. For the purposes of analysis in the long-memory limit, we can therefore consider $\mathbf{s}$ to be defined by the simplified equation

$$\left[\alpha m^{-1}n\mathbf{A}^T + (1-\alpha)n^{-1}\mathbf{E}\right]\mathbf{s} = \mathbf{s} \; . \tag{S4}$$

In the next time step, we will have

$$\left[\alpha m^{-1}n(\mathbf{A}^T + \delta\mathbf{A}^T) + (1-\alpha)n^{-1}\mathbf{E}\right](\mathbf{s} + \delta\mathbf{s}) = \mathbf{s} + \delta\mathbf{s} \; .$$

Expanding and canceling yields

$$\left[\alpha m^{-1}n\mathbf{A}^T + (1-\alpha)n^{-1}\mathbf{E}\right]\delta\mathbf{s} + \alpha m^{-1}n(\delta\mathbf{A}^T)\mathbf{s} + o(1-\lambda) = \delta\mathbf{s} \; .$$

The term $o(1-\lambda)$ includes terms involving the product $(\delta\mathbf{A}^T)(\delta\mathbf{s})$, and relies on the fact that $\delta\mathbf{s}$ is a smooth function of $\mathbf{A}$. Rearranging and dropping the asymptotic term, we obtain, in the long memory limit,

$$\left[\mathbf{I} - \alpha m^{-1}n\mathbf{A}^T - (1-\alpha)n^{-1}\mathbf{E}\right]\delta\mathbf{s} = \alpha m^{-1}n(\delta\mathbf{A}^T)\mathbf{s} \; . \tag{S5}$$

This expression gives an implicit representation of $\mathbf{f}$ via the relation $\mathbf{f}(\mathbf{s},\mathbf{A}) = \lim_{\lambda\to 1}\frac{\mathbb{E}[\delta\mathbf{s}]}{1-\lambda}$. We can therefore enforce $\mathbf{f}(\mathbf{s},\mathbf{A}) = \mathbf{0}$ by setting $\mathbb{E}[\delta\mathbf{s}] = \mathbf{0}$, obtaining the necessary condition $\mathbb{E}[\delta\mathbf{A}^T]\mathbf{s} = \mathbf{0}$ for roots of $\mathbf{f}$. Expanding this condition yields,

$$\mathbf{0} = \mathbb{E}[\delta\mathbf{A}^T]\mathbf{s} = (1-\lambda)(\mathbf{G}^T - \mathbf{A}^T)\mathbf{s} \; .$$

Inserting Eq. (S4) and rearranging yields the nonlinear system

$$\left[\mathbf{G}^T + \alpha^{-1}(1-\alpha)n^{-2}\mathbf{E}\right]\mathbf{s} = \alpha^{-1}n^{-1}\mathbf{s} \; . \tag{S6}$$

The largest eigenvalue of the matrix on the lefthand side is $\alpha^{-1}n^{-1}$. This allows us to numerically solve Eq. (S6) iteratively, by alternating between solving for $\mathbf{s}$ via a standard eigenvalue solver and updating $\mathbf{G}$ with the new value of $\mathbf{s}$. This is the method implemented in the accompanying software and used to generate equilibria in Fig. 3.

In order to derive the linear stability criterion, we divide both sides of Eq. (S5) by $1-\lambda$ and differentiate with respect to $\mathbf{s}$, obtaining

$$\left[\mathbf{I} - \alpha m^{-1}n\mathbf{A}^T - (1-\alpha)n^{-1}\mathbf{E}\right]\mathbf{J}(\mathbf{s}) = \alpha m^{-1}n\frac{\partial}{\partial\mathbf{s}}\left[\mathbf{G}^T\mathbf{s} - \mathbf{A}^T\mathbf{s}\right] \; .$$

After inserting Eq. (S4) and simplifying, we have

$$\begin{aligned}\left[\mathbf{I}-\alpha m^{-1}n\mathbf{A}^T-(1-\alpha)n^{-1}\mathbf{E}\right]\mathbf{J}(\mathbf{s}) &= \alpha m^{-1}n\frac{\partial}{\partial\mathbf{s}}\left[\mathbf{G}^T\mathbf{s}-\alpha^{-1}mn^{-1}\mathbf{s}+\alpha^{-1}(1-\alpha)mn^{-2}\mathbf{E}\mathbf{s}\right]\\ &= \alpha m^{-1}n\frac{\partial}{\partial\mathbf{s}}\left[\mathbf{G}^T\mathbf{s}-\alpha^{-1}mn^{-1}\mathbf{s}\right]\ .\end{aligned}$$

The second line follows from the normalization of $\mathbf{s}$, which implies that $\mathbf{E}\mathbf{s}=n\mathbf{e}$, a constant vector which does not depend on $\mathbf{s}$. Differentiating the righthand side then yields

$$\begin{aligned}\left[\mathbf{I}-\alpha m^{-1}n\mathbf{A}^T-(1-\alpha)n^{-1}\mathbf{E}\right]\mathbf{J}(\mathbf{s}) &= \alpha m^{-1}n\left[\mathbf{G}^T+(\mathbf{e}^T\mathbf{s})mn^{-1}\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{s}}\right]-\mathbf{I}\\ &= \alpha m^{-1}n\left[\mathbf{G}^T+m\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{s}}\right]-\mathbf{I}\ .\end{aligned}$$

Evaluated at the egalitarian solution $\mathbf{s}_0=\mathbf{e}$, this becomes

$$\left[\mathbf{I}-\alpha m^{-1}n\mathbf{A}^T-(1-\alpha)n^{-1}\mathbf{E}\right]\mathbf{J}(\mathbf{s}_0)=\alpha m^{-1}n^{-1}\mathbf{E}+\alpha\mathbf{M}(\mathbf{s}_0;\boldsymbol{\beta})-\mathbf{I}\ .$$

