Supplementary Materials for "Improved doubly robust estimation in learning optimal individualized treatment rules"

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The supplementary materials contain five appendices. Appendix A provides proofs of Lemmas 1-3 and gives the asymptotic variance of various estimators under correctly specified propensity score model. Appendix B presents a general formula for the asymptotic variance of semiparametric M-estimators. This is a direct extension of Theorem 2 in Chen et al. (2003). Appendix C presents details of the Theorem 1 as well as the proofs. Appendix D presents details of the Theorem 2 as well as the proofs. Additional simulation results are reported in Appendix E.

Appendix A: Proofs and derivations in Section 2 of the main paper

A.1 Proof of Lemma 1

Proof. We show that when the propensity score is correct, i.e., $\pi(A, \boldsymbol{X}) = \pi_0(A, \boldsymbol{X})$, the class of influence functions corresponding to estimators of the form (2) have the following expression

$$\frac{YI\{A = d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} - \frac{I\{A = d(\boldsymbol{X})\} - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^*\} - V(d).$$

The technical details are as follows. Let $\tilde{\boldsymbol{\beta}}$ denote an intermediate value between $\hat{\boldsymbol{\beta}}$ and

$\boldsymbol{\beta}^*$, and $Q_{\boldsymbol{\beta}}(\boldsymbol{X}, A; \boldsymbol{\beta}) \triangleq \partial Q(\boldsymbol{X}, A; \boldsymbol{\beta})/\partial \boldsymbol{\beta}$,

$$\begin{aligned}
&\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\beta}}) - V(d)\right\} \\
= \;& \sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{Y_i I\{A_i = d(\boldsymbol{X}_i)\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} - \frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \hat{\boldsymbol{\beta}}\}\right] - V(d)\right\} \\
= \;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{Y_i I\{A_i = d(\boldsymbol{X}_i)\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} - \frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \boldsymbol{\beta}^*\} - V(d)\right. \\
& \left. - \frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \tilde{\boldsymbol{\beta}}\} \cdot (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*)\right] \\
= \;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{Y_i I\{A_i = d(\boldsymbol{X}_i)\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} - \frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \boldsymbol{\beta}^*\} - V(d)\right] + o_p(1)
\end{aligned}$$

The third equality in the above comes from the fact that

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \tilde{\boldsymbol{\beta}}\} \cdot (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) \\
= \;& \sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) \cdot \frac{1}{n}\sum_{i=1}^{n}\frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \tilde{\boldsymbol{\beta}}\}
\end{aligned}$$

Since $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) = O_p(1)$, and note that

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\frac{I\{A_i = d(\boldsymbol{X}_i)\} - \pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \tilde{\boldsymbol{\beta}}\} \\
\xrightarrow{p} \;& E\Big[\frac{I\{A = d(\boldsymbol{X})\} - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^*\}\Big] \\
= \;& E\Big\{E\Big[\frac{I\{A = d(\boldsymbol{X})\} - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^*\}\Big|\boldsymbol{X}\Big]\Big\} = 0,
\end{aligned}$$

we know that the whole term converges in probability to 0, and is thus negligible. □

A.2 Derivation of term (I) in (4)

$$
\begin{aligned}
& \operatorname{Var}\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - \frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right] \\
= \; & E\left(\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - \frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right]^2 \middle| \boldsymbol{X}\right) \\
= \; & E\left\{\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]^2 Q^2\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right\} \\
& -2E\left(\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right] Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right) \\
& +E\left(\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right]^2 \middle| \boldsymbol{X}\right).
\end{aligned}
$$

We know that

$$
\begin{aligned}
& E\left\{\frac{I\{A=d(\boldsymbol{X})\} - 2\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}I\{A=d(\boldsymbol{X})\} + \pi_0^2\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0^2\{d(\boldsymbol{X}),\boldsymbol{X}\}} \cdot Q^2\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right\} \\
= \; & \frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q^2\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\},
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left(\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right] Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right) \\
= \; & E\left(\frac{YI\{A=d(\boldsymbol{X})\}\left[1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}\right]}{\pi_0^2\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right) \\
& -E\left(\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q_0\{\boldsymbol{X},d(\boldsymbol{X})\} \cdot Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} \middle| \boldsymbol{X}\right) \\
= \; & \frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q_0\{\boldsymbol{X},d(\boldsymbol{X})\} \cdot Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E\left(\operatorname{Var}\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}-\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\}\,\middle|\,\boldsymbol{X}\right]\right) \\
= \; & E\left[\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q^2\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\}\right]+E\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}-Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right]^2 \\
& -2E\left[\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\cdot Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\}\right].
\end{aligned}
$$

A.3 Proof of Lemma 2

Proof. When the propensity score is correct, i.e., $\pi(A,\boldsymbol{X})=\pi_0(A,\boldsymbol{X})$, we have shown that $\hat{\boldsymbol{\beta}}^{\text{opt1}}\xrightarrow{p}\boldsymbol{\beta}^{\text{opt}}$. It is straightforward to show that

$$
\begin{aligned}
& \mathbb{P}_n\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q\{\boldsymbol{X},d(\boldsymbol{X});\hat{\boldsymbol{\beta}}^{\text{opt1}}\}\right] \\
= \; & \mathbb{P}_n\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^{\text{opt}}\}\right]+o_p(1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\text{opt1}}) &= \mathbb{P}_n\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}-\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q\{\boldsymbol{X},d(\boldsymbol{X});\hat{\boldsymbol{\beta}}^{\text{opt1}}\}\right] \\
&= \mathbb{P}_n\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]-\mathbb{P}_n\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^{\text{opt}}\}\right]+o_p(1),
\end{aligned}
$$

which converges to

$$
V(d)-E\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^{\text{opt}}\}\right]=V(d).
$$

Here we have used the fact that $E\left[I\{A=d(\boldsymbol{X})\}\mid\boldsymbol{X}\right]=P\{A=d(\boldsymbol{X})\mid\boldsymbol{X}\}=\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}$.

When the outcome model is correct, i.e., $Q(\boldsymbol{X},A;\boldsymbol{\beta}_0)=Q_0(\boldsymbol{X},A)$ for some $\boldsymbol{\beta}_0$, but the propensity score may not be, $\pi(A, \boldsymbol{X}) \neq \pi_0(A, \boldsymbol{X})$, we have shown that $\hat{\boldsymbol{\beta}}^{\text{opt1}} \xrightarrow{p} \boldsymbol{\beta}_0$.

$$\begin{aligned}
&\mathbb{P}_n\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q\{\boldsymbol{X}, d(\boldsymbol{X}); \hat{\boldsymbol{\beta}}^{\text{opt1}}\}\right] \\
= \;&\mathbb{P}_n\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_0\}\right] \\
&+\mathbb{P}_n\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \tilde{\boldsymbol{\beta}}\} \cdot (\hat{\boldsymbol{\beta}}^{\text{opt1}}-\boldsymbol{\beta}_0)\right],
\end{aligned}$$

where $\tilde{\boldsymbol{\beta}}$ is an intermediate value between $\hat{\boldsymbol{\beta}}^{\text{opt1}}$ and $\boldsymbol{\beta}_0$. The second term is $o_p(1)$ after assuming that $E\left[\frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_0\}\right]$ is finite. Thus, $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\text{opt1}})$ equals

$$\mathbb{P}_n\left[\frac{YI\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} - \frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_0\}\right] + o_p(1),$$

which converges to

$$E\left(\frac{I\{A=d(\boldsymbol{X})\}\left[Y-Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\}\right]}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}\right) + E\left[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\}\right].$$

The first term in the above equation is zero because $E\left[I\{A=d(\boldsymbol{X})\} \cdot Y \mid \boldsymbol{X}, A\right] = I\{A = d(\boldsymbol{X})\}Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\}$. The second term is equal to $V(d)$. Thus, $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\text{opt1}}) \xrightarrow{p} V(d)$ when the outcome model is correctly specified.

Since $\hat{\boldsymbol{\beta}}^{\text{opt1}} \xrightarrow{p} \boldsymbol{\beta}^{\text{opt}}$ when the propensity score is correct, by definition of $\boldsymbol{\beta}^{\text{opt}}$, it is trivial that $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\text{opt1}})$ achieves the smallest variance among class of estimators (2). □

A.4 Proof of Lemma 3

We first introduce the 'generalized product kernels' from Racine and Li (2004). For the ith subject, let $\boldsymbol{X}_i = (\boldsymbol{X}_i^d, \boldsymbol{X}_i^c)$, where $\boldsymbol{X}_i^d$ denote a $k \times 1$ vector of discrete variables and $\boldsymbol{X}_i^c \in \mathbb{R}^p$ denote the remaining continuous variables. We use $X_{t,i}^d$ to denote the tth component of $\boldsymbol{X}_i^d$. For any particular $\boldsymbol{x}^d$ whose tth component is x_t^d, Racine and Li (2004) defined the following kernel

$$l(X_{t,i}^d, x_t^d, \lambda) = \begin{cases} 1 & \text{if } X_{t,i}^d = x_t^d \\ \lambda & \text{if } X_{t,i}^d \neq x_t^d. \end{cases}$$

Then the product kernel for the discrete variables can be defined as $L(\boldsymbol{X}_i^d, \boldsymbol{x}^d, \lambda) = \prod_{t=1}^k l(X_{t,i}^d, x_t^d, \lambda)$. Let $W\{\cdot\}$ denote the kernel function associated with the continuous variables $\boldsymbol{x}^c$ and h to denote the smoothing parameters for the continuous variables. The product kernel for both continuous and discrete variables is $K_{h,ix} = W_{h,ix} L_{\lambda,ix}$, where $W_{h,ix} = h^{-p} W\{(\boldsymbol{X}_i^c - \boldsymbol{x}^c)/h\}$ and $L_{\lambda,ix} = L(\boldsymbol{X}_i^d, \boldsymbol{x}^d, \lambda)$.

The Q function $Q_0(\boldsymbol{X}, 1) = E(Y \mid \boldsymbol{X}, A = 1)$ and $Q_0(\boldsymbol{X}, -1) = E(Y \mid \boldsymbol{X}, A = -1)$ will be estimated by

$$\widehat{Q}_0(\boldsymbol{X}, 1) = \frac{\sum_{i=1}^n Y_i I(A_i = 1) K_{h,ix}}{\sum_{i=1}^n I(A_i = 1) K_{h,ix}}, \quad \widehat{Q}_0(\boldsymbol{X}, -1) = \frac{\sum_{i=1}^n Y_i I(A_i = -1) K_{h,ix}}{\sum_{i=1}^n I(A_i = -1) K_{h,ix}}.$$

It is well known that for kernel regression with only continuous variables, the estimator is consistent as long as $h \to 0$ and $nh^p \to \infty$ as $n \to \infty$. When all the variables are discrete, we need $\lambda \to 0$. Thus, $\widehat{Q}_0(\boldsymbol{X}, 1)$ and $\widehat{Q}_0(\boldsymbol{X}, -1)$ are consistent estimators for $Q_0(\boldsymbol{X}, 1)$ and $Q_0(\boldsymbol{X}, -1)$ if two sets of conditions are satisfied.

Recall that $\hat{\boldsymbol{\beta}}^{\text{opt2}}$ is the solution to

$$\begin{aligned} (*) - \mathbb{P}_n \Bigg(& \frac{I\{A = d(\boldsymbol{X})\} - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \\ & \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \Big[\widehat{Q}_0\{\boldsymbol{X}, d(\boldsymbol{X})\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \Bigg) = 0. \end{aligned} \tag{S.1}$$

where

$$(*) = \mathbb{P}_n \left(\frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \right).$$

When the propensity score is correct, the left-hand side of (S.1) converges in probability to

$$
\begin{aligned}
& E\left(\frac{I\{A=d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Y-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right)\\
= \;& E\left(\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right).
\end{aligned}
$$

Hence, $\hat{\boldsymbol{\beta}}^{\text{opt2}} \xrightarrow{p} \boldsymbol{\beta}^{\text{opt}}$. On the other hand, based on the fact that $\widehat{Q}_0\{\boldsymbol{X},d(\boldsymbol{X})\}$ is consistent for $Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}$, when the outcome regression model is correct but propensity score may not be, the left-hand side of (S.1) converges in probability to

$$
\begin{aligned}
& E\Bigg(\frac{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}\left[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}\right]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\\
& -\frac{\left[\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}\right]\left[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}\right]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Bigg)\\
=& E\left(\frac{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right)
\end{aligned}
$$

which equals 0 when $\boldsymbol{\beta}=\boldsymbol{\beta}_0$. Thus, $\hat{\boldsymbol{\beta}}^{\text{opt2}} \xrightarrow{p} \boldsymbol{\beta}_0$. These two facts ensure that $\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\text{opt2}})$ is doubly robust and achieves the smallest asymptotic variance among the class of DR estimators. That is to say, $\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\text{opt2}})$ is improved DR.