To complete the argument, we note that, at the egalitarian solution of our model dynamics, $\mathbf{A}=n^{-2}\mathbf{E}$. Inserting and simplifying, we have

$$\left[\mathbf{I}-\alpha m^{-1}n^{-1}\mathbf{E}\right]\mathbf{J}(\mathbf{s}_0)=\alpha n^{-1}m^{-1}\mathbf{E}+\alpha n\mathbf{M}(\mathbf{s}_0;\boldsymbol{\beta})-\mathbf{I}\ .$$

Provided that $\alpha<1$, the premultiplying matrix on the lefthand side is invertible, and $\left[\mathbf{I}-\alpha m^{-1}n^{-1}\mathbf{E}\right]^{-1}=\mathbf{I}+\alpha(m-\alpha)^{-1}n^{-1}\mathbf{E}$. This matrix has a single eigenvalue $1+\alpha(m-\alpha)^{-1}$ with eigenvector $\mathbf{e}$, and additional eigenvalues equal to unity in orthogonal directions. We then have

$$\mathbf{J}(\mathbf{s}_0)=\alpha m^{-1}(1+\alpha(m-\alpha)^{-1})\mathbf{E}+\alpha n\left[\mathbf{I}+\alpha(m-\alpha)^{-1}n^{-1}\mathbf{E}\right]\mathbf{M}(\mathbf{s}_0;\boldsymbol{\beta})-\mathbf{I}\ .$$

In the PageRank-Linear model, $\mathbf{M}(\mathbf{s}_0;\boldsymbol{\beta})=\beta n^{-1}(\mathbf{I}-n^{-1}\mathbf{E})$, and we therefore have

$$\mathbf{J}(\mathbf{s}_0)=\alpha m^{-1}(1+\alpha(m-\alpha)^{-1})\mathbf{E}+\alpha\beta\left[\mathbf{I}+\alpha(m-\alpha)^{-1}n^{-1}\mathbf{E}\right](\mathbf{I}-n^{-1}\mathbf{E})-\mathbf{I}\ .$$

We can now read off the eigenvalues of $\mathbf{J}(\mathbf{s}_0)$ analytically. The eigenvector $\mathbf{e}$ has eigenvalue -1, while any vector orthogonal to $\mathbf{e}$ has eigenvalue $\alpha\beta-1$. This latter eigenvalue is strictly negative if and only if $\beta<\frac{1}{\alpha}$, as was to be shown. ☐

C. SpringRank Scores. We return to the general formalism of score functions and features introduced at the beginning of this section.

A SpringRank vector $\mathbf{s}$ for a matrix $\mathbf{A}$ with regularization $\alpha\in\mathbb{R}$ is a solution to the linear system

$$\left[\mathbf{D}^i+\mathbf{D}^o-(\mathbf{A}+\mathbf{A}^T)+\alpha\mathbf{I}\right]\mathbf{s}=\mathbf{d}^i-\mathbf{d}^o. \quad [S7]$$

where, $\mathbf{d}^i=\mathbf{e}^T\mathbf{A}$, $\mathbf{d}^o=\mathbf{A}^T\mathbf{e}$, $\mathbf{D}^i=\mathrm{diag}(\mathbf{d}^i)$, and $\mathbf{D}^o=\mathrm{diag}(\mathbf{d}^o)$. When $\alpha>0$, Eq. (**??**) is invertible and $\mathbf{s}$ is therefore unique. Thus, throughout this section we will assume that $\alpha>0$, and correspondingly refer to $\mathbf{s}$ as "the" SpringRank vector of $\mathbf{A}$. It is convenient to define $\mathbf{L}_\alpha=\mathbf{D}^i+\mathbf{D}^o-(\mathbf{A}+\mathbf{A}^T)+\alpha\mathbf{I}$ and $\boldsymbol{\Lambda}=\mathbf{D}^i-\mathbf{D}^o$, in which case the SpringRank relation reads $\mathbf{L}_\alpha\mathbf{s}=\boldsymbol{\Lambda}\mathbf{e}$.

Theorem S3 (Stable Egalitarianism with SpringRank Scores)**.** *When σ is the SpringRank map, the vector $\mathbf{s}_0=\mathbf{0}$ is a fixed point of $\mathbf{f}$, and is the only egalitarian fixed point of the dynamics. This fixed point is linearly stable in the long-memory limit if and only if the matrix*

$$\mathbf{M}(\mathbf{0};\boldsymbol{\beta})-2n^{-1}(\mathbf{I}-n^{-1}\mathbf{E})$$

has eigenvalues strictly smaller than $\frac{\alpha n}{m}$.

We will break the proof into a series of three lemmas. The first lemma calculates the analytical form of $\mathbf{f}$. The second shows that $\mathbf{s}_0=\mathbf{0}$ is the unique egalitarian fixed point of the long-memory limiting dynamics $\mathbf{f}$. The third gives the criterion for linear stability.

Lemma S1. *The deterministic approximant $\mathbf{f}$ for the SpringRank vector is given by*

$$\mathbf{f}(\mathbf{s},\mathbf{A})=\mathbf{s}+\mathbf{L}_\alpha^{-1}\left(-\alpha\mathbf{s}-m\left(n^{-1}\mathbf{L}_\mathbf{G}\mathbf{s}-(n^{-1}\mathbf{e}-\boldsymbol{\gamma})\right)\right)\ , \quad [S8]$$

where $\mathbf{L}_\mathbf{G}=\boldsymbol{\Gamma}+n^{-1}\mathbf{I}-(\mathbf{G}+\mathbf{G}^T)$.

Proof. Let us fix an implicit time step t. Here and below, we use the notational template $\delta M = M(t+1) - M(t)$ to refer to increments in various quantities under the dynamics Eq. (**??**). For example, $\delta \mathbf{A} = \mathbf{A}(t+1) - \mathbf{A}(t)$ refers to the increment in $\mathbf{A}$ under the dynamics. We compute directly

$$\begin{aligned}\delta \mathbf{A} &= (\lambda - 1)(\mathbf{A} - \boldsymbol{\Delta}) \\ \delta \mathbf{D}^o &= (\lambda - 1)(\mathbf{D}^o - \mathrm{diag}(\boldsymbol{\Delta}\mathbf{e})) \\ \delta \mathbf{D}^i &= (\lambda - 1)(\mathbf{D}^i - \mathrm{diag}(\boldsymbol{\Delta}^T\mathbf{e}))\ .\end{aligned}$$

We can also explicitly write out formulae for the increments in $\mathbf{L}_\alpha$ and $\boldsymbol{\Lambda}$:

$$\begin{aligned}\delta \boldsymbol{\Lambda} &= \delta \mathbf{D}^i - \delta \mathbf{D}^o \\ &= (\lambda - 1)\left[\mathbf{D}^i - \mathbf{D}^o + \mathrm{diag}((\boldsymbol{\Delta} - \boldsymbol{\Delta}^T)\mathbf{e})\right] \\ &= (\lambda - 1)\left[\boldsymbol{\Lambda} + \mathrm{diag}((\boldsymbol{\Delta} - \boldsymbol{\Delta}^T)\mathbf{e})\right]\ , \qquad \text{[S9]} \\ \delta \mathbf{L}_\alpha &= \delta \mathbf{D}^i + \delta \mathbf{D}^o - (\delta \mathbf{A} + \delta \mathbf{A}^T) \\ &= (\lambda - 1)\left[\mathbf{D}^i + \mathbf{D}^o - \mathrm{diag}(\boldsymbol{\Delta}^T\mathbf{e} + \boldsymbol{\Delta}\mathbf{e}) - (\mathbf{A} + \mathbf{A}^T) + \boldsymbol{\Delta} + \boldsymbol{\Delta}^T\right] \\ &= (\lambda - 1)\left[\mathbf{L} - \mathrm{diag}(\boldsymbol{\Delta}^T\mathbf{e} + \boldsymbol{\Delta}\mathbf{e}) + \boldsymbol{\Delta} + \boldsymbol{\Delta}^T\right] \\ &\triangleq (\lambda - 1)\left[\mathbf{L} - \mathbf{L}_{\boldsymbol{\Delta}}\right]\ , \qquad \text{[S10]}\end{aligned}$$

where we have given a name to the Laplacian $\mathbf{L}_{\boldsymbol{\Delta}} = \mathrm{diag}(\boldsymbol{\Delta}^T\mathbf{e} + \boldsymbol{\Delta}\mathbf{e}) - \boldsymbol{\Delta}^T - \boldsymbol{\Delta}$ of $\boldsymbol{\Delta}$. Note that $\delta \mathbf{L}_\alpha$ does not depend on α, and we therefore simply write $\delta \mathbf{L} = \delta \mathbf{L}_\alpha$.

We can now formulate a simple condition for equilibrium in expectation. We have

$$(\mathbf{L}_\alpha + \delta \mathbf{L})(\mathbf{s} + \delta \mathbf{s}) = (\boldsymbol{\Lambda} + \delta \boldsymbol{\Lambda})\mathbf{e}\ .$$

Subtracting the SpringRank relation $\mathbf{L}_\alpha \mathbf{s} = \boldsymbol{\Lambda}\mathbf{e}$ from each side of this expression, we obtain

$$(\mathbf{L}_\alpha + \delta \mathbf{L})\delta \mathbf{s} = (\delta \boldsymbol{\Lambda})\mathbf{e} - (\delta \mathbf{L})\mathbf{s}\ .$$

Since $\delta \mathbf{L} = O(1-\lambda)$, the lefthand matrix is invertible in for small λ provided that $\alpha > 0$. We therefore obtain

$$\begin{aligned}\delta \mathbf{s} &= \left(\mathbf{L}_\alpha^{-1} + O(1-\lambda)\right)((\delta \boldsymbol{\Lambda})\mathbf{e} - (\delta \mathbf{L})\mathbf{s}) \\ &= \mathbf{L}_\alpha^{-1}((\delta \boldsymbol{\Lambda})\mathbf{e} - (\delta \mathbf{L})\mathbf{s}) + O((1-\lambda)^2)\ .\end{aligned}$$

The term $O((1-\lambda)^2)$ arises from the product of $O(1-\lambda)$ and the copy of $(\lambda - 1)$ within $\delta \boldsymbol{\Lambda}$ and $\delta \mathbf{L}$. Taking expectations,

$$\mathbb{E}[\delta \mathbf{s}] = \mathbf{L}_\alpha^{-1}(\mathbb{E}[\delta \boldsymbol{\Lambda}]\mathbf{e} - \mathbb{E}[\delta \mathbf{L}]\mathbf{s}) + O((1-\lambda)^2)\ .$$

We next insert the expressions Eq. (S9) and Eq. (S10) and use the fact that $\mathbb{E}[\boldsymbol{\Delta}] = m\mathbf{G}$. This gives

$$\mathbb{E}[\delta \mathbf{s}] = (1-\lambda)\mathbf{L}_\alpha^{-1}\left([\mathbf{L} - m\mathbf{L}_{\mathbf{G}}]\,\mathbf{s} - \left[\boldsymbol{\Lambda} + m \cdot \mathrm{diag}((\mathbf{G} - \mathbf{G}^T)\mathbf{e})\right]\mathbf{e}\right) + O((1-\lambda)^2)\ .$$

We can simplify this expression by recalling that $(\mathbf{L} + \alpha \mathbf{I})\mathbf{s} = \boldsymbol{\Lambda}\mathbf{e}$ by definition, as well as the identities $\mathbf{G}\mathbf{e} = n^{-1}\mathbf{e}$ and $\mathbf{G}^T\mathbf{e} = \boldsymbol{\gamma}$. Inserting these identities and simplifying yields

$$= (1-\lambda)\mathbf{L}_\alpha^{-1}\left(-\alpha \mathbf{s} - m\left(\mathbf{L}_{\mathbf{G}}\mathbf{s} + (n^{-1}\mathbf{e} - \boldsymbol{\gamma})\right)\right) + O((1-\lambda)^2)\ .$$