A.5 Influence functions of (1) under correctly specified propensity score model

The detailed derivations are the following. When the propensity score model is correctly specified,

$$
\begin{aligned}
&\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}})-V(d)\right\}\\
=&\ \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\frac{Y_iI\{A_i=d(\boldsymbol{X}_i)\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\hat{\boldsymbol{\gamma}}\}}-\frac{I\{A_i=d(\boldsymbol{X}_i)\}-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\hat{\boldsymbol{\gamma}}\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\hat{\boldsymbol{\gamma}}\}}Q\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\hat{\boldsymbol{\beta}}\}\right]-V(d)\right)\\
=&\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{Y_iI\{A_i=d(\boldsymbol{X}_i)\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\hat{\boldsymbol{\gamma}}\}}-\frac{I\{A_i=d(\boldsymbol{X}_i)\}-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\hat{\boldsymbol{\gamma}}\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\hat{\boldsymbol{\gamma}}\}}Q\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}^*\}-V(d)\right]+o_p(1)
\end{aligned}
$$

Now we expand $\hat{\boldsymbol{\gamma}}$ about $\boldsymbol{\gamma}_0$ to obtain

$$
\begin{aligned}
\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}})-V(d)\right\}=&\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0,\boldsymbol{\beta}^*)\\
&+\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\partial\tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_n^*,\boldsymbol{\beta}^*)}{\partial\boldsymbol{\gamma}^\top}\right\}\sqrt{n}(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0)+o_p(1),
\end{aligned}
\tag{S.2}
$$

where $\boldsymbol{\gamma}_n^*$ is some intermediate value between $\hat{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}_0$. Since under regularity conditions, $\boldsymbol{\gamma}_n^*$ converges in probability to $\boldsymbol{\gamma}_0$, we obtain

$$
\frac{1}{n}\sum_{i=1}^{n}\frac{\partial\tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_n^*,\boldsymbol{\beta}^*)}{\partial\boldsymbol{\gamma}^\top}\xrightarrow{p}E\left\{\frac{\partial\tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0,\boldsymbol{\beta}^*)}{\partial\boldsymbol{\gamma}^\top}\right\}. \tag{S.3}
$$

Using standard results from finite-dimensional parametric models, we know that

$$
\sqrt{n}(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[E\left\{S_{\boldsymbol{\gamma}}(A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0)S_{\boldsymbol{\gamma}}^\top(A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0)\right\}\right]^{-1}S_{\boldsymbol{\gamma}}(A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0)+o_p(1). \tag{S.4}
$$

Combing equations (S.4) and (S.3), we deduce that (S.2) is

$$\begin{aligned}
&\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}})-V(d)\right\}\\
=\;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Bigg(\tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0,\boldsymbol{\beta}^*)\\
+\;& E\Big\{\frac{\partial\tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0,\boldsymbol{\beta}^*)}{\partial\boldsymbol{\gamma}^\top}\Big\}\Big[E\left\{S_{\boldsymbol{\gamma}}(A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0)S_{\boldsymbol{\gamma}}^\top(A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0)\right\}\Big]^{-1}S_{\boldsymbol{\gamma}}(A_i,\boldsymbol{X}_i,\boldsymbol{\gamma}_0)\Bigg)+o_p(1).
\end{aligned}$$

Thus, the influence function is

$$\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}^*)+E\Big\{\frac{\partial\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}^*)}{\partial\boldsymbol{\gamma}^\top}\Big\}\Big[E\left\{S_{\boldsymbol{\gamma}}(A,\boldsymbol{X},\boldsymbol{\gamma}_0)S_{\boldsymbol{\gamma}}^\top(A,\boldsymbol{X},\boldsymbol{\gamma}_0)\right\}\Big]^{-1}S_{\boldsymbol{\gamma}}(A,\boldsymbol{X},\boldsymbol{\gamma}_0).$$

A.6 Variance of (7) and its minimizer

We know that $\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}=I\{d(\boldsymbol{X})=1\}\pi\{1,\boldsymbol{X};\boldsymbol{\gamma}\}+I\{d(\boldsymbol{X})=-1\}\pi\{-1,\boldsymbol{X};\boldsymbol{\gamma}\}$. Hence,

$$\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}=\frac{\partial\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{\partial\boldsymbol{\gamma}}=I\{d(\boldsymbol{X})=1\}\cdot\frac{\pi\{1,\boldsymbol{X};\boldsymbol{\gamma}\}}{\partial\boldsymbol{\gamma}}-I\{d(\boldsymbol{X})=-1\}\cdot\frac{\pi\{1,\boldsymbol{X};\boldsymbol{\gamma}\}}{\partial\boldsymbol{\gamma}}.$$

To suppress notations, we define $R\triangleq I\{A=d(\boldsymbol{X})\}$. Observe that

$$\begin{aligned}
S_{\boldsymbol{\gamma}}(A,\boldsymbol{X},\boldsymbol{\gamma}) &= I(A=1)\frac{\pi_{\boldsymbol{\gamma}}(1,\boldsymbol{X};\boldsymbol{\gamma})}{\pi(1,\boldsymbol{X};\boldsymbol{\gamma})}-I(A=-1)\frac{\pi_{\boldsymbol{\gamma}}(1,\boldsymbol{X};\boldsymbol{\gamma})}{1-\pi(1,\boldsymbol{X};\boldsymbol{\gamma})}\\
&= I\{A=d(\boldsymbol{X})\}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}-I\{A\neq d(\boldsymbol{X})\}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\\
&= \frac{R-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}\left[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}\right]}\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}.
\end{aligned}$$

The influence function (7) can be rewritten as

$$\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}-\frac{R-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\}+\Gamma_0(\boldsymbol{\beta}^*)\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]-V(d).$$

We use the formula $\mathrm{Var}(\cdot) = E\{\mathrm{Var}(\cdot \mid X)\} + \mathrm{Var}\{E(\cdot \mid X)\}$. It is obvious that

$$\begin{aligned} & E\left(\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - \frac{R-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]\middle|\boldsymbol{X}\right) \\ = \; & E\left[\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\middle|\boldsymbol{X}\right] = Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}. \end{aligned}$$

Hence, the second term is $\mathrm{Var}\left[Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right]$, not relevant to $\boldsymbol{\beta}^*$.

$$\begin{aligned} & \mathrm{Var}\left(\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - \frac{R-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]\middle|\boldsymbol{X}\right) \\ = \; & E\Bigg[\Bigg(\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\} \\ & -\frac{R-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]\Bigg)^2 \mid \boldsymbol{X}\Bigg] \\ = \; & B_1 + B_2 + B_3, \end{aligned}$$

where

$$B_1 = E\left(\left[\frac{R-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]^2\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]^2\middle|\boldsymbol{X}\right),$$

$$\begin{aligned} B_2 = -2E\Bigg(&\frac{R-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot\left[\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right] \\ & \cdot\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]\Bigg|\boldsymbol{X}\Bigg), \end{aligned}$$

$$B_3 = E\left(\left[\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}\right]^2\middle|\boldsymbol{X}\right).$$

After some calculations, it can be shown that

$$B_1 = \frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0}\cdot\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right]^2,$$

$$B_2 = -2 \cdot \frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} \left[Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \right].$$

Therefore, the first term $E\{\text{Var}(\cdot \mid X)\}$ equals

$$E\left(\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \left[Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \right]^2 \right)$$
$$-2E\left(\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} \left[Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^*\} + \Gamma_0(\boldsymbol{\beta}^*)\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \right] \right)$$
$$+E\left(\left[\frac{RY}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} - Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} \right]^2 \right).$$

By taking the derivative with respect to $\boldsymbol{\beta}^*$ and set it equal to 0, we know that the variance is minimized by the solution to the following equation.

$$E\left(\frac{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} + \Phi_0(\boldsymbol{\beta})\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \right] \cdot \left[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - \Gamma_0(\boldsymbol{\beta})\Sigma^{-1}_{\boldsymbol{\gamma\gamma},0} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}_0\}}{1-\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}} \right] \right) = 0. \tag{S.5}$$

In a slight abuse of notation, we use $\boldsymbol{\beta}^{\text{opt}}$ to denote this minimizer.

A.7 Improved DR property of $\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{\mathbf{opt3}})$

When the model for propensity score is correctly specified, by observing that $\hat{\boldsymbol{\gamma}} \xrightarrow{p} \boldsymbol{\gamma}_0$ and $\widehat{\Sigma}_{\boldsymbol{\gamma\gamma}}, \widehat{\Gamma}(\boldsymbol{\beta}), \widehat{\Phi}(\boldsymbol{\beta})$ converge to $\Sigma_{\boldsymbol{\gamma\gamma},0}, \Gamma_0(\boldsymbol{\beta}), \Phi_0(\boldsymbol{\beta})$, respectively, it is straightforward to show that the left-hand side of (9) converges in probability to the left-hand side of (S.5), thus $\hat{\boldsymbol{\beta}}^{\text{opt3}} \xrightarrow{p} \boldsymbol{\beta}^{\text{opt}}$. On the other hand, when the outcome model is correct but the propensity score is not, $\hat{\boldsymbol{\gamma}} \xrightarrow{p} \boldsymbol{\gamma}^*$, the left-hand side of (9) converges to

$$E\left(\frac{\pi_0\{d(\boldsymbol{X}),\boldsymbol{X}\}\left[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}\right]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}} \left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} + \Phi_*(\boldsymbol{\beta})\Sigma^{-1}_{\boldsymbol{\gamma\gamma},*} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}} \right] \cdot \left[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - \Gamma_*(\boldsymbol{\beta})\Sigma^{-1}_{\boldsymbol{\gamma\gamma},*} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}} \right] \right), \tag{S.6}$$

where $\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},*} = E\left\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}, \boldsymbol{\gamma}^*)S_{\boldsymbol{\gamma}}^\top(A, \boldsymbol{X}, \boldsymbol{\gamma}^*)\right\}$, $\Gamma_*(\boldsymbol{\beta}) = -E\left\{\partial\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}^*, \boldsymbol{\beta})/\partial\boldsymbol{\gamma}^\top\right\}$, $\Phi_*(\boldsymbol{\beta}) = -E\left\{\partial^2\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}^*, \boldsymbol{\beta})/\partial\boldsymbol{\gamma}^\top\partial\boldsymbol{\beta}\right\}$. By noting that $\Gamma_*(\boldsymbol{\beta}_0) = 0$, (S.6) equals 0 when $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, thus, $\hat{\boldsymbol{\beta}}^{\text{opt3}} \xrightarrow{p} \boldsymbol{\beta}_0$. Hence, $\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{\text{opt3}})$ is improved DR.

Appendix B: Asymptotic normality of semiparametric M-estimators

Chen et al. (2003) established the asymptotic theory of semiparametric M-estimators when there is one infinite dimensional parameter. In this Section, we extend their results to the case where there are two infinite dimensional parameters.

Suppose that there exists a vector-valued function $\boldsymbol{m}$ such that $E\left\{\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0(\cdot, \boldsymbol{\theta}), h_0(\cdot, \boldsymbol{\theta}))\right\} = \boldsymbol{0}$ at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. We denote $\boldsymbol{\theta}_0 \in \Theta$ and $g_0, h_0 \in \mathcal{H}$ as the true unknown finite and infinite dimensional parameters. Assume that $g_0(\cdot, \boldsymbol{\theta})$ and $h_0(\cdot, \boldsymbol{\theta})$ are functions of $\boldsymbol{U}$, possibly indexed by $\boldsymbol{\theta}$. We usually suppress the arguments of the functions g_0, h_0 for notational convenience, i.e., $(\boldsymbol{\theta}, g, h) \equiv (\boldsymbol{\theta}, g(\cdot, \boldsymbol{\theta}), h(\cdot, \boldsymbol{\theta}))$, $(\boldsymbol{\theta}, g_0, h_0) \equiv (\boldsymbol{\theta}, g_0(\cdot, \boldsymbol{\theta}), h_0(\cdot, \boldsymbol{\theta}))$. We assume that $\mathcal{H}$ is a vector space of functions endowed with a pseudo-metric $||\cdot||_{\mathcal{H}}$. For example, when $\mathcal{H}$ is a class of continuous functions, we can take $||g||_{\mathcal{H}} = \sup_{\boldsymbol{\theta}} ||g(\cdot, \boldsymbol{\theta})||_\infty = \sup_{\boldsymbol{\theta}} \sup_{\boldsymbol{U}} |g(\boldsymbol{U}, \boldsymbol{\theta})|$. Furthermore, let us define $\boldsymbol{M}(\boldsymbol{\theta}, g, h) = E\left\{\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta}, g, h)\right\}$ and $\boldsymbol{M}_n(\boldsymbol{\theta}, g, h) = n^{-1}\sum_{i=1}^n \boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta}, g, h)$.

Let us denote $\Theta_\delta \equiv \{\boldsymbol{\theta} \in \Theta : ||\boldsymbol{\theta} - \boldsymbol{\theta}_0|| \le \delta\}$, $\mathcal{G}_\delta \equiv \{g \in \mathcal{H} : ||g - g_0||_{\mathcal{H}} \le \delta\}$ and $\mathcal{H}_\delta \equiv \{h \in \mathcal{H} : ||h - h_0||_{\mathcal{H}} \le \delta\}$ for some $\delta > 0$. For any $(\boldsymbol{\theta}, g, h) \in \Theta_\delta \times \mathcal{G}_\delta \times \mathcal{H}_\delta$, we denote the ordinary derivative of $\boldsymbol{M}(\boldsymbol{\theta}, g, h)$ with respect to $\boldsymbol{\theta}$ as $\boldsymbol{\Gamma}_1(\boldsymbol{\theta}, g, h)$, which satisfies

$$\begin{aligned}\boldsymbol{\Gamma}_1(\boldsymbol{\theta}, g, h)(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}) &= \lim_{\tau\to 0}\Big\{\boldsymbol{M}(\boldsymbol{\theta} + \tau(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}), g(\cdot, \boldsymbol{\theta} + \tau(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta})), h(\cdot, \boldsymbol{\theta} + \tau(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}))) \\ &\quad - \boldsymbol{M}(\boldsymbol{\theta}, g(\cdot, \boldsymbol{\theta}), h(\cdot, \boldsymbol{\theta}))\Big\}/\tau\end{aligned}$$

for all $\bar{\boldsymbol{\theta}} \in \Theta$. In the following discussion, we assume that the dimension of $\boldsymbol{M}(\boldsymbol{\theta}, g, h)$ is the same as the dimension of $\boldsymbol{\theta}$, hence $\boldsymbol{\Gamma}_1(\boldsymbol{\theta}, g, h)$ is a square matrix.

As in Chen et al. (2003), we say that $\boldsymbol{M}(\boldsymbol{\theta}, g, h)$ is pathwise differentiable at $g \in \mathcal{G}_\delta$ in the direction $[\bar{g} - g]$ if $\{g + \tau(\bar{g} - g) : \tau \in [0, 1]\} \subset \mathcal{H}$ and

$$\lim_{\tau \to 0} \{\boldsymbol{M}(\boldsymbol{\theta}, g(\cdot, \boldsymbol{\theta}) + \tau(\bar{g}(\cdot, \boldsymbol{\theta}) - g(\cdot, \boldsymbol{\theta})), h(\cdot, \boldsymbol{\theta})) - \boldsymbol{M}(\boldsymbol{\theta}, g(\cdot, \boldsymbol{\theta}), h(\cdot, \boldsymbol{\theta}))\} / \tau$$

exists; we denote the limit $\boldsymbol{\Gamma}_2(\boldsymbol{\theta}, g, h)[\bar{g} - g]$. Similarly, if

$$\lim_{\tau \to 0} \left\{\boldsymbol{M}(\boldsymbol{\theta}, g(\cdot, \boldsymbol{\theta}), h(\cdot, \boldsymbol{\theta}) + \tau(\bar{h}(\cdot, \boldsymbol{\theta}) - h(\cdot, \boldsymbol{\theta}))) - \boldsymbol{M}(\boldsymbol{\theta}, g(\cdot, \boldsymbol{\theta}), h(\cdot, \boldsymbol{\theta}))\right\} / \tau$$

exists; we denote the limit $\boldsymbol{\Gamma}_3(\boldsymbol{\theta}, g, h)[\bar{h} - h]$. These functional derivatives capture the effect of the nonparametric estimation of g_0, h_0 on the variability of $\hat{\boldsymbol{\theta}}$.

Theorem S1. *Suppose that* $\boldsymbol{\theta}_0$ *satisfies* $\boldsymbol{M}(\boldsymbol{\theta}_0, g_0, h_0) = \mathbf{0}$*, that* $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = o_p(1)$*, and that*

(1.1) $||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| = \inf_{\boldsymbol{\theta} \in \Theta_\delta} ||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h})|| + o_p(1/\sqrt{n})$.