We now construct $\mathbf{f}$, obtaining Since $\mathbb{E}[\delta \mathbf{s}] = \mathbb{E}[\sigma(\lambda \mathbf{A} + (1-\lambda)\boldsymbol{\Delta})]$, we can write

$$\begin{aligned}\mathbf{f}(\mathbf{s}, \mathbf{A}) &= \mathbf{s} + \lim_{\lambda \to 1} \frac{\mathbb{E}[\delta \mathbf{s}]}{1-\lambda} \\ &= \mathbf{s} - \mathbf{L}_\alpha^{-1}\left[\alpha \mathbf{s} + m\left(\mathbf{L}_{\mathbf{G}}\mathbf{s} + (n^{-1}\mathbf{e} - \boldsymbol{\gamma})\right)\right]\ ,\end{aligned}$$

as was to be shown. □

Lemma S2. *When σ is the SpringRank map, the vector $\mathbf{s}_0 = \mathbf{0}$ is a root of $\mathbf{f}$, and is the only egalitarian fixed point.*

Proof. To show that $\mathbf{s}_0 = \mathbf{0}$ is a fixed point of $\mathbf{f}$, it suffices to insert this solution into Eq. (S8) and simplify, noting that, when $\mathbf{s} = \mathbf{0}$, $\boldsymbol{\gamma} = n^{-1}\mathbf{e}$. To show that it is the unique egalitarian root realizable as a SpringRank score, suppose that $s\mathbf{e}$ were a SpringRank score for some $s \neq 0$. Inserting this into Eq. (**??**) and using the fact that $\mathbf{e}$ is a zero eigenvector of the unregularized Laplacian, we would have

$$\alpha s \mathbf{e} = \mathbf{d}^i - \mathbf{d}^o\ .$$

The total in-degree must equal the total out-degree. Pre-multiplying by $\mathbf{e}$ therefore zeros out the righthand, leaving:

$$\alpha s \mathbf{e}^T\mathbf{e} = \alpha s n = 0\ ,$$

which is a contradiction unless $s = 0$. □

Lemma S3. *The egalitarian root* $\mathbf{s} = \mathbf{0}$ *is a linearly stable root of the SpringRank dynamics in the long-memory limit if and only if the matrix*

$$\mathbf{M}(\mathbf{0};\beta) - 2n^{-1}(\mathbf{I} - n^{-1}\mathbf{E})$$

has eigenvalues strictly smaller than $\frac{\alpha}{m}$.

Proof. We need to compute $\mathbf{J}(\mathbf{s}_0)$, the Jacobian matrix of $\mathbf{f}$ at $\mathbf{s}_0 = \mathbf{0}$. The fixed point will be stable provided that $\mathbf{J}(\mathbf{s}_0)$ has strictly negative eigenvalues. To compute this Jacobian, we compute derivatives in Eq. (S8). Doing so and applying the product rule, we have

$$\frac{\partial f(\mathbf{s})}{\partial \mathbf{s}} = \mathbf{I} - \mathbf{L}_\alpha^{-1}\left(\alpha\mathbf{I} + m\left(n^{-1}\frac{\partial(\mathbf{L}_\mathbf{G}\mathbf{s})}{\partial\mathbf{s}} - \frac{\partial\gamma}{\partial\mathbf{s}}\right)\right) .$$

We calculate $\frac{\partial \mathbf{L}_\mathbf{G}}{\partial \mathbf{s}}$ in Equation (S11), now obtaining

$$\frac{\partial f(\mathbf{s})}{\partial \mathbf{s}} = \mathbf{I} - \mathbf{L}_\alpha^{-1}\left(\alpha\mathbf{I} + m\left(n^{-1}\left[\mathbf{L}_\mathbf{G} + \boldsymbol{\Sigma}\frac{\partial\gamma}{\partial\mathbf{s}} - \frac{\partial\gamma}{\partial\mathbf{s}}(\mathbf{S}^T + (\mathbf{e}^T\mathbf{s})\mathbf{I})\right] - \frac{\partial\gamma}{\partial\mathbf{s}}\right)\right) .$$

Evaluating this expression at $\mathbf{s} = \mathbf{0}$, we have

$$\mathbf{J}(\mathbf{0}) = -\mathbf{L}_\alpha^{-1}\left(\alpha\mathbf{I} + m\left(n^{-1}\mathbf{L}_\mathbf{G} - \frac{\partial\gamma(\mathbf{0})}{\partial\mathbf{s}}\right)\right) ,$$

where $\mathbf{L}_\mathbf{G}$ must also be evaluated at $\mathbf{s} = \mathbf{0}$. We have $\mathbf{G}(\mathbf{0}) = n^{-1}\mathbf{E}$, which implies $\mathbf{L}_\mathbf{G} = 2(\mathbf{I} - n^{-1}\mathbf{E})$. We insert this expression and the formula for $\frac{\partial\gamma}{\partial\mathbf{s}}$ given in Eq. (S2), obtaining

$$\mathbf{J}(\mathbf{0}) = -\mathbf{L}_\alpha^{-1}\left[\alpha\mathbf{I} + mn^{-1}(\mathbf{I} - n^{-1}\mathbf{E})\left(2\mathbf{I} - \sum_{i=1}^{n}\sum_{\ell=1}^{k}\beta_\ell\frac{\partial\phi_i^\ell(\mathbf{s}_0)}{\partial\mathbf{s}}\right)\right] .$$

Since $\mathbf{L}_\alpha$ is symmetric and positive-definite, $\mathbf{L}_\alpha^{-1}$ is as well. The stability of the egalitarian fixed point is therefore determined by the eigenvalues of the matrix inside the brackets. Multiplying by nm^{-1}, we find that a necessary and sufficient condition is that the matrix

$$(\mathbf{I} - n^{-1}\mathbf{E})\left(2\mathbf{I} - \sum_{i=1}^{n}\sum_{\ell=1}^{k}\beta_\ell\frac{\partial\phi_i^\ell(\mathbf{s}_0)}{\partial\mathbf{s}}\right) = \mathbf{M}(\mathbf{0};\beta) - 2n^{-1}(\mathbf{I} - n^{-1}\mathbf{E})$$

have eigenvalues no larger than $\frac{\alpha}{m}$, completing the proof. ☐

Corrolary S2. *In the SpringRank-Linear model,* $\mathbf{s}_0 = \mathbf{0}$ *is a linearly stable fixed point of* $\mathbf{f}$ *if and only if* $\beta < 2 + \frac{\alpha n}{m}$.

Proof. It suffices to specialize Theorem S3 to the case of linear features. In particular, we have $\mathbf{M}(\mathbf{0};\beta) = \beta n^{-1}(\mathbf{I} - n^{-1}\mathbf{E})$. We therefore require that the matrix

$$\beta n^{-1}(\mathbf{I} - n^{-1}\mathbf{E}) - 2n^{-1}(\mathbf{I} - n^{-1}\mathbf{E}) = n^{-1}(\beta - 2)(\mathbf{I} - n^{-1}\mathbf{E})$$

have eigenvalues smaller than $\frac{\alpha}{m}$. We can compute the eigenvalues of this matrix analytically – there is a zero eigenvalue corresponding to the vector $\mathbf{e}$. Then, any vector $\mathbf{v} \perp \mathbf{e}$ is also an eigenvector with eigenvalue $n^{-1}(\beta - 2)$. We therefore require $n^{-1}(\beta - 2) < \frac{\alpha}{m}$, or $\beta < 2 + \frac{\alpha n}{m}$, completing the argument. ☐

Lemma S4. *We have*

$$\frac{\partial \mathbf{L}_\mathbf{G}\mathbf{s}}{\partial\mathbf{s}} = \mathbf{L}_\mathbf{G} + \boldsymbol{\Sigma}\frac{\partial\gamma}{\partial\mathbf{s}} - \frac{\partial\gamma}{\partial\mathbf{s}}(\mathbf{S}^T + (\mathbf{e}^T\mathbf{s})\mathbf{I}) . \qquad \text{[S11]}$$

Proof. We first compute the derivatives $\frac{\partial(\mathbf{G}\mathbf{s})}{\partial\mathbf{s}}$ and $\frac{\partial(\mathbf{G}^T\mathbf{s})}{\partial\mathbf{s}}$. The ith component of $\mathbf{G}\mathbf{s}$ is $v_i = \sum_j \gamma_j s_j$. The product rule for scalar functions of vectors gives the ith row of the derivative:

$$\frac{\partial \mathbf{G}\mathbf{s}_i}{\partial\mathbf{s}} = \sum_j \gamma_j\mathbf{e}_j + \sum_j s_j\frac{\partial\gamma_j}{\partial\mathbf{s}} = \boldsymbol{\gamma} + \sum_j s_j\frac{\partial\gamma_j}{\partial\mathbf{s}} .$$

Written in matrix notation, the first term is $\mathbf{G}$. To write the second term in matrix form, note that we need to multiply $\frac{\partial\gamma}{\partial\mathbf{s}}$ by the matrix each of whose columns is a copy of $\mathbf{s}$. This matrix is $\mathbf{S}^T$. We therefore obtain

$$\frac{\partial(\mathbf{G}\mathbf{s})}{\partial\mathbf{s}} = \mathbf{G} + \frac{\partial\gamma}{\partial\mathbf{s}}\mathbf{S}^T \quad .$$

To compute the second derivative, note that $\mathbf{G}^T\mathbf{s} = \boldsymbol{\gamma}(\mathbf{e}^T\mathbf{s})$, with ith component $\gamma_i\mathbf{e}^T\mathbf{s}$. Using the product rule for scalar functions of vectors, we have

$$\frac{\partial}{\partial \mathbf{s}}\gamma_i\mathbf{e}^T\mathbf{s} = \gamma_i\mathbf{e} + (\mathbf{e}^T\mathbf{s})\frac{\partial \gamma_i}{\partial \mathbf{s}}.$$

The first term will become the matrix whose ith row is γ_i, i.e. $\mathbf{G}^T$. This yields

$$\frac{\partial(\mathbf{G}^T\mathbf{s})}{\partial \mathbf{s}} = \mathbf{G}^T + (\mathbf{e}^T\mathbf{s})\frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{s}} .$$

Combining these expressions yields our formula for $\frac{\partial \mathbf{L}_\mathbf{G}\mathbf{s}}{\partial \mathbf{s}}$:

$$\begin{aligned}\frac{\partial \mathbf{L}_\mathbf{G}\mathbf{s}}{\partial \mathbf{s}} &= \frac{\partial}{\partial \mathbf{s}}\left[\boldsymbol{\Gamma}\mathbf{s} + \mathbf{s} - \mathbf{G}\mathbf{s} - \mathbf{G}^T\mathbf{s}\right] \\ &= \boldsymbol{\Gamma} + \boldsymbol{\Sigma}\frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{s}} + \mathbf{I} - \left(\mathbf{G} + \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{s}}\mathbf{S}^T + \mathbf{G}^T + (\mathbf{e}^T\mathbf{s})\frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{s}}\right) \\ &= \mathbf{L}_\mathbf{G} + \boldsymbol{\Sigma}\frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{s}} - \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{s}}(\mathbf{S}^T + (\mathbf{e}^T\mathbf{s})\mathbf{I}) ,\end{aligned}$$

as was to be shown. □

2. Parameter Estimation

Throughout this section, we use the shorthand $\{\mathbf{A}(t)\} = \{\mathbf{A}(t)\}_{t=0}^{\tau}$ to refer to temporal sequences of matrices up to fixed time τ. We now describe a simple maximum-likelihood model for learning the parameter $\boldsymbol{\beta}$ from a sequence of observations $\{\boldsymbol{\Delta}(t)\}$. By construction, $\boldsymbol{\Delta}(\tau)$ depends on the sequence of state matrices $\{\mathbf{A}(t)\}$ only through the most recent state $\mathbf{A}(\tau)$. We may therefore factor the probability of observing the data given a set of undetermined parameters as:

$$\mathbb{P}(\{\boldsymbol{\Delta}(t)\}; \mathbf{A}(0), \lambda, \boldsymbol{\beta}) = \prod_{t=0}^{\tau}\mathbb{P}(\boldsymbol{\Delta}(t); \mathbf{A}(t), \boldsymbol{\beta}) .$$

While the parameter λ has disappeared from the righthand side, this expression is nevertheless implicitly a function of λ since the value of $\mathbf{A}(\tau)$ given $\mathbf{A}(\tau-1)$ and $\boldsymbol{\Delta}(\tau-1)$ depends on λ.