(1.2) *(i) The ordinary derivative* $\boldsymbol{\Gamma}_1(\boldsymbol{\theta}, g_0, h_0)$ *exists for* $\boldsymbol{\theta} \in \Theta_\delta$*, and is continuous at* $\boldsymbol{\theta} = \boldsymbol{\theta}_0$*; (ii) the matrix* $\boldsymbol{\Gamma}_1 \equiv \boldsymbol{\Gamma}_1(\boldsymbol{\theta}_0, g_0, h_0)$ *is of full rank.*

(1.3) *For all* $\boldsymbol{\theta} \in \Theta_\delta$*, the pathwise derivatives* $\boldsymbol{\Gamma}_2(\boldsymbol{\theta}, g_0, h_0)[g - g_0]$ *and* $\boldsymbol{\Gamma}_3(\boldsymbol{\theta}, g_0, h_0)[h - h_0]$ *exist in all directions* $[g - g_0], [h - h_0] \in \mathcal{H}$*; and for all* $(\boldsymbol{\theta}, g, h) \in \Theta_{\delta_n} \times \mathcal{G}_{\delta_n} \times \mathcal{H}_{\delta_n}$ *with a positive sequence* $\delta_n = o(1)$*: (i)* $||\boldsymbol{M}(\boldsymbol{\theta}, g, h) - \boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0) - \boldsymbol{\Gamma}_2(\boldsymbol{\theta}, g_0, h_0)[g - g_0] - \boldsymbol{\Gamma}_3(\boldsymbol{\theta}, g_0, h_0)[h - h_0]|| \leq c\,\{||g - g_0||_{\mathcal{H}}^2 + ||h - h_0||_{\mathcal{H}}^2\}$ *for a constant* $c \geq 0$*; (ii)* $||\boldsymbol{\Gamma}_2(\boldsymbol{\theta}, g_0, h_0)[g - g_0] - \boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[g - g_0]|| \leq o(1)\delta_n$*; (iii)* $||\boldsymbol{\Gamma}_3(\boldsymbol{\theta}, g_0, h_0)[h - h_0] - \boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[h - h_0]|| \leq o(1)\delta_n$.

(1.4) $\hat{g}, \hat{h} \in \mathcal{H}$ *with probability tending to one; and* $||\hat{g} - g_0||_{\mathcal{H}} = o_p(n^{-1/4})$*,* $||\hat{h} - h_0||_{\mathcal{H}} = o_p(n^{-1/4})$.

(1.5) *For all sequences of positive numbers* $\{\delta_n\}$ *with* $\delta_n = o(1)$*,*

$$\sup_{(\boldsymbol{\theta}, g, h) \in \Theta_{\delta_n} \times \mathcal{G}_{\delta_n} \times \mathcal{H}_{\delta_n}} \frac{\sqrt{n}||\boldsymbol{M}_n(\boldsymbol{\theta}, g, h) - \boldsymbol{M}(\boldsymbol{\theta}, g, h) - \boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0)||}{1 + \sqrt{n}\,\{||\boldsymbol{M}_n(\boldsymbol{\theta}, g, h)|| + ||\boldsymbol{M}(\boldsymbol{\theta}, g, h)||\}} = o_p(1).$$

(1.6) For some finite matrix $\boldsymbol{V}_1$,

$$\sqrt{n}\left\{\boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0) + \boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[\hat{g} - g_0] + \boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[\hat{h} - h_0]\right\} \xrightarrow{D} N(\boldsymbol{0}, \boldsymbol{V}_1).$$

Then, $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} N(\boldsymbol{0}, \boldsymbol{\Omega})$, *where* $\boldsymbol{\Omega} \equiv \boldsymbol{\Gamma}_1^{-1}\boldsymbol{V}_1\left\{\boldsymbol{\Gamma}_1^{-1}\right\}^{\top}$.

Proof. The proof is very similar to that of Theorem 2 in Chen et al. (2003) and Theorem 3.3 in Pakes and Pollard (1989).

We first establish $\sqrt{n}$-consistency of $\hat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}_0$. We choose a positive sequence $\delta_n = o(1)$ such that $\Pr(||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0|| \geq \delta_n, ||\hat{g} - g_0||_{\mathcal{H}} \geq \delta_n, ||\hat{h} - h_0||_{\mathcal{H}} \geq \delta_n) \to 0$. By condition (1.2), there exists a constant $C > 0$ such that $||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0||C$ is bounded by $||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)||$ with probability tending to one; this in turn is bounded above by

$$\begin{aligned}
&||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| + ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) + \boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0)|| \\
&+||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| + O_p(n^{-1/2}) \qquad \text{(S.7)}
\end{aligned}$$

by the triangle inequality and condition (1.6). By condition (1.3), (1.4) and (1.6),

$$\begin{aligned}
& ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| \\
\leq\ & ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) - \boldsymbol{\Gamma}_2(\hat{\boldsymbol{\theta}}, g_0, h_0)[\hat{g} - g_0] - \boldsymbol{\Gamma}_3(\hat{\boldsymbol{\theta}}, g_0, h_0)[\hat{h} - h_0]|| \\
& +||\boldsymbol{\Gamma}_2(\hat{\boldsymbol{\theta}}, g_0, h_0)[\hat{g} - g_0] - \boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[\hat{g} - g_0]|| + ||\boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[\hat{g} - g_0]|| \\
& +||\boldsymbol{\Gamma}_3(\hat{\boldsymbol{\theta}}, g_0, h_0)[\hat{h} - h_0] - \boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[\hat{h} - h_0]|| + ||\boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[\hat{h} - h_0]|| \\
\leq\ & c\left\{||\hat{g} - g_0||_{\mathcal{H}}^2 + ||\hat{h} - h_0||_{\mathcal{H}}^2\right\} + ||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0|| \times o_p(1) + O_p(n^{-1/2}) \\
=\ & o_p(n^{-1/2}) + ||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0|| \times o_p(1) + O_p(n^{-1/2}) \\
\leq\ & ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| \times o_p(1) + O_p(n^{-1/2}). \qquad \text{(S.8)}
\end{aligned}$$

Using the above fact and by condition (1.5),

$$\begin{aligned}
&||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) + \boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0)|| \\
\leq\ & o_p(1) \times \left\{n^{-1/2} + ||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| + ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})||\right\} \\
\leq\ & o_p(1) \times \left\{n^{-1/2} + ||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| + ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| + ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| \times o_p(1) + O_p(n^{-1/2})\right\} \\
=\ & o_p(n^{-1/2}) + o_p(1) \times \left\{||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| + ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| \times (1 + o_p(1))\right\}. \qquad \text{(S.9)}
\end{aligned}$$

Combine (S.7), (S.8), (S.9) and condition (1.1), it implies that

$$\begin{aligned}
||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| \times \{1 - o_p(1)\} &\leq o_p(n^{-1/2}) + ||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h})|| \times \{1 + o_p(1)\} + O_p(n^{-1/2}) \\
&\leq \inf_{\boldsymbol{\theta} \in \Theta_\delta} ||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h})|| \times \{1 + o_p(1)\} + O_p(n^{-1/2}). \quad \text{(S.10)}
\end{aligned}$$

By assuming that $\boldsymbol{M}(\boldsymbol{\theta}, g, h)$ is continuous in g and h at $g = g_0$, $h = h_0$, we have $||\boldsymbol{M}(\boldsymbol{\theta}, \hat{g}, \hat{h}) - \boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0)|| = o_p(1)$. Again under conditions (1.3)-(1.6), we have that

$$\begin{aligned}
||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h})|| &\leq ||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h}) - \boldsymbol{M}(\boldsymbol{\theta}, \hat{g}, \hat{h}) - \boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0)|| + ||\boldsymbol{M}(\boldsymbol{\theta}, \hat{g}, \hat{h}) - \boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0)|| \\
&\quad + ||\boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0)|| + ||\boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0)|| \\
&\leq o_p(n^{-1/2}) + o_p(1) \times \left\{||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h})|| + ||\boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0)|| \times (1 + o_p(1))\right\} \\
&\quad + ||\boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0)|| + O_p(n^{-1/2}).
\end{aligned}$$

With $\boldsymbol{M}(\boldsymbol{\theta}_0, g_0, h_0) = \mathbf{0}$, we have $||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h})|| \times \{1 - o_p(1)\} \leq o_p(1) \times ||\boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0) - \boldsymbol{M}(\boldsymbol{\theta}_0, g_0, h_0)|| + O_p(n^{-1/2})$ where all the $o_p(1), O_p(n^{-1/2})$ holds uniformly with respect to $\boldsymbol{\theta} \in \Theta_\delta$.

$$\begin{aligned}
\inf_{\boldsymbol{\theta} \in \Theta_\delta} ||\boldsymbol{M}_n(\boldsymbol{\theta}, \hat{g}, \hat{h})|| &\leq o_p(1) \times \inf_{\boldsymbol{\theta} \in \Theta_\delta} ||\boldsymbol{M}(\boldsymbol{\theta}, g_0, h_0) - \boldsymbol{M}(\boldsymbol{\theta}_0, g_0, h_0)|| + O_p(n^{-1/2}) \\
&= O_p(n^{-1/2}).
\end{aligned}$$

This and (S.10) imply that $||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| \leq O_p(n^{-1/2})$. Hence, $||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0|| C \leq ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0)|| \leq O_p(n^{-1/2})$.

Next we establish the asymptotic normality property. We only sketch the main steps here. Define the linearization $\mathcal{L}_n(\boldsymbol{\theta}) = \boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0) + \boldsymbol{\Gamma}_1(\boldsymbol{\theta} - \boldsymbol{\theta}_0) + \boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[\hat{g} - g_0] + \boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[\hat{h} - h_0]$. By conditions (1.2)-(1.5) and the root-n rate results above,

$$\begin{aligned}
& ||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \mathcal{L}_n(\hat{\boldsymbol{\theta}})|| \\
= \ & ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) + \boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) + \boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) - \mathcal{L}_n(\hat{\boldsymbol{\theta}})|| \\
\leq \ & ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) - \boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[\hat{g} - g_0] - \boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[\hat{h} - h_0]|| \\
& + ||\boldsymbol{M}(\hat{\boldsymbol{\theta}}, g_0, h_0) - \boldsymbol{\Gamma}_1(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)|| + ||\boldsymbol{M}_n(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}(\hat{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0)|| \\
= \ & o_p(n^{-1/2}).
\end{aligned}$$

Let us define $\bar{\boldsymbol{\theta}}$ as the minimizer of $||\mathcal{L}_n(\cdot)||$, it is straightforward to show that $\bar{\boldsymbol{\theta}}$ satisfies

$$\sqrt{n}(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = -\boldsymbol{\Gamma}_1^{-1}\sqrt{n}\left\{\boldsymbol{M}_n(\boldsymbol{\theta}_0, g_0, h_0) + \boldsymbol{\Gamma}_2(\boldsymbol{\theta}_0, g_0, h_0)[\hat{g} - g_0] + \boldsymbol{\Gamma}_3(\boldsymbol{\theta}_0, g_0, h_0)[\hat{h} - h_0]\right\}.$$

Similarly, $||\boldsymbol{M}_n(\bar{\boldsymbol{\theta}}, \hat{g}, \hat{h}) - \mathcal{L}_n(\bar{\boldsymbol{\theta}})|| = o_p(n^{-1/2})$. A little more work shows that $\sqrt{n}(\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}) = o_p(1)$. Hence, $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{\Omega})$, where $\boldsymbol{\Omega} \equiv \boldsymbol{\Gamma}_1^{-1}\boldsymbol{V}_1\left\{\boldsymbol{\Gamma}_1^{-1}\right\}^{\top}$. □

Appendix C: Details and proof of Theorem 1

Let us write $\boldsymbol{U} = (Y, A, \boldsymbol{X})$ and let $\boldsymbol{\theta}$ be the collection of unknown parameters involved in obtaining the estimators for $V(d)$. For $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\mathrm{LS}})$ and $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\mathrm{opt1}})$, the estimator for $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}$, can be obtained by solving a set of M-estimating equations $\sum_{i=1}^n \boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta}) = \mathbf{0}$ (Stefanski and Boos, 2002), where the last element of $\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta})$ corresponds to the estimating equation for $V(d)$. We denote $\boldsymbol{\theta}_0$ as the value that satisfies $E\{\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta}_0)\} = \mathbf{0}$. Following standard M-estimation theory, $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{B}(\boldsymbol{\theta}_0)^{-1}\boldsymbol{C}(\boldsymbol{\theta}_0)\{\boldsymbol{B}(\boldsymbol{\theta}_0)^{-1}\}^{\top})$, where $\boldsymbol{B}(\boldsymbol{\theta}) = E\{\partial \boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta})/\partial \boldsymbol{\theta}^{\top}\}$ and $\boldsymbol{C}(\boldsymbol{\theta}) = E\{\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta})\boldsymbol{m}^{\top}(\boldsymbol{U}_i, \boldsymbol{\theta})\}$. Therefore, the asymptotic variance of $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\mathrm{LS}})$ and $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\mathrm{opt1}})$ is in the last, rightmost diagonal entry of the corresponding matrix $\boldsymbol{B}(\boldsymbol{\theta}_0)^{-1}\boldsymbol{C}(\boldsymbol{\theta}_0)\{\boldsymbol{B}(\boldsymbol{\theta}_0)^{-1}\}^\top$. For $\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\text{opt2}})$, the estimator for $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}$, can be obtained by solving $\sum_{i=1}^n \boldsymbol{m}(\boldsymbol{U}_i,\boldsymbol{\theta},\hat{g},\hat{h}) = \boldsymbol{0}$ where $\hat{g},\hat{h}$ are nonparametric estimators of some infinite dimensional parameters. In this case, the infinite dimensional parameters are the conditional expectations of Y given $(\boldsymbol{X},A)$. We utilize semiparametric M-estimation theory to calculate the asymptotic variance of $\hat{\boldsymbol{\theta}}$.

In the following discussions, we assume that $\boldsymbol{\gamma}$ is a q-dimensional vector, i.e., $\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_q)$ and $\boldsymbol{\beta}$ is a s-dimensional vector, i.e., $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_s)$. We use $\boldsymbol{0}_{a\times b}$ to denote a zero matrix with a rows and b columns. Sometimes we omit the dimension when there is no confusion. In addition, let us define $Q_{\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta}) = \partial^2 Q(\boldsymbol{X},A;\boldsymbol{\beta})/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top$, and let $g_0(\boldsymbol{X}) \triangleq E(Y \mid \boldsymbol{X}, A=1)$, $h_0(\boldsymbol{X}) \triangleq E(Y \mid \boldsymbol{X}, A=-1)$ denote the true infinite dimensional parameters.