Let us write out a typical factor on the righthand side. Let $\mathbf{k}_i = \boldsymbol{\Delta}_{i\cdot}$, and let $K_i = \mathbf{e}^T\mathbf{k}_i$. Then,

$$\mathbb{P}(\boldsymbol{\Delta}(t); \mathbf{A}(\tau), \boldsymbol{\beta}) = \prod_{i=1}^{n}\left(\frac{K_i}{\prod_{j=1}^{n}k_{ij}!}\prod_{j=1}^{n}(\gamma_{ij}(t))^{k_{ij}}\right) .$$

Taking logarithms and collecting terms that do not depend on $\boldsymbol{\beta}$ or λ into a constant $C(t)$, we obtain

$$\log \mathbb{P}(\boldsymbol{\Delta}(t); \mathbf{A}(t), \boldsymbol{\beta}) = \sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}(t)\log\gamma_{ij}(t) + C(t).$$

The log-likelihood of the full sequence is then

$$\mathcal{L}(\lambda, \boldsymbol{\beta}; \{\boldsymbol{\Delta}(t)\}, \mathbf{A}(0)) \triangleq \log \mathbb{P}(\{\boldsymbol{\Delta}(t)\}; \mathbf{A}(0), \lambda, \boldsymbol{\beta}) = \sum_{t=0}^{\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}(t)\log\gamma_{ij}(t) + C,$$

where $C = \sum_{t=0}^{\tau}C(t)$. The dependence on $\boldsymbol{\beta}$ appears through γ_{ij}.

The maximum likelihood approach encourages us to choose as parameter estimates $\hat{\lambda}$ and $\hat{\boldsymbol{\beta}}$ the values

$$\hat{\lambda}, \hat{\boldsymbol{\beta}} = \underset{\lambda, \boldsymbol{\beta}}{\operatorname{argmax}}\, \mathcal{L}(\lambda, \boldsymbol{\beta}; \{\boldsymbol{\Delta}(t)\}, \mathbf{A}(0)) . \qquad \text{[S12]}$$

Standard theory of maximum likelihood in exponential families implies that $\mathcal{L}$ is convex in $\boldsymbol{\beta}$ for any fixed λ. This implies that, when $\hat{\lambda}$ is known, we can solve for $\hat{\boldsymbol{\beta}}$ via standard first- or second-order optimization methods. Let $\mathcal{L}^*(\lambda; \{\boldsymbol{\Delta}(t)\}, \mathbf{A}(0))$ be the optimized loglikelihood for fixed λ. We then complete the maximum likelihood scheme by optimizing $\mathcal{L}^*$ with respect to λ, which our accompanying software does via a customized hill-climbing algorithm. In general, $\mathcal{L}^*$ may fail to be convex as a function of λ, and we therefore perform multiple runs with different initial values of λ in order to find the global maximum.

3. Additional Model Traces

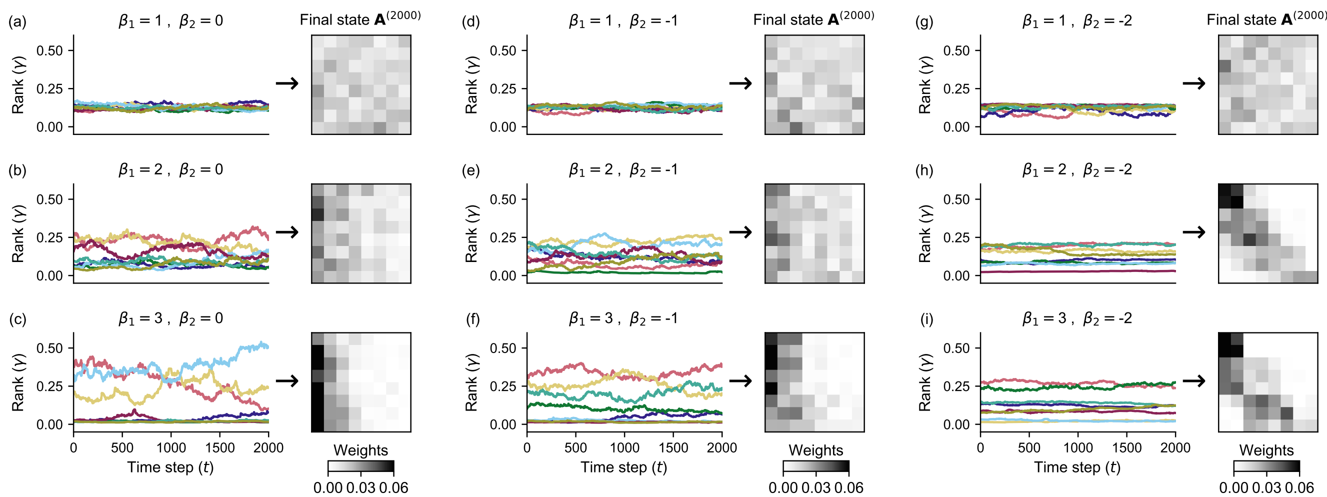

Fig. S1. Example dynamics of the model. Populations of $n = 8$ agents were simulated for 2000 time steps using the SpringRank score with linear and quadratic features, varying the preference parameters β_1 and β_2 as indicated in the panels. The memory parameter was fixed at $\lambda = 0.995$. In each panel, the plot on the left shows the simulated rank vector $\boldsymbol{\gamma}$ over time; different colors track the ranks of different agents. The heatmap on the right shows the adjacency matrix $\mathbf{A}$ at time step $t = 2000$ for the corresponding parameter values.