Theorem 1. *(Asymptotic normality when propensity score model is fully specified). The unknown parameters are $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, V(d))^\top$. When either the propensity score or the outcome model is correct,*

$$\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\beta}}^{LS}) - V(d)\right\} \xrightarrow{D} N(0, U_1(\boldsymbol{\theta}_0^{LS})),$$

where $U_1(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{LS}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{11}^{LS}(\boldsymbol{\theta})\{\boldsymbol{B}_1^{LS}(\boldsymbol{\theta})\}^{-1}\boldsymbol{B}_2^\top(\boldsymbol{\theta}) - 2\boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{LS}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{12}^{LS}(\boldsymbol{\theta}) + \boldsymbol{C}_{22}(\boldsymbol{\theta})$. *Here,*

$$\begin{aligned}
\boldsymbol{B}_1^{LS}(\boldsymbol{\theta}) &= E\left[Q_{\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})\{Y - Q(\boldsymbol{X},A;\boldsymbol{\beta})\} - Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})Q_{\boldsymbol{\beta}}^\top(\boldsymbol{X},A;\boldsymbol{\beta})\right],\\
\boldsymbol{B}_2(\boldsymbol{\theta}) &= E\left[\left(1 - \frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\right)Q_{\boldsymbol{\beta}}^\top\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right],\\
\boldsymbol{C}_{11}^{LS}(\boldsymbol{\theta}) &= E\left[\{Y - Q(\boldsymbol{X},A;\boldsymbol{\beta})\}^2 Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})Q_{\boldsymbol{\beta}}^\top(\boldsymbol{X},A;\boldsymbol{\beta})\right],\\
\boldsymbol{C}_{12}^{LS}(\boldsymbol{\theta}) &= E\Big(\frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\{Y - Q(\boldsymbol{X},A;\boldsymbol{\beta})\}^2 Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})\\
&\quad + \{Y - Q(\boldsymbol{X},A;\boldsymbol{\beta})\}\left[Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\} - V(d)\right]Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})\Big),\\
\boldsymbol{C}_{22}(\boldsymbol{\theta}) &= E\left(\frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Y - Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right] + Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\} - V(d)\right)^2.
\end{aligned}$$

The true values are $\boldsymbol{\theta}_0^{LS} = (\boldsymbol{\beta}_{LS}^{*\top}, V(d))^\top$ *where* $\boldsymbol{\beta}_{LS}^*$ *satisfies* $E\left[Q_{\boldsymbol{\beta}}(\boldsymbol{X}, A; \boldsymbol{\beta}_{LS}^*)\left\{Y - Q(\boldsymbol{X}, A; \boldsymbol{\beta}_{LS}^*)\right\}\right] = \mathbf{0}$. *For the improved DR estimator with IPW estimating equation,*

$$\sqrt{n}\left\{\widehat{V}(d; \hat{\boldsymbol{\beta}}^{opt1}) - V(d)\right\} \xrightarrow{D} N(0, U_2(\boldsymbol{\theta}_0^{opt1})),$$

where $U_2(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{opt1}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{11}^{opt1}(\boldsymbol{\theta})\{\boldsymbol{B}_1^{opt1}(\boldsymbol{\theta})\}^{-1}\boldsymbol{B}_2^\top(\boldsymbol{\theta}) - 2\boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{opt1}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{12}^{opt1}(\boldsymbol{\theta}) + \boldsymbol{C}_{22}(\boldsymbol{\theta})$. *Here,*

$$\begin{aligned}
\boldsymbol{B}_1^{opt1}(\boldsymbol{\theta}) &= E\Bigg\{\frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \\
&\quad \cdot \Big(\Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Big] Q_{\boldsymbol{\beta}\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} Q_{\boldsymbol{\beta}}^\top\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Big)\Bigg\}, \\
\boldsymbol{C}_{11}^{opt1}(\boldsymbol{\theta}) &= E\Bigg(\frac{I\{A = d(\boldsymbol{X})\}\left[1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}\right]^2}{\pi^4\{d(\boldsymbol{X}), \boldsymbol{X}\}}\Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Big]^2 \\
&\quad \cdot Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} Q_{\boldsymbol{\beta}}^\top\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Bigg), \\
\boldsymbol{C}_{12}^{opt1}(\boldsymbol{\theta}) &= E\Bigg(\frac{I\{A = d(\boldsymbol{X})\}\left[1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}\right]}{\pi^2\{d(\boldsymbol{X}), \boldsymbol{X}\}}\Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \\
&\quad \cdot \left[\frac{Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} + Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - V(d)\right]\Bigg).
\end{aligned}$$

The true values are $\boldsymbol{\theta}_0^{opt1} = (\boldsymbol{\beta}_{opt1}^{*\top}, V(d))^\top$ *where* $\boldsymbol{\beta}_{opt1}^*$ *satisfies*

$$E\left(\frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}\Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_{opt1}^*\}\Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_{opt1}^*\}\right) = \mathbf{0}.$$

For the improved DR estimator with augmented IPW estimating equation,

$$\sqrt{n}\left\{\widehat{V}(d; \hat{\boldsymbol{\beta}}^{opt2}) - V(d)\right\} \xrightarrow{D} N(0, U_3(\boldsymbol{\theta}_0^{opt2})),$$

where $U_3(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{opt2}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{11}^{opt2}(\boldsymbol{\theta})\{\boldsymbol{B}_1^{opt2}(\boldsymbol{\theta})\}^{-1}\boldsymbol{B}_2^\top(\boldsymbol{\theta}) - 2\boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{opt2}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{12}^{opt2}(\boldsymbol{\theta}) + \boldsymbol{C}_{22}(\boldsymbol{\theta})$. *Here,*

$$\boldsymbol{B}_1^{opt2}(\boldsymbol{\theta}) = E\left\{\partial \boldsymbol{m}_1 / \partial \boldsymbol{\beta}^\top\right\}, \quad \boldsymbol{C}_{11}^{opt2}(\boldsymbol{\theta}) = E\left\{(\boldsymbol{m}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2)(\boldsymbol{m}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2)^\top\right\},$$

$$\boldsymbol{C}_{12}^{opt2}(\boldsymbol{\theta}) = E\left\{(\boldsymbol{m}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2)m_2\right\}.$$

Here we suppress the notations by writing $\boldsymbol{m}_1 \equiv \boldsymbol{m}_1(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0)$*,* $m_2 \equiv m_2(\boldsymbol{U}_i, \boldsymbol{\theta})$*,* $\boldsymbol{q}_1 \equiv \boldsymbol{q}_1(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0)$*,* $\boldsymbol{q}_2 \equiv \boldsymbol{q}_2(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0)$ *where*

$$\begin{aligned}
\boldsymbol{m}_1(\boldsymbol{U}, \boldsymbol{\theta}, g_0, h_0) &= \frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \\
&\quad - \Bigg(\frac{I\{A = d(\boldsymbol{X})\} - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \\
&\quad \Big[g_0(\boldsymbol{X}) I\{d(\boldsymbol{X}) = 1\} + h_0(\boldsymbol{X}) I\{d(\boldsymbol{X}) = -1\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} \Bigg), \\
m_2(\boldsymbol{U}, \boldsymbol{\theta}) &= \frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} [Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}] + Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - V(d), \\
\boldsymbol{q}_1(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0) &= -\{Y_i - g_0(\boldsymbol{X}_i)\} \frac{I(A_i = 1) f_{\boldsymbol{X}}(\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{X}_i, 1)} \cdot \frac{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\} - \pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} \cdot \\
&\quad \frac{1 - \pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} I\{d(\boldsymbol{X}_i) = 1\} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \boldsymbol{\beta}\}, \\
\boldsymbol{q}_2(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0) &= -\{Y_i - h_0(\boldsymbol{X}_i)\} \frac{I(A_i = -1) f_{\boldsymbol{X}}(\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{X}_i, -1)} \cdot \frac{\pi_0\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\} - \pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} \cdot \\
&\quad \frac{1 - \pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}} I\{d(\boldsymbol{X}_i) = -1\} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \boldsymbol{\beta}\},
\end{aligned}$$

where $f_{\boldsymbol{X},A}(\cdot,\cdot)$ *and* $f_{\boldsymbol{X}}(\cdot)$ *denote the density of* $(\boldsymbol{X}, A)$ *and* $\boldsymbol{X}$*, respectively. The true values are* $\boldsymbol{\theta}_0^{opt2} = ({\boldsymbol{\beta}_{opt2}^*}^\top, V(d))^\top$ *where* $\boldsymbol{\beta}_{opt2}^*$ *satisfies*

$$\begin{aligned}
&E\Bigg(\frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_{opt2}^*\}\Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_{opt2}^*\} \Bigg) \\
&- E\Bigg(\frac{I\{A = d(\boldsymbol{X})\} - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \\
&\Big[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_{opt2}^*\}\Big] Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}_{opt2}^*\} \Bigg) = \boldsymbol{0}.
\end{aligned}$$

Consistent estimators of U_1, U_2, U_3 can be constructed by replacing the expectation E with the empirical measure $\mathbb{P}_n$ in the above quantities.

Remark 1. *When the propensity score is correctly specified, $\pi(A, \boldsymbol{X}) = \pi_0(A, \boldsymbol{X})$, note that $\boldsymbol{B}_2(\boldsymbol{\theta}) = \mathbf{0}$, hence $U_1(\boldsymbol{\theta}) = U_2(\boldsymbol{\theta}) = U_3(\boldsymbol{\theta}) = \boldsymbol{C}_{22}(\boldsymbol{\theta})$, which is equal to*

$$
\begin{aligned}
&E\left(\frac{I\{A = d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}\left[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\right] + Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - V(d)\right)^2 \\
=&E\left[\frac{1 - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q^2\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\right] + E\left[\frac{Y I\{A = d(\boldsymbol{X})\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} - Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\}\right]^2 \\
&- 2E\left[\frac{1 - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}} Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} \cdot Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\right] + \mathit{Var}\left[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\}\right].
\end{aligned}
$$

Furthermore, note that $\boldsymbol{\beta}^*_{opt1} = \boldsymbol{\beta}^*_{opt2} = \boldsymbol{\beta}^{opt}$ which minimizes the above quantity. This confirms that when the propensity score is correct but outcome model incorrect, $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{opt1})$ and $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{opt2})$ have the same asymptotic variance, which is smaller than that of $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{LS})$. Though, in small sample size, $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{opt2})$ is preferred since it utilizes all the available data and is much more stable. When both models are correct, by noting that $\boldsymbol{\beta}^{opt} = \boldsymbol{\beta}_0 = \boldsymbol{\beta}^*_{LS}$, all three estimators have the same asymptotic variance.

Proof. For the usual DR estimator $\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\mathrm{LS}})$, the estimating equation $\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta})$ is given by

$$
\boldsymbol{m}(\boldsymbol{U}_i, \boldsymbol{\theta}) = \begin{pmatrix} Q_{\boldsymbol{\beta}}(\boldsymbol{X}_i, A_i; \boldsymbol{\beta})\left\{Y_i - Q(\boldsymbol{X}_i, A_i; \boldsymbol{\beta})\right\} \\ \dfrac{I\{A_i = d(\boldsymbol{X}_i)\}}{\pi\{d(\boldsymbol{X}_i), \boldsymbol{X}_i\}}\left[Y_i - Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \boldsymbol{\beta}\}\right] + Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i); \boldsymbol{\beta}\} - V(d) \end{pmatrix}.
$$

When either the propensity score or the outcome model is correct, the true values are $\boldsymbol{\theta}_0^{\mathrm{LS}} = ({\boldsymbol{\beta}^*_{\mathrm{LS}}}^\top, V(d))^\top$ where $\boldsymbol{\beta}^*_{\mathrm{LS}}$ satisfies $E\left[Q_{\boldsymbol{\beta}}(\boldsymbol{X}, A; \boldsymbol{\beta}^*_{\mathrm{LS}})\left\{Y - Q(\boldsymbol{X}, A; \boldsymbol{\beta}^*_{\mathrm{LS}})\right\}\right] = \mathbf{0}$. By M-estimation theory, $\hat{\boldsymbol{\theta}}^{\mathrm{LS}} \triangleq (\{\hat{\boldsymbol{\beta}}^{\mathrm{LS}}\}^\top, \widehat{V}(d; \hat{\boldsymbol{\beta}}^{\mathrm{LS}}))^\top$ is asymptotically normal,

$$
\sqrt{n}(\hat{\boldsymbol{\theta}}^{\mathrm{LS}} - \boldsymbol{\theta}_0^{\mathrm{LS}}) \xrightarrow{D} N(0, \{\boldsymbol{B}^{\mathrm{LS}}(\boldsymbol{\theta}_0^{\mathrm{LS}})\}^{-1}\boldsymbol{C}^{\mathrm{LS}}(\boldsymbol{\theta}_0^{\mathrm{LS}})[\{\boldsymbol{B}^{\mathrm{LS}}(\boldsymbol{\theta}_0^{\mathrm{LS}})\}^{-1}]^\top),
$$

where

$$\boldsymbol{B}^{\mathrm{LS}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{B}_1^{\mathrm{LS}}(\boldsymbol{\theta}) & \mathbf{0}_{s\times 1} \\ \boldsymbol{B}_2(\boldsymbol{\theta}) & -1 \end{pmatrix} \qquad \boldsymbol{C}^{\mathrm{LS}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{C}_{11}^{\mathrm{LS}}(\boldsymbol{\theta}) & \boldsymbol{C}_{12}^{\mathrm{LS}}(\boldsymbol{\theta}) \\ \{\boldsymbol{C}_{12}^{\mathrm{LS}}(\boldsymbol{\theta})\}^\top & \boldsymbol{C}_{22}(\boldsymbol{\theta}) \end{pmatrix},$$

with all the quantities defined in the theorem. After some algebra, we obtain

$$\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\mathrm{LS}}) - V(d)\right\} \xrightarrow{D} N(0, U_1(\boldsymbol{\theta}_0^{\mathrm{LS}})),$$

where $U_1(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{\mathrm{LS}}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{11}^{\mathrm{LS}}(\boldsymbol{\theta})\{\boldsymbol{B}_1^{\mathrm{LS}}(\boldsymbol{\theta})\}^{-1}\boldsymbol{B}_2^\top(\boldsymbol{\theta}) - 2\boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{\mathrm{LS}}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{12}^{\mathrm{LS}}(\boldsymbol{\theta}) + \boldsymbol{C}_{22}(\boldsymbol{\theta})$.

For the improved DR estimator with IPW estimating equation $\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\mathrm{opt1}})$, $\boldsymbol{m}(\boldsymbol{U}_i,\boldsymbol{\theta})$ is given by

$$\boldsymbol{m}(\boldsymbol{U}_i,\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{I\{A_i = d(\boldsymbol{X}_i)\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}} \cdot \dfrac{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}\Big[Y_i - Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i);\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i, d(\boldsymbol{X}_i);\boldsymbol{\beta}\} \\ \dfrac{I\{A_i = d(\boldsymbol{X}_i)\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}\left[Y_i - Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i);\boldsymbol{\beta}\}\right] + Q\{\boldsymbol{X}_i, d(\boldsymbol{X}_i);\boldsymbol{\beta}\} - V(d) \end{pmatrix}.$$

The true values are $\boldsymbol{\theta}_0^{\mathrm{opt1}} = ({\boldsymbol{\beta}^*_{\mathrm{opt1}}}^\top, V(d))^\top$ where $\boldsymbol{\beta}^*_{\mathrm{opt1}}$ satisfies

$$E\left(\frac{I\{A = d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}} \cdot \frac{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Y - Q\{\boldsymbol{X}, d(\boldsymbol{X});\boldsymbol{\beta}^*_{\mathrm{opt1}}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X});\boldsymbol{\beta}^*_{\mathrm{opt1}}\}\right) = \mathbf{0}.$$

Again by M-estimation theory, $\hat{\boldsymbol{\theta}}^{\mathrm{opt1}} \triangleq (\{\hat{\boldsymbol{\beta}}^{\mathrm{opt1}}\}^\top, \widehat{V}(d;\hat{\boldsymbol{\beta}}^{\mathrm{opt1}}))^\top$ is asymptotically normal,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}^{\mathrm{opt1}} - \boldsymbol{\theta}_0^{\mathrm{opt1}}) \xrightarrow{D} N(0, \{\boldsymbol{B}^{\mathrm{opt1}}(\boldsymbol{\theta}_0^{\mathrm{opt1}})\}^{-1}\boldsymbol{C}^{\mathrm{opt1}}(\boldsymbol{\theta}_0^{\mathrm{opt1}})[\{\boldsymbol{B}^{\mathrm{opt1}}(\boldsymbol{\theta}_0^{\mathrm{opt1}})\}^{-1}]^\top),$$

where

$$\boldsymbol{B}^{\mathrm{opt1}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{B}_1^{\mathrm{opt1}}(\boldsymbol{\theta}) & \mathbf{0}_{s\times 1} \\ \boldsymbol{B}_2(\boldsymbol{\theta}) & -1 \end{pmatrix} \qquad \boldsymbol{C}^{\mathrm{opt1}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{C}_{11}^{\mathrm{opt1}}(\boldsymbol{\theta}) & \boldsymbol{C}_{12}^{\mathrm{opt1}}(\boldsymbol{\theta}) \\ \{\boldsymbol{C}_{12}^{\mathrm{opt1}}(\boldsymbol{\theta})\}^\top & \boldsymbol{C}_{22}(\boldsymbol{\theta}) \end{pmatrix},$$

with all the quantities defined in the theorem. The rest is by some simple algebra.

For the improved DR estimator with *augmented* IPW estimating equation $\widehat{V}(d;\hat{\boldsymbol{\beta}}^{\text{opt2}})$, the estimating equation is $\boldsymbol{m}^{\text{opt2}}(\boldsymbol{U}_i,\boldsymbol{\theta},g,h) = (\boldsymbol{m}_1(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)^\top, m_2(\boldsymbol{U}_i,\boldsymbol{\theta}))^\top$ where

$$\begin{aligned}
\boldsymbol{m}_1(\boldsymbol{U},\boldsymbol{\theta},g,h) &= \frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot\frac{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Y-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\} \\
&\quad -\Bigg(\frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot\frac{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot \\
&\quad \Big[g(\boldsymbol{X})I\{d(\boldsymbol{X})=1\}+h(\boldsymbol{X})I\{d(\boldsymbol{X})=-1\}-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\Bigg), \\
m_2(\boldsymbol{U},\boldsymbol{\theta}) &= \frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\left[Y-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right]+Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}-V(d).
\end{aligned}$$

The true values are $\boldsymbol{\theta}_0^{\text{opt2}} = ({\boldsymbol{\beta}^*_{\text{opt2}}}^\top, V(d))^\top$ where $\boldsymbol{\beta}^*_{\text{opt2}}$ satisfies

$$\begin{aligned}
&E\Bigg(\frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot\frac{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\Big[Y-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*_{\text{opt2}}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*_{\text{opt2}}\}\Bigg) \\
&-E\Bigg(\frac{I\{A=d(\boldsymbol{X})\}-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot\frac{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X}\}}\cdot \\
&\Big[Q_0\{\boldsymbol{X},d(\boldsymbol{X})\}-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*_{\text{opt2}}\}\Big]Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}^*_{\text{opt2}}\}\Bigg) = \mathbf{0}.
\end{aligned}$$

Let us define $\boldsymbol{M}^{\text{opt2}}(\boldsymbol{\theta},g,h) = E\left\{\boldsymbol{m}^{\text{opt2}}(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)\right\}$ and $\boldsymbol{M}_n^{\text{opt2}}(\boldsymbol{\theta},g,h) = n^{-1}\sum_{i=1}^n \boldsymbol{m}^{\text{opt2}}(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)$. Now we calculate all the ordinary and functional derivatives in Theorem S1. The ordinary derivative of $\boldsymbol{M}^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)$ with respect to $\boldsymbol{\theta}$ is

$$\boldsymbol{\Gamma}_1^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0) = \begin{pmatrix} \boldsymbol{B}_1^{\text{opt2}}(\boldsymbol{\theta}) & \mathbf{0}_{s\times 1} \\ \boldsymbol{B}_2(\boldsymbol{\theta}) & -1 \end{pmatrix}$$

where $\boldsymbol{B}_1^{\text{opt2}}(\boldsymbol{\theta}) = E\left\{\partial \boldsymbol{m}_1(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0)/\partial\boldsymbol{\beta}^\top\right\}$. The functional derivatives are

$$\boldsymbol{\Gamma}_2^{\text{opt2}}(\boldsymbol{\theta},g,h)[\bar{g}-g] = \begin{pmatrix} \boldsymbol{\Gamma}_{21}^{\text{opt2}}(\boldsymbol{\theta},g,h)[\bar{g}-g] \\ 0 \end{pmatrix} \quad \boldsymbol{\Gamma}_3^{\text{opt2}}(\boldsymbol{\theta},g,h)[\bar{h}-h] = \begin{pmatrix} \boldsymbol{\Gamma}_{31}^{\text{opt2}}(\boldsymbol{\theta},g,h)[\bar{h}-h] \\ 0 \end{pmatrix}$$

where

$$\mathbf{\Gamma}_{21}^{\text{opt2}}(\boldsymbol{\theta}, g, h)[\bar{g} - g] = E\Bigg(- \frac{I\{A = d(\boldsymbol{X})\} - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \{\bar{g}(\boldsymbol{X}) - g(\boldsymbol{X})\} I\{d(\boldsymbol{X}) = 1\} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Bigg),$$

$$\mathbf{\Gamma}_{31}^{\text{opt2}}(\boldsymbol{\theta}, g, h)[\bar{h} - h] = E\Bigg(- \frac{I\{A = d(\boldsymbol{X})\} - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi\{d(\boldsymbol{X}), \boldsymbol{X}\}} \cdot \{\bar{h}(\boldsymbol{X}) - h(\boldsymbol{X})\} I\{d(\boldsymbol{X}) = -1\} Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\}\Bigg).$$

Verification of the conditions (1.1)-(1.3) in Theorem S1 is standard. Condition (1.4) can be easily verified using well-known results on kernel estimation. By Section 4 of Chen et al. (2003), condition (1.5) is met since $\boldsymbol{m}^{\text{opt2}}(\boldsymbol{U}, \boldsymbol{\theta}, g, h)$ is (pointwise) Lipschitz continuous with respect to $(\boldsymbol{\theta}, g, h)$. Now we investigate the asymptotic properties of $\sqrt{n}\mathbf{\Gamma}_{21}^{\text{opt2}}(\boldsymbol{\theta}, g_0, h_0)[\hat{g} - g_0]$ and $\sqrt{n}\mathbf{\Gamma}_{31}^{\text{opt2}}(\boldsymbol{\theta}, g_0, h_0)[\hat{h} - h_0]$ where $\hat{g}, \hat{h}$ are kernel regression estimators. Notice that

$$\mathbf{\Gamma}_{21}^{\text{opt2}}(\boldsymbol{\theta}, g_0, h_0)[\hat{g} - g_0] = \int \Bigg(- \frac{\pi_0\{d(\boldsymbol{x}), \boldsymbol{x}\} - \pi\{d(\boldsymbol{x}), \boldsymbol{x}\}}{\pi\{d(\boldsymbol{x}), \boldsymbol{x}\}} \cdot \frac{1 - \pi\{d(\boldsymbol{x}), \boldsymbol{x}\}}{\pi\{d(\boldsymbol{x}), \boldsymbol{x}\}} \cdot \{\hat{g}(\boldsymbol{x}) - g_0(\boldsymbol{x})\} I\{d(\boldsymbol{x}) = 1\} Q_{\boldsymbol{\beta}}\{\boldsymbol{x}, d(\boldsymbol{x}); \boldsymbol{\beta}\}\Bigg) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}.$$

where $f_{\boldsymbol{X}}(\cdot)$ is the density function of $\boldsymbol{X}$. Recall that

$$\hat{g}(\boldsymbol{x}) = \frac{\sum_{i=1}^n K_{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{X}_i) I(A_i = 1) Y_i}{\sum_{i=1}^n K_{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{X}_i) I(A_i = 1)}.$$

By remark 3.3 in Ichimura and Lee (2010), we know that

$$\hat{g}(\boldsymbol{x}) - g_0(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^n \frac{Y_i - E(Y \mid \boldsymbol{X} = \boldsymbol{X}_i, A = 1)}{f_{\boldsymbol{X},A}(\boldsymbol{x}, 1)} K_{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{X}_i) I(A_i = 1) + o_p(n^{-1/2})$$

where $f_{\boldsymbol{X},A}(\cdot,\cdot)$ is the joint density of $(\boldsymbol{X},A)$. Therefore,

$$\begin{aligned}\boldsymbol{\Gamma}_{21}^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)[\hat{g}-g_0] =& \frac{1}{n}\sum_{i=1}^{n}\Bigg[\{Y_i-g_0(\boldsymbol{X}_i)\}\,I(A_i=1)\cdot\int\Bigg(-\frac{\pi_0\{d(\boldsymbol{x}),\boldsymbol{x}\}-\pi\{d(\boldsymbol{x}),\boldsymbol{x}\}}{\pi\{d(\boldsymbol{x}),\boldsymbol{x}\}}\cdot \\ & \frac{1-\pi\{d(\boldsymbol{x}),\boldsymbol{x}\}}{\pi\{d(\boldsymbol{x}),\boldsymbol{x}\}}\cdot\frac{K_{\boldsymbol{H}}(\boldsymbol{x}-\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{x},1)}I\{d(\boldsymbol{x})=1\}Q_{\boldsymbol{\beta}}\{\boldsymbol{x},d(\boldsymbol{x});\boldsymbol{\beta}\}\Bigg)f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}\Bigg]\end{aligned}$$

plus an $o_p(n^{-1/2})$ term. Similar to Example 1 in Chen et al. (2003), by using standard change of variables and Taylor expansion we have

$$\boldsymbol{\Gamma}_{21}^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)[\hat{g}-g_0] = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{q}_1(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0)+o_p(n^{-1/2})$$

where

$$\begin{aligned}\boldsymbol{q}_1(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0) \;=\;& -\{Y_i-g_0(\boldsymbol{X}_i)\}\,\frac{I(A_i=1)f_{\boldsymbol{X}}(\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{X}_i,1)}\cdot\frac{\pi_0\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}\cdot \\ & \frac{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}I\{d(\boldsymbol{X}_i)=1\}Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}\}.\end{aligned}$$

Similarly,

$$\boldsymbol{\Gamma}_{31}^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)[\hat{h}-h_0] = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{q}_2(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0)+o_p(n^{-1/2})$$

where

$$\begin{aligned}\boldsymbol{q}_2(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0) \;=\;& -\{Y_i-h_0(\boldsymbol{X}_i)\}\,\frac{I(A_i=-1)f_{\boldsymbol{X}}(\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{X}_i,-1)}\cdot\frac{\pi_0\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}\cdot \\ & \frac{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\}}I\{d(\boldsymbol{X}_i)=-1\}Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}\}.\end{aligned}$$

Combining the above results, we have

$$\sqrt{n}\left\{\boldsymbol{M}_n^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)+\boldsymbol{\Gamma}_2^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)[\hat{g}-g_0]+\boldsymbol{\Gamma}_3^{\text{opt2}}(\boldsymbol{\theta},g_0,h_0)[\hat{h}-h_0]\right\}\xrightarrow{D}N(\mathbf{0},\boldsymbol{C}^{\text{opt2}}(\boldsymbol{\theta})),$$

where

$$\boldsymbol{C}^{\text{opt2}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{C}_{11}^{\text{opt2}}(\boldsymbol{\theta}) & \boldsymbol{C}_{12}^{\text{opt2}}(\boldsymbol{\theta}) \\ \{\boldsymbol{C}_{12}^{\text{opt2}}(\boldsymbol{\theta})\}^\top & \boldsymbol{C}_{22}(\boldsymbol{\theta}) \end{pmatrix},$$

$$\boldsymbol{C}_{11}^{\text{opt2}}(\boldsymbol{\theta}) = E\left\{(\boldsymbol{m}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2)(\boldsymbol{m}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2)^\top\right\}, \quad \boldsymbol{C}_{12}^{\text{opt2}}(\boldsymbol{\theta}) = E\left\{(\boldsymbol{m}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2)m_2\right\}.$$

Here $\boldsymbol{m}_1, m_2, \boldsymbol{q}_1, \boldsymbol{q}_2$ are shorthand for $\boldsymbol{m}_1(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0)$, $m_2(\boldsymbol{U}_i, \boldsymbol{\theta})$, $\boldsymbol{q}_1(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0)$, $\boldsymbol{q}_2(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0)$. Based on Theorem S1, $\hat{\boldsymbol{\theta}}^{\text{opt2}} \triangleq (\{\hat{\boldsymbol{\beta}}^{\text{opt2}}\}^\top, \widehat{V}(d; \hat{\boldsymbol{\beta}}^{\text{opt2}}))^\top$ is asymptotically normal,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}^{\text{opt2}} - \boldsymbol{\theta}_0^{\text{opt2}}) \xrightarrow{D} N(0, \boldsymbol{\Omega}^{\text{opt2}}(\boldsymbol{\theta}_0^{\text{opt2}})),$$

where $\boldsymbol{\Omega}^{\text{opt2}}(\boldsymbol{\theta}) = \boldsymbol{\Gamma}_1^{\text{opt2}}(\boldsymbol{\theta}, g_0, h_0)^{-1} \boldsymbol{C}^{\text{opt2}}(\boldsymbol{\theta}) \{\boldsymbol{\Gamma}_1^{\text{opt2}}(\boldsymbol{\theta}, g_0, h_0)^{-1}\}^\top$. By some algebra, we obtain

$$\sqrt{n}\left\{\widehat{V}(d; \hat{\boldsymbol{\beta}}^{\text{opt2}}) - V(d)\right\} \xrightarrow{D} N(0, U_3(\boldsymbol{\theta}_0^{\text{opt2}})),$$

where $U_3(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{\text{opt2}}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{11}^{\text{opt2}}(\boldsymbol{\theta})\{\boldsymbol{B}_1^{\text{opt2}}(\boldsymbol{\theta})\}^{-1}\boldsymbol{B}_2^\top(\boldsymbol{\theta}) - 2\boldsymbol{B}_2(\boldsymbol{\theta})\{\boldsymbol{B}_1^{\text{opt2}}(\boldsymbol{\theta})\}^{-1}\boldsymbol{C}_{12}^{\text{opt2}}(\boldsymbol{\theta}) + \boldsymbol{C}_{22}(\boldsymbol{\theta})$.