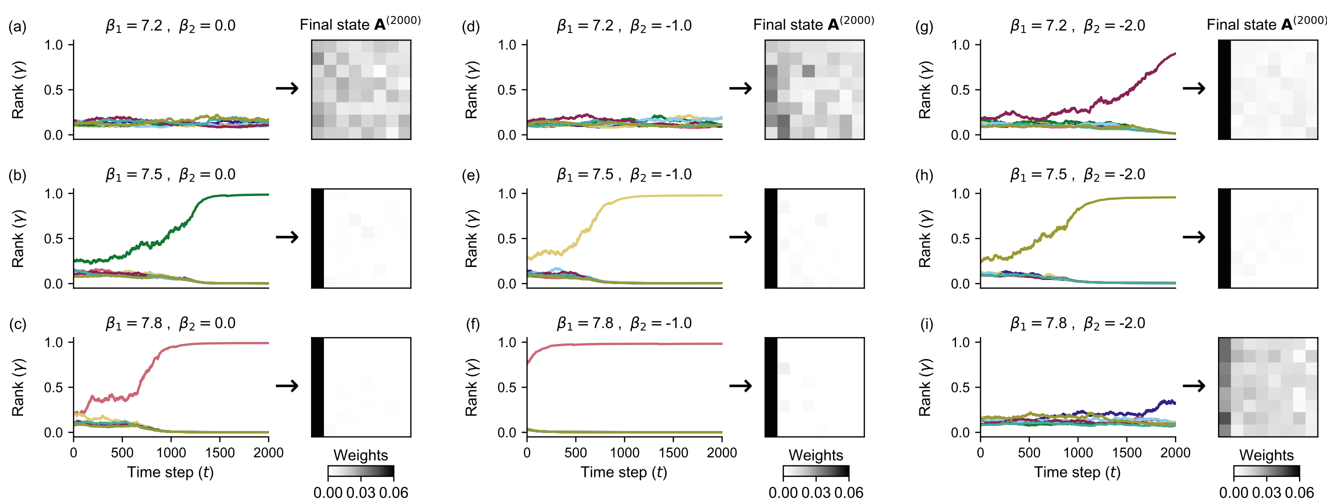

Fig. S2. As in Fig. S1, using the PageRank score function.

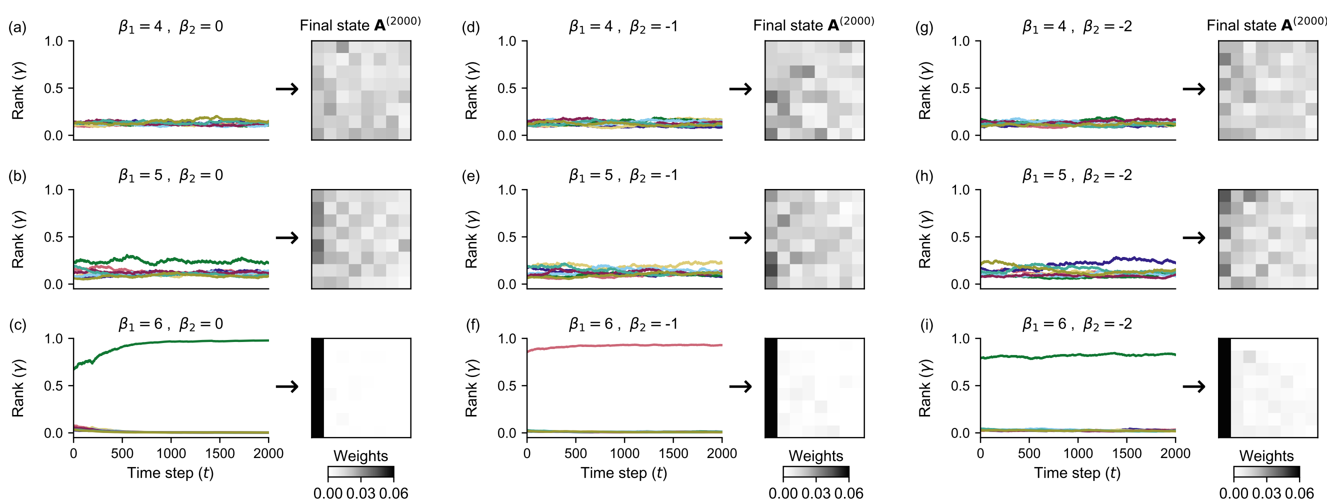

Fig. S3. As in Fig. S1, using the Root-Degree score function.