□

Appendix D: Details and proof of Theorem 2

To investigate asymptotic properties of $\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{\text{opt3}})$ and $\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{\text{opt4}})$, we introduce additional parameters $\boldsymbol{\zeta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\psi}^\top, \boldsymbol{\phi}^\top)^\top$ to account for $\widehat{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$, $\widehat{\Gamma}(\boldsymbol{\beta})$, $\widehat{\Phi}(\boldsymbol{\beta})$. Here, $\boldsymbol{\alpha}$ corresponds to $-E\{\partial\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}, \boldsymbol{\beta})/\partial\boldsymbol{\gamma}\}$, $\boldsymbol{\psi} = (\boldsymbol{\psi}_1^\top, \ldots, \boldsymbol{\psi}_q^\top)^\top$ where $\boldsymbol{\psi}_j$ corresponds to the j-th column of $E\left\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}, \boldsymbol{\gamma})S_{\boldsymbol{\gamma}}^\top(A, \boldsymbol{X}, \boldsymbol{\gamma})\right\}$, and $\boldsymbol{\phi} = (\boldsymbol{\phi}_1^\top, \ldots, \boldsymbol{\phi}_q^\top)^\top$ where $\boldsymbol{\phi}_j$ corresponds to $-E\{\partial^2\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}, \boldsymbol{\beta})/\partial\gamma_j\partial\boldsymbol{\beta}\}$. We use $[\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_q]$ to represent a matrix with j-th column being $\boldsymbol{\phi}_j$, similarly for $[\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q]$. In addition, define $S_{\boldsymbol{\gamma}\boldsymbol{\gamma}}(A, \boldsymbol{X}; \boldsymbol{\gamma}) \triangleq \partial S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}; \boldsymbol{\gamma})/\partial\boldsymbol{\gamma}^\top$, let r be the dimension of $\boldsymbol{\zeta}$, and let $\text{col}_j\{\cdot\}$ denote the j-th column of a matrix.

Theorem 2. *(Asymptotic normality when there is nuisance parameter in the propensity score model). To obtain $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{LS})$, the unknown parameters are $\boldsymbol{\theta}=(\boldsymbol{\gamma}^\top,\boldsymbol{\beta}^\top,V(d))^\top$. When either the propensity score or the outcome model is correct,*

$$\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{LS})-V(d)\right\}\xrightarrow{D} N(0,U_4(\boldsymbol{\theta}_0^{LS2})),$$

where

$$\begin{aligned}U_4(\boldsymbol{\theta}) =&\boldsymbol{D}_2^{LS2}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{LS2}(\boldsymbol{\theta})\}^{-1}\boldsymbol{F}_{11}^{LS2}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{LS2}(\boldsymbol{\theta})\}^{-1}\{\boldsymbol{D}_2^{LS2}(\boldsymbol{\theta})\}^\top\\ &-2\boldsymbol{D}_2^{LS2}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{LS2}(\boldsymbol{\theta})\}^{-1}\boldsymbol{F}_{12}^{LS2}(\boldsymbol{\theta})+\boldsymbol{F}_{22}(\boldsymbol{\theta}).\end{aligned}$$

Here,

$$\boldsymbol{D}_1^{LS2}(\boldsymbol{\theta})=\begin{pmatrix}\boldsymbol{J}_1(\boldsymbol{\theta}) & \mathbf{0}_{q\times s}\\ \mathbf{0}_{s\times q} & \boldsymbol{J}_2(\boldsymbol{\theta})\end{pmatrix},\quad \boldsymbol{D}_2^{LS2}(\boldsymbol{\theta})=\begin{pmatrix}\boldsymbol{J}_3(\boldsymbol{\theta}) & \boldsymbol{J}_4(\boldsymbol{\theta})\end{pmatrix},$$

$$\boldsymbol{F}_{11}^{LS2}(\boldsymbol{\theta})=\begin{pmatrix}\boldsymbol{J}_5(\boldsymbol{\theta}) & \boldsymbol{J}_6(\boldsymbol{\theta})\\ \{\boldsymbol{J}_6(\boldsymbol{\theta})\}^\top & \boldsymbol{J}_7(\boldsymbol{\theta})\end{pmatrix},\quad \boldsymbol{F}_{12}^{LS2}(\boldsymbol{\theta})=\begin{pmatrix}\boldsymbol{J}_8(\boldsymbol{\theta})\\ \boldsymbol{J}_9(\boldsymbol{\theta})\end{pmatrix},\quad \boldsymbol{F}_{22}(\boldsymbol{\theta})=E\left\{\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma},\boldsymbol{\beta})\right\}^2,$$

where

$$\begin{aligned}
\boldsymbol{J}_1(\boldsymbol{\theta}) &= E\left\{S_{\boldsymbol{\gamma}\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma})\right\},\\
\boldsymbol{J}_2(\boldsymbol{\theta}) &= E\left[Q_{\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})\left\{Y-Q(\boldsymbol{X},A;\boldsymbol{\beta})\right\}-Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})Q_{\boldsymbol{\beta}}^\top(\boldsymbol{X},A;\boldsymbol{\beta})\right],\\
\boldsymbol{J}_3(\boldsymbol{\theta}) &= E\left\{\partial\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma},\boldsymbol{\beta})/\partial\boldsymbol{\gamma}^\top\right\},\\
\boldsymbol{J}_4(\boldsymbol{\theta}) &= E\left[\left(1-\frac{I\{A=d(\boldsymbol{X})\}}{\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\right)Q_{\boldsymbol{\beta}}^\top\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}\right],\\
\boldsymbol{J}_5(\boldsymbol{\theta}) &= E\left\{S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma})S_{\boldsymbol{\gamma}}^\top(A,\boldsymbol{X};\boldsymbol{\gamma})\right\},\\
\boldsymbol{J}_6(\boldsymbol{\theta}) &= E\left[S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma})Q_{\boldsymbol{\beta}}^\top(\boldsymbol{X},A;\boldsymbol{\beta})\left\{Y-Q(\boldsymbol{X},A;\boldsymbol{\beta})\right\}\right],\\
\boldsymbol{J}_7(\boldsymbol{\theta}) &= E\left[\left\{Y-Q(\boldsymbol{X},A;\boldsymbol{\beta})\right\}^2Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})Q_{\boldsymbol{\beta}}^\top(\boldsymbol{X},A;\boldsymbol{\beta})\right],\\
\boldsymbol{J}_8(\boldsymbol{\theta}) &= E\left\{S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma})\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma},\boldsymbol{\beta})\right\},\\
\boldsymbol{J}_9(\boldsymbol{\theta}) &= E\left[Q_{\boldsymbol{\beta}}(\boldsymbol{X},A;\boldsymbol{\beta})\left\{Y-Q(\boldsymbol{X},A;\boldsymbol{\beta})\right\}\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma},\boldsymbol{\beta})\right].
\end{aligned}$$

The true values are $\boldsymbol{\theta}_0^{LS2} = (\boldsymbol{\gamma}^{*\top}, {\boldsymbol{\beta}_{LS}^{*}}^\top, V(d))^\top$ *where* $\boldsymbol{\gamma}^*$ *satisfies* $E\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}; \boldsymbol{\gamma}^*)\} = \mathbf{0}$ *and* $\boldsymbol{\beta}_{LS}^*$ *satisfies* $E[Q_{\boldsymbol{\beta}}(\boldsymbol{X}, A; \boldsymbol{\beta}_{LS}^*)\{Y - Q(\boldsymbol{X}, A; \boldsymbol{\beta}_{LS}^*)\}] = \mathbf{0}$.

To obtain $\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{opt3})$, *the unknown parameters are* $\boldsymbol{\theta} = (\boldsymbol{\gamma}^\top, \boldsymbol{\zeta}^\top, \boldsymbol{\beta}^\top, V(d))^\top$. *When either the propensity score or the outcome model is correct,*

$$\sqrt{n}\left\{\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{opt3}) - V(d)\right\} \xrightarrow{D} N(0, U_5(\boldsymbol{\theta}_0^{opt3}))$$

where

$$\begin{aligned} U_5(\boldsymbol{\theta}) =& \boldsymbol{D}_2^{opt3}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{opt3}(\boldsymbol{\theta})\}^{-1}\boldsymbol{F}_{11}^{opt3}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{opt3}(\boldsymbol{\theta})\}^{-1}\{\boldsymbol{D}_2^{opt3}(\boldsymbol{\theta})\}^\top \\ &- 2\boldsymbol{D}_2^{opt3}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{opt3}(\boldsymbol{\theta})\}^{-1}\boldsymbol{F}_{12}^{opt3}(\boldsymbol{\theta}) + \boldsymbol{F}_{22}(\boldsymbol{\theta}). \end{aligned}$$

Here,

$$\boldsymbol{D}_1^{opt3}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{\theta}) & \mathbf{0}_{q\times r} & \mathbf{0}_{q\times s} \\ E\left\{\partial \boldsymbol{m}_3/\partial\boldsymbol{\gamma}^\top\right\} & E\left\{\partial \boldsymbol{m}_3/\partial\boldsymbol{\zeta}^\top\right\} & E\left\{\partial \boldsymbol{m}_3/\partial\boldsymbol{\beta}^\top\right\} \\ E\left\{\partial \boldsymbol{m}_4/\partial\boldsymbol{\gamma}^\top\right\} & E\left\{\partial \boldsymbol{m}_4/\partial\boldsymbol{\zeta}^\top\right\} & E\left\{\partial \boldsymbol{m}_4/\partial\boldsymbol{\beta}^\top\right\} \end{pmatrix}, \quad \boldsymbol{D}_2^{opt3}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_3(\boldsymbol{\theta}) & \mathbf{0}_{1\times r} & \boldsymbol{J}_4(\boldsymbol{\theta}) \end{pmatrix},$$

$$\boldsymbol{F}_{11}^{opt3}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_5(\boldsymbol{\theta}) & E\left\{S_{\boldsymbol{\gamma}}\boldsymbol{m}_3^\top\right\} & E\left\{S_{\boldsymbol{\gamma}}\boldsymbol{m}_4^\top\right\} \\ E\left\{\boldsymbol{m}_3 S_{\boldsymbol{\gamma}}^\top\right\} & E\left\{\boldsymbol{m}_3\boldsymbol{m}_3^\top\right\} & E\left\{\boldsymbol{m}_3\boldsymbol{m}_4^\top\right\} \\ E\left\{\boldsymbol{m}_4 S_{\boldsymbol{\gamma}}^\top\right\} & E\left\{\boldsymbol{m}_4\boldsymbol{m}_3^\top\right\} & E\left\{\boldsymbol{m}_4\boldsymbol{m}_4^\top\right\} \end{pmatrix}, \quad \boldsymbol{F}_{12}^{opt3}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_8(\boldsymbol{\theta}) \\ E\left\{\boldsymbol{m}_3\tilde{\varphi}\right\} \\ E\left\{\boldsymbol{m}_4\tilde{\varphi}\right\} \end{pmatrix}.$$

Here $S_{\boldsymbol{\gamma}}$, $\boldsymbol{m}_3$, $\boldsymbol{m}_4$, $\tilde{\varphi}$ *are shorthand for* $S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}; \boldsymbol{\gamma})$, $\boldsymbol{m}_3(\boldsymbol{U}, \boldsymbol{\theta})$, $\boldsymbol{m}_4(\boldsymbol{U}, \boldsymbol{\theta})$, $\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}, \boldsymbol{\beta})$ *where*

$$\boldsymbol{m}_3(\boldsymbol{U}, \boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{\alpha} + \partial\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}, \boldsymbol{\beta})/\partial\boldsymbol{\gamma} \\ \boldsymbol{\psi}_1 - col_1\left\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}, \boldsymbol{\gamma})S_{\boldsymbol{\gamma}}^\top(A, \boldsymbol{X}, \boldsymbol{\gamma})\right\} \\ \dots \\ \boldsymbol{\psi}_q - col_q\left\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}, \boldsymbol{\gamma})S_{\boldsymbol{\gamma}}^\top(A, \boldsymbol{X}, \boldsymbol{\gamma})\right\} \\ \boldsymbol{\phi}_1 + \partial^2\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}, \boldsymbol{\beta})/\partial\gamma_1\partial\boldsymbol{\beta} \\ \dots \\ \boldsymbol{\phi}_q + \partial^2\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}, \boldsymbol{\beta})/\partial\gamma_q\partial\boldsymbol{\beta} \end{pmatrix},$$

$$\boldsymbol{m}_4(\boldsymbol{U},\boldsymbol{\theta}) = \frac{R\left[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}\right]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}+[\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\right]$$
$$\cdot\left[Y-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}-\boldsymbol{\alpha}^\top[\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\right].$$

The true parameters are $\boldsymbol{\theta}_0^{opt3}=(\boldsymbol{\gamma}^{*\top},\boldsymbol{\zeta}_{opt3}^{*}{}^\top,\boldsymbol{\beta}_{opt3}^{*}{}^\top,V(d))^\top$ *where* $(\boldsymbol{\zeta}_{opt3}^*,\boldsymbol{\beta}_{opt3}^*)$ *is the solution to the following set of equations:*

$$\boldsymbol{\alpha}+E\left\{\partial\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}^*,\boldsymbol{\beta})/\partial\boldsymbol{\gamma}\right\}=\boldsymbol{0},$$
$$[\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]-E\left\{S_{\boldsymbol{\gamma}}(A,\boldsymbol{X},\boldsymbol{\gamma}^*)S_{\boldsymbol{\gamma}}^\top(A,\boldsymbol{X},\boldsymbol{\gamma}^*)\right\}=\boldsymbol{0},$$
$$[\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q]+E\left\{\partial^2\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}^*,\boldsymbol{\beta})/\partial\boldsymbol{\gamma}^\top\partial\boldsymbol{\beta}\right\}=\boldsymbol{0},$$
$$E\left(\frac{R\left[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}\right]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}+[\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}\right]\right.$$
$$\left.\cdot\left[Y-Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}-\boldsymbol{\alpha}^\top[\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}^*\}}\right]\right)=\boldsymbol{0}. \tag{S.11}$$

To obtain $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt4})$*, the unknown parameters are* $\boldsymbol{\theta}=(\boldsymbol{\gamma}^\top,\boldsymbol{\zeta}^\top,\boldsymbol{\beta}^\top,V(d))^\top$*. When either the propensity score or the outcome model is correct,*

$$\sqrt{n}\left\{\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt4})-V(d)\right\}\xrightarrow{D}N(0,U_6(\boldsymbol{\theta}_0^{opt4}))$$

where

$$U_6(\boldsymbol{\theta}) = \boldsymbol{D}_2^{opt4}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{opt4}(\boldsymbol{\theta})\}^{-1}\boldsymbol{F}_{11}^{opt4}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{opt4}(\boldsymbol{\theta})\}^{-1}\{\boldsymbol{D}_2^{opt4}(\boldsymbol{\theta})\}^\top$$
$$-2\boldsymbol{D}_2^{opt4}(\boldsymbol{\theta})\{\boldsymbol{D}_1^{opt4}(\boldsymbol{\theta})\}^{-1}\boldsymbol{F}_{12}^{opt4}(\boldsymbol{\theta})+\boldsymbol{F}_{22}(\boldsymbol{\theta}).$$