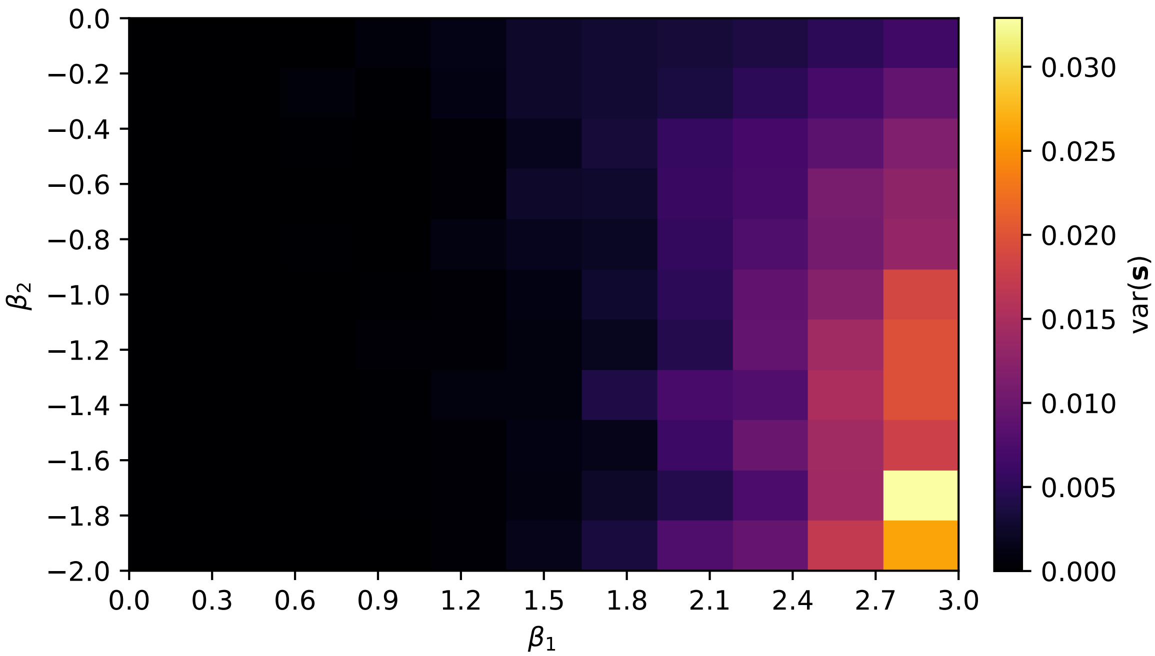

Fig. S4. Plot of the variance in the rank vector $\mathbf{s}$ over the final 500 iterations of a series of simulations with $n = 8$ and $\lambda = 0.995$ (as in Fig. 2). The parameters β_1 and β_2 are allowed to vary. Higher variances correspond to more strongly hierarchical states.

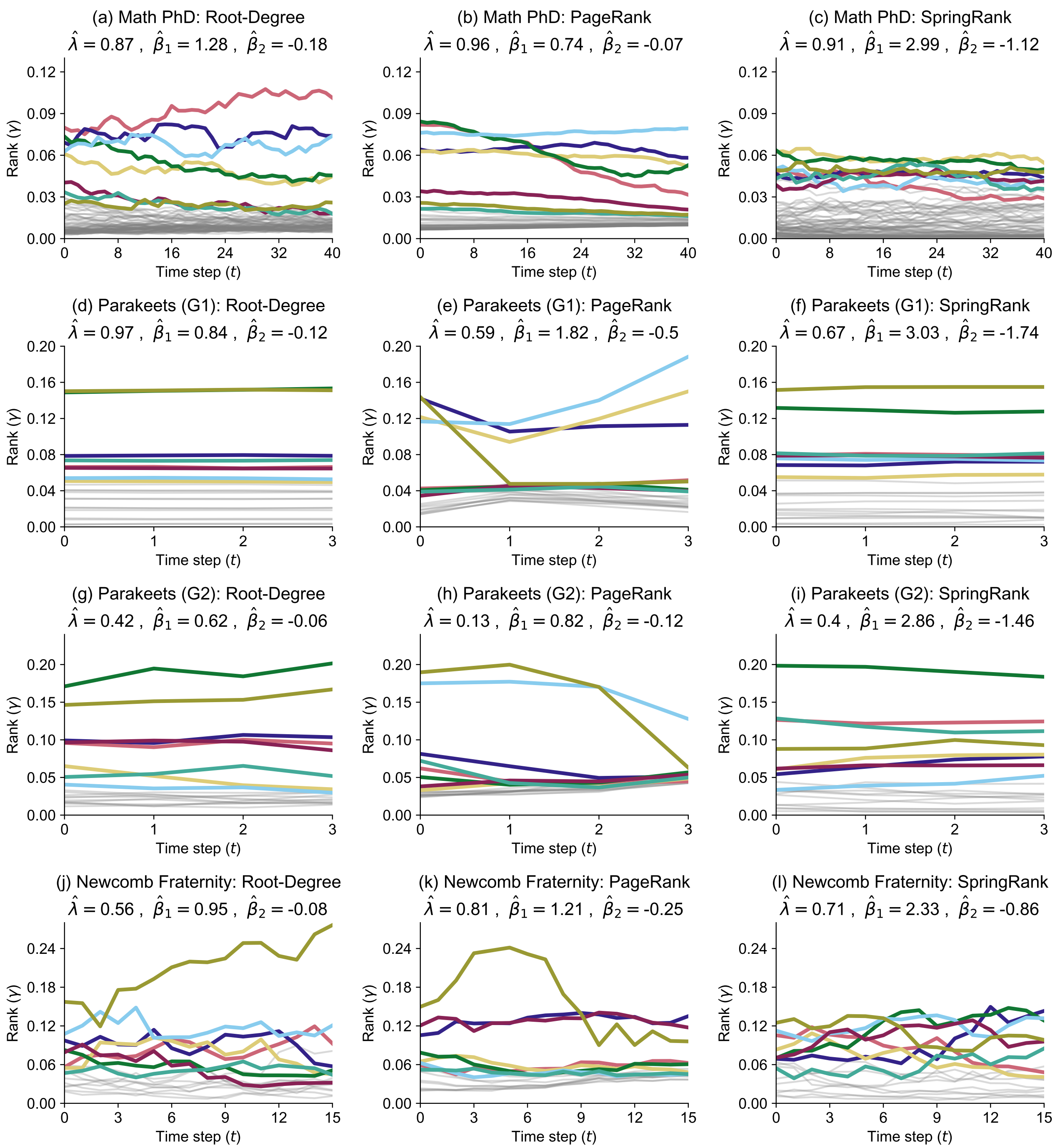

Fig. S5. Simulated dynamics of the model using inferred parameters $\hat{\lambda}$, $\hat{\beta}_1$, $\hat{\beta}_2$ in Table 1. The value of m for each row of panels corresponds to the average number of updates per time step in the corresponding data set, indicated in the panel title ($m = 150$ for Math PhD, $m = 279$ for Parakeets (G1), $m = 320$ for Parakeets (G2), and $m = 85$ for Newcomb Fraternity). Furthermore, the simulations in each row were initialized using the network at the relevant initial time step in the corresponding data set: the network of endorsements aggregated up to year 1960 for the Math PhD data set, and the network at time step 0 in each of the Parakeet and Newcomb Fraternity data sets. The traces in color correspond to nodes that rank among the top 8 on average over time; those in light gray track all other nodes. Other parameters: $\alpha_p = 0.85$, $\alpha_s = 10^{-8}$.

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