Here,

$$\boldsymbol{D}_1^{opt4}(\boldsymbol{\theta})=\begin{pmatrix}\boldsymbol{J}_1(\boldsymbol{\theta}) & \boldsymbol{0}_{q\times r} & \boldsymbol{0}_{q\times s}\\ E\left\{\partial\boldsymbol{m}_3/\partial\boldsymbol{\gamma}^\top\right\} & E\left\{\partial\boldsymbol{m}_3/\partial\boldsymbol{\zeta}^\top\right\} & E\left\{\partial\boldsymbol{m}_3/\partial\boldsymbol{\beta}^\top\right\}\\ E\left\{\partial\boldsymbol{m}_5/\partial\boldsymbol{\gamma}^\top\right\} & E\left\{\partial\boldsymbol{m}_5/\partial\boldsymbol{\zeta}^\top\right\} & E\left\{\partial\boldsymbol{m}_5/\partial\boldsymbol{\beta}^\top\right\}\end{pmatrix},$$

$$\boldsymbol{D}_2^{opt4}(\boldsymbol{\theta}) = \boldsymbol{D}_2^{opt3}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_3(\boldsymbol{\theta}) & 0_{1\times r} & \boldsymbol{J}_4(\boldsymbol{\theta}) \end{pmatrix},$$

$$\boldsymbol{F}_{11}^{opt4}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_5(\boldsymbol{\theta}) & E\left\{S_{\boldsymbol{\gamma}}\boldsymbol{m}_3^\top\right\} & E\left\{S_{\boldsymbol{\gamma}}(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)^\top\right\} \\ E\left\{\boldsymbol{m}_3 S_{\boldsymbol{\gamma}}^\top\right\} & E\left\{\boldsymbol{m}_3\boldsymbol{m}_3^\top\right\} & E\left\{\boldsymbol{m}_3(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)^\top\right\} \\ E\left\{(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)S_{\boldsymbol{\gamma}}^\top\right\} & E\left\{(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)\boldsymbol{m}_3^\top\right\} & E\left\{(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)^\top\right\} \end{pmatrix}$$

$$\boldsymbol{F}_{12}^{opt4}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_8(\boldsymbol{\theta})^\top & E(\tilde{\varphi}\boldsymbol{m}_3^\top) & E\left\{\tilde{\varphi}(\boldsymbol{m}_5+\boldsymbol{q}_3+\boldsymbol{q}_4)^\top\right\} \end{pmatrix}^\top.$$

Here we suppress the notations by writing $\boldsymbol{m}_5 \equiv \boldsymbol{m}_5(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0)$, $\boldsymbol{q}_3 \equiv \boldsymbol{q}_3(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0)$, $\boldsymbol{q}_4 \equiv \boldsymbol{q}_4(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0)$ *where*

$$\begin{aligned} \boldsymbol{m}_5(\boldsymbol{U},\boldsymbol{\theta},g_0,h_0) = \boldsymbol{m}_4(\boldsymbol{U},\boldsymbol{\theta}) - \Bigg(& \frac{[R-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}]\,[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}} \\ & \cdot\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\} + [\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\right] \\ & \cdot\Bigg[g_0(\boldsymbol{X})I\{d(\boldsymbol{X})=1\} + h_0(\boldsymbol{X})I\{d(\boldsymbol{X})=-1\} \\ & - Q\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\} - \boldsymbol{\alpha}^\top[\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\Bigg]\Bigg). \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_3(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0) = & -\{Y_i - g_0(\boldsymbol{X}_i)\}\,\frac{I(A_i=1)f_{\boldsymbol{X}}(\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{X}_i,1)}\cdot\frac{\pi_0\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\} - \pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}} \\ & \cdot\frac{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}\cdot I\{d(\boldsymbol{X}_i)=1\} \\ & \cdot\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}\} + [\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}\right], \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_4(\boldsymbol{U}_i,\boldsymbol{\theta},g_0,h_0) = & -\{Y_i - h_0(\boldsymbol{X}_i)\}\,\frac{I(A_i=-1)f_{\boldsymbol{X}}(\boldsymbol{X}_i)}{f_{\boldsymbol{X},A}(\boldsymbol{X}_i,-1)}\cdot\frac{\pi_0\{d(\boldsymbol{X}_i),\boldsymbol{X}_i\} - \pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}} \\ & \cdot\frac{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}\cdot I\{d(\boldsymbol{X}_i)=-1\} \\ & \cdot\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}\} + [\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}\right]. \end{aligned}$$

The true parameters are $\boldsymbol{\theta}_0^{opt4} = (\boldsymbol{\gamma}^{*\top}, {\boldsymbol{\zeta}^*_{opt4}}^\top, {\boldsymbol{\beta}^*_{opt4}}^\top, V(d))^\top$ *where* $(\boldsymbol{\zeta}^*_{opt4}, \boldsymbol{\beta}^*_{opt4})$ *is the solution to the following set of equations:*

$$\begin{gathered}
\boldsymbol{\alpha} + E\left\{\partial\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}^*, \boldsymbol{\beta})/\partial\boldsymbol{\gamma}\right\} = \mathbf{0}, \\
[\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q] - E\left\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}, \boldsymbol{\gamma}^*) S_{\boldsymbol{\gamma}}^\top(A, \boldsymbol{X}, \boldsymbol{\gamma}^*)\right\} = \mathbf{0}, \\
[\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_q] + E\left\{\partial^2\tilde{\varphi}(Y, A, \boldsymbol{X}, \boldsymbol{\gamma}^*, \boldsymbol{\beta})/\partial\boldsymbol{\gamma}^\top\partial\boldsymbol{\beta}\right\} = \mathbf{0},
\end{gathered}$$

$$\begin{aligned}
&(***) - E\Bigg(\frac{[R - \pi\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}]\,[1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}]}{\pi^2\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}} \\
&\cdot \left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} + [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_q][\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q]^{-1} \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}}{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}}\right] \\
&\cdot \left[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}\} - \boldsymbol{\alpha}^\top[\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q]^{-1} \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}}{1 - \pi\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}^*\}}\right]\Bigg) = \mathbf{0},
\end{aligned}$$

where $(***)$ *is the left hand side of* (S.11).

Remark 2. *When* $\pi(A, \boldsymbol{X}; \boldsymbol{\gamma})$ *is correctly specified, i.e.* $\pi(A, \boldsymbol{X}; \boldsymbol{\gamma}_0) = \pi_0(A, \boldsymbol{X})$ *for some* $\boldsymbol{\gamma}_0$*, it is obvious that* $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}_0$*. Observe that* $\boldsymbol{J}_4(\boldsymbol{\theta}) = \mathbf{0}_{1\times s}$ *for any* $\boldsymbol{\theta}$*. Hence,*

$$\boldsymbol{D}_2^{opt3}(\boldsymbol{\theta}) = \boldsymbol{D}_2^{opt4}(\boldsymbol{\theta}) = \begin{pmatrix}\boldsymbol{J}_3(\boldsymbol{\theta}) & \mathbf{0}_{1\times r} & \mathbf{0}_{1\times s}\end{pmatrix}.$$

By some algebra, it can be shown that in this case,

$$U_4(\boldsymbol{\theta}) = U_5(\boldsymbol{\theta}) = U_6(\boldsymbol{\theta}) = \boldsymbol{J}_3(\boldsymbol{\theta})\boldsymbol{J}_1^{-1}(\boldsymbol{\theta})\boldsymbol{J}_5(\boldsymbol{\theta})\boldsymbol{J}_1^{-1}(\boldsymbol{\theta})\boldsymbol{J}_3^\top(\boldsymbol{\theta}) - 2\boldsymbol{J}_3(\boldsymbol{\theta})\boldsymbol{J}_1^{-1}(\boldsymbol{\theta})\boldsymbol{J}_8(\boldsymbol{\theta}) + \boldsymbol{F}_{22}(\boldsymbol{\theta}).$$

In a slight abuse of notation, it is easy to show that $\boldsymbol{\beta}^*_{opt3} = \boldsymbol{\beta}^*_{opt4} = \boldsymbol{\beta}^{opt}$ *where* $\boldsymbol{\beta}^{opt}$ *satisfies*

$$\begin{aligned}
&E\left(\frac{1 - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}{\pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^{opt}\} + \Phi_0(\boldsymbol{\beta}^{opt})\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}_0\}}{1 - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}\right]\right. \\
&\left.\cdot \left[Q_0\{\boldsymbol{X}, d(\boldsymbol{X})\} - Q\{\boldsymbol{X}, d(\boldsymbol{X}); \boldsymbol{\beta}^{opt}\} - \Gamma_0(\boldsymbol{\beta}^{opt})\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1} \cdot \frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}), \boldsymbol{X}; \boldsymbol{\gamma}_0\}}{1 - \pi_0\{d(\boldsymbol{X}), \boldsymbol{X}\}}\right]\right) = \mathbf{0}.
\end{aligned}$$

In addition, note that

$$\boldsymbol{J}_1(\boldsymbol{\theta}_0^{LS2}) = \boldsymbol{J}_1(\boldsymbol{\theta}_0^{opt3}) = \boldsymbol{J}_1(\boldsymbol{\theta}_0^{opt4}) = E\left\{S_{\boldsymbol{\gamma}\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma}_0)\right\} = -E\left\{S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma}_0)S_{\boldsymbol{\gamma}}^{\top}(A,\boldsymbol{X};\boldsymbol{\gamma}_0)\right\} = -\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}$$

$$\boldsymbol{J}_3(\boldsymbol{\theta}_0^{LS2}) = E\left\{\partial\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}_{LS}^*)/\partial\boldsymbol{\gamma}^{\top}\right\} = -\Gamma_0(\boldsymbol{\beta}_{LS}^*), \quad \boldsymbol{J}_3(\boldsymbol{\theta}_0^{opt3}) = \boldsymbol{J}_3(\boldsymbol{\theta}_0^{opt4}) = -\Gamma_0(\boldsymbol{\beta}^{opt}),$$

$$\boldsymbol{J}_5(\boldsymbol{\theta}_0^{LS2}) = \boldsymbol{J}_5(\boldsymbol{\theta}_0^{opt3}) = \boldsymbol{J}_5(\boldsymbol{\theta}_0^{opt4}) = \Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}.$$

Hence, the asymptotic variance of $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{LS})$ *is*

$$\begin{aligned}&\Gamma_0(\boldsymbol{\beta}_{LS}^*)\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1}\Gamma_0^{\top}(\boldsymbol{\beta}_{LS}^*) - 2\Gamma_0(\boldsymbol{\beta}_{LS}^*)\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1}E\{S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma}_0)\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}_{LS}^*)\} + E\tilde{\varphi}^2(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}_{LS}^*)\\ &= E\left\{\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}_{LS}^*) - \Gamma_0(\boldsymbol{\beta}_{LS}^*)\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1}S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma}_0)\right\}^2\end{aligned}$$

The asymptotic variance is the same for $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt3})$ *and* $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt4})$*, which equals to*

$$E\left\{\tilde{\varphi}(Y,A,\boldsymbol{X},\boldsymbol{\gamma}_0,\boldsymbol{\beta}^{opt}) - \Gamma_0(\boldsymbol{\beta}^{opt})\Sigma_{\boldsymbol{\gamma}\boldsymbol{\gamma},0}^{-1}S_{\boldsymbol{\gamma}}(A,\boldsymbol{X};\boldsymbol{\gamma}_0)\right\}^2.$$

The above two quantities are the variance of the influence function (7) evaluated at different $\boldsymbol{\beta}$ *values. By definition of* $\boldsymbol{\beta}^{opt}$*, when the propensity score is correct but outcome model incorrect,* $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt3})$ *and* $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt4})$ *have the same asymptotic variance, which is smaller than that of* $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{LS})$*. Though, in small sample size,* $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{opt4})$ *is preferred since it produces much more stable estimates. When both models are correct, note that* $\boldsymbol{\beta}^{opt} = \boldsymbol{\beta}_0 = \boldsymbol{\beta}_{LS}^*$*, all estimators are asymptotically equivalent.*

Proof. For the usual DR estimator $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{\mathrm{LS}})$, the parameters are $\boldsymbol{\theta} = (\boldsymbol{\gamma}^{\top},\boldsymbol{\beta}^{\top},V(d))^{\top}$ and the estimating equation is given by

$$\boldsymbol{m}(\boldsymbol{U}_i,\boldsymbol{\theta}) = \begin{pmatrix} S_{\boldsymbol{\gamma}}(A_i,\boldsymbol{X}_i;\boldsymbol{\gamma}) \\ Q_{\boldsymbol{\beta}}(\boldsymbol{X}_i,A_i;\boldsymbol{\beta})\left\{Y_i - Q(\boldsymbol{X}_i,A_i;\boldsymbol{\beta})\right\} \\ \dfrac{I\{A_i = d(\boldsymbol{X}_i)\}}{\pi\{d(\boldsymbol{X}_i),\boldsymbol{X}_i;\boldsymbol{\gamma}\}}\left[Y_i - Q\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}\}\right] + Q\{\boldsymbol{X}_i,d(\boldsymbol{X}_i);\boldsymbol{\beta}\} - V(d) \end{pmatrix}.$$

When at least one model is correct, $\boldsymbol{\theta}_0^{\mathrm{LS2}} = (\boldsymbol{\gamma}^{*\top}, {\boldsymbol{\beta}_{\mathrm{LS}}^{*}}^{\top}, V(d))^\top$ where $\boldsymbol{\gamma}^*$ satisfies $E\left\{S_{\boldsymbol{\gamma}}(A, \boldsymbol{X}; \boldsymbol{\gamma}^*)\right\} = \mathbf{0}$ and $\boldsymbol{\beta}_{\mathrm{LS}}^{*}$ satisfies $E\left[Q_{\boldsymbol{\beta}}(\boldsymbol{X}, A; \boldsymbol{\beta}_{\mathrm{LS}}^{*})\left\{Y - Q(\boldsymbol{X}, A; \boldsymbol{\beta}_{\mathrm{LS}}^{*})\right\}\right] = \mathbf{0}$. By M-estimation theory, $\hat{\boldsymbol{\theta}}^{\mathrm{LS2}} \triangleq (\hat{\boldsymbol{\gamma}}^\top, \{\hat{\boldsymbol{\beta}}^{\mathrm{LS}}\}^\top, \widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{\mathrm{LS}}))^\top$ is asymptotically normal,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}^{\mathrm{LS2}} - \boldsymbol{\theta}_0^{\mathrm{LS2}}) \xrightarrow{D} N(0, \{\boldsymbol{D}^{\mathrm{LS2}}(\boldsymbol{\theta}_0^{\mathrm{LS2}})\}^{-1} \boldsymbol{F}^{\mathrm{LS2}}(\boldsymbol{\theta}_0^{\mathrm{LS2}})[\{\boldsymbol{D}^{\mathrm{LS2}}(\boldsymbol{\theta}_0^{\mathrm{LS2}})\}^{-1}]^\top),$$

where

$$\boldsymbol{D}^{\mathrm{LS2}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{D}_1^{\mathrm{LS2}}(\boldsymbol{\theta}) & \mathbf{0}_{(s+q)\times 1} \\ \boldsymbol{D}_2^{\mathrm{LS2}}(\boldsymbol{\theta}) & -1 \end{pmatrix}, \qquad \boldsymbol{F}^{\mathrm{LS2}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{F}_{11}^{\mathrm{LS2}}(\boldsymbol{\theta}) & \boldsymbol{F}_{12}^{\mathrm{LS2}}(\boldsymbol{\theta}) \\ \{\boldsymbol{F}_{12}^{\mathrm{LS2}}(\boldsymbol{\theta})\}^\top & \boldsymbol{F}_{22}(\boldsymbol{\theta}) \end{pmatrix},$$

with all the quantities defined in the Theorem. The rest is by some algebra.

For $\widehat{V}(d; \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\beta}}^{\mathrm{opt3}})$, the unknown parameters are $\boldsymbol{\theta} = (\boldsymbol{\gamma}^\top, \boldsymbol{\zeta}^\top, \boldsymbol{\beta}^\top, V(d))^\top$. The estimating equation is

$$\boldsymbol{m}^{\mathrm{opt3}}(\boldsymbol{U}_i, \boldsymbol{\theta}) = \begin{pmatrix} S_{\boldsymbol{\gamma}}(A_i, \boldsymbol{X}_i; \boldsymbol{\gamma}) \\ \boldsymbol{m}_3(\boldsymbol{U}_i, \boldsymbol{\theta}) \\ \boldsymbol{m}_4(\boldsymbol{U}_i, \boldsymbol{\theta}) \\ \tilde{\varphi}(Y_i, A_i, \boldsymbol{X}_i, \boldsymbol{\gamma}, \boldsymbol{\beta}) \end{pmatrix},$$

where $\boldsymbol{m}_3(\boldsymbol{U}_i, \boldsymbol{\theta})$, $\boldsymbol{m}_4(\boldsymbol{U}_i, \boldsymbol{\theta})$ are defined in the Theorem. By M-estimation theory,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}^{\mathrm{opt3}} - \boldsymbol{\theta}_0^{\mathrm{opt3}}) \xrightarrow{D} N(0, \{\boldsymbol{D}^{\mathrm{opt3}}(\boldsymbol{\theta}_0^{\mathrm{opt3}})\}^{-1} \boldsymbol{F}^{\mathrm{opt3}}(\boldsymbol{\theta}_0^{\mathrm{opt3}})[\{\boldsymbol{D}^{\mathrm{opt3}}(\boldsymbol{\theta}_0^{\mathrm{opt3}})\}^{-1}]^\top),$$

where

$$\boldsymbol{D}^{\mathrm{opt3}}(\boldsymbol{\theta}) = E\left\{\partial \boldsymbol{m}^{\mathrm{opt3}}(\boldsymbol{U}_i, \boldsymbol{\theta})/\partial \boldsymbol{\theta}^\top\right\} = \begin{pmatrix} \boldsymbol{D}_1^{\mathrm{opt3}}(\boldsymbol{\theta}) & \mathbf{0}_{(q+r+s)\times 1} \\ \boldsymbol{D}_2^{\mathrm{opt3}}(\boldsymbol{\theta}) & -1 \end{pmatrix},$$

$$\boldsymbol{F}^{\mathrm{opt3}}(\boldsymbol{\theta}) = E\left\{\boldsymbol{m}^{\mathrm{opt3}}(\boldsymbol{U}_i, \boldsymbol{\theta}) \boldsymbol{m}^{\mathrm{opt3}}(\boldsymbol{U}_i, \boldsymbol{\theta})^\top\right\} = \begin{pmatrix} \boldsymbol{F}_{11}^{\mathrm{opt3}}(\boldsymbol{\theta}) & \boldsymbol{F}_{12}^{\mathrm{opt3}}(\boldsymbol{\theta}) \\ \{\boldsymbol{F}_{12}^{\mathrm{opt3}}(\boldsymbol{\theta})\}^\top & \boldsymbol{F}_{22}(\boldsymbol{\theta}) \end{pmatrix},$$

with all the quantities defined in the Theorem. The rest is by some algebra.

For $\widehat{V}(d;\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\beta}}^{\text{opt4}})$, the unknown parameters are $\boldsymbol{\theta}=(\boldsymbol{\gamma}^\top,\boldsymbol{\zeta}^\top,\boldsymbol{\beta}^\top,V(d))^\top$. The estimating equation is

$$\boldsymbol{m}^{\text{opt4}}(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)=\begin{pmatrix} S_{\boldsymbol{\gamma}}(A_i,\boldsymbol{X}_i;\boldsymbol{\gamma}) \\ \boldsymbol{m}_3(\boldsymbol{U}_i,\boldsymbol{\theta}) \\ \boldsymbol{m}_5(\boldsymbol{U}_i,\boldsymbol{\theta},g,h) \\ \tilde{\varphi}(Y_i,A_i,\boldsymbol{X}_i,\boldsymbol{\gamma},\boldsymbol{\beta}) \end{pmatrix},$$

where $\boldsymbol{m}_5(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)$ is defined in the Theorem. Let us define $\boldsymbol{M}^{\text{opt4}}(\boldsymbol{\theta},g,h)=E\left\{\boldsymbol{m}^{\text{opt4}}(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)\right\}$ and $\boldsymbol{M}_n^{\text{opt4}}(\boldsymbol{\theta},g,h)=n^{-1}\sum_{i=1}^n \boldsymbol{m}^{\text{opt4}}(\boldsymbol{U}_i,\boldsymbol{\theta},g,h)$. Now we calculate the ordinary and functional derivatives in Theorem S1. The ordinary derivative of $\boldsymbol{M}^{\text{opt4}}(\boldsymbol{\theta},g_0,h_0)$ with respect to $\boldsymbol{\theta}$ is

$$\boldsymbol{\Gamma}_1^{\text{opt4}}(\boldsymbol{\theta},g_0,h_0)=\begin{pmatrix} \boldsymbol{D}_1^{\text{opt4}}(\boldsymbol{\theta}) & \boldsymbol{0}_{(q+r+s)\times 1} \\ \boldsymbol{D}_2^{\text{opt4}}(\boldsymbol{\theta}) & -1 \end{pmatrix}.$$

In addition,

$$\boldsymbol{\Gamma}_2^{\text{opt4}}(\boldsymbol{\theta},g,h)[\bar{g}-g]=\begin{pmatrix} \boldsymbol{0}_{1\times q} \\ \boldsymbol{0}_{1\times r} \\ \boldsymbol{\Gamma}_{21}^{\text{opt4}}(\boldsymbol{\theta},g,h)[\bar{g}-g] \\ 0 \end{pmatrix} \quad \boldsymbol{\Gamma}_3^{\text{opt4}}(\boldsymbol{\theta},g,h)[\bar{h}-h]=\begin{pmatrix} \boldsymbol{0}_{1\times q} \\ \boldsymbol{0}_{1\times r} \\ \boldsymbol{\Gamma}_{31}^{\text{opt4}}(\boldsymbol{\theta},g,h)[\bar{h}-h] \\ 0 \end{pmatrix}$$

where

$$\begin{aligned}\boldsymbol{\Gamma}_{21}^{\text{opt4}}(\boldsymbol{\theta},g,h)[\bar{g}-g]=E\Bigg(&-\{\bar{g}(\boldsymbol{X})-g(\boldsymbol{X})\}I\{d(\boldsymbol{X})=1\}\frac{[R-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}]\,[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}} \\ &\cdot\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}+[\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\right]\Bigg),\\ \boldsymbol{\Gamma}_{31}^{\text{opt4}}(\boldsymbol{\theta},g,h)[\bar{h}-h]=E\Bigg(&-\{\bar{h}(\boldsymbol{X})-h(\boldsymbol{X})\}I\{d(\boldsymbol{X})=-1\}\frac{[R-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}]\,[1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}]}{\pi^2\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}} \\ &\cdot\left[Q_{\boldsymbol{\beta}}\{\boldsymbol{X},d(\boldsymbol{X});\boldsymbol{\beta}\}+[\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_q][\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_q]^{-1}\frac{\pi_{\boldsymbol{\gamma}}\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}{1-\pi\{d(\boldsymbol{X}),\boldsymbol{X};\boldsymbol{\gamma}\}}\right]\Bigg).\end{aligned}$$

Using similar arguments, we have

$$\boldsymbol{\Gamma}_{21}^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0)[\hat{g} - g_0] = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{q}_3(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0) + o_p(n^{-1/2})$$
$$\boldsymbol{\Gamma}_{31}^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0)[\hat{h} - h_0] = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{q}_4(\boldsymbol{U}_i, \boldsymbol{\theta}, g_0, h_0) + o_p(n^{-1/2})$$

where $\boldsymbol{q}_3$ and $\boldsymbol{q}_4$ are defined previously. Combining the above results, we have

$$\sqrt{n}\left\{\boldsymbol{M}_n^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0) + \boldsymbol{\Gamma}_2^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0)[\hat{g} - g_0] + \boldsymbol{\Gamma}_3^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0)[\hat{h} - h_0]\right\} \xrightarrow{D} N(\mathbf{0}, \boldsymbol{F}^{\text{opt4}}(\boldsymbol{\theta})),$$

where

$$\boldsymbol{F}^{\text{opt4}}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{F}_{11}^{\text{opt4}}(\boldsymbol{\theta}) & \boldsymbol{F}_{12}^{\text{opt4}}(\boldsymbol{\theta}) \\ \{\boldsymbol{F}_{12}^{\text{opt4}}(\boldsymbol{\theta})\}^\top & \boldsymbol{F}_{22}(\boldsymbol{\theta}) \end{pmatrix},$$

Based on Theorem S1,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}^{\text{opt4}} - \boldsymbol{\theta}_0^{\text{opt4}}) \xrightarrow{D} N(0, \boldsymbol{\Omega}^{\text{opt4}}(\boldsymbol{\theta}_0^{\text{opt4}})),$$

where $\boldsymbol{\Omega}^{\text{opt4}}(\boldsymbol{\theta}) = \boldsymbol{\Gamma}_1^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0)^{-1} \boldsymbol{F}^{\text{opt4}}(\boldsymbol{\theta})\{\boldsymbol{\Gamma}_1^{\text{opt4}}(\boldsymbol{\theta}, g_0, h_0)^{-1}\}^\top$. The rest of the proof follows by some simple algebra. □

Appendix E: Additional simulation results

Simulation results for Scenario 1 when $n = 100$ or 500 are shown in Figure S1. Simulation results for Scenario 2 when $n = 100$ or 500 are shown in Figure S2. We draw the same conclusions as in the main paper, our proposed method Aug-Improved-DR outperformed other competing methods evidenced by larger value functions and smaller variance in value functions. Table S1 and S2 displays the MSE of different methods in terms of estimating $\boldsymbol{\eta}$. Again, Aug-Improved-DR has superior performance evidenced by smaller MSE.

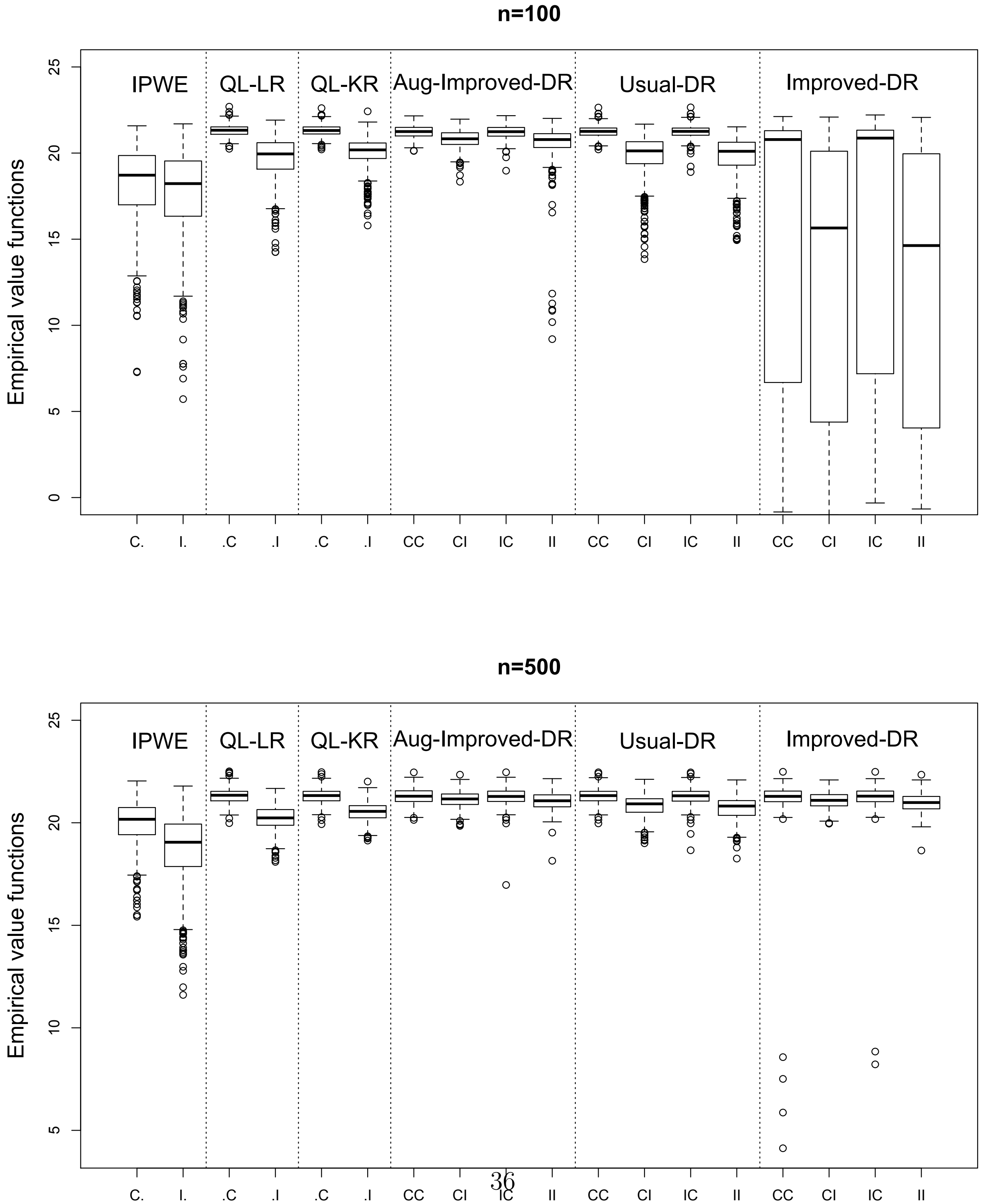

Figure S1: Simulation results for Scenario 1. Value functions over 500 replications. The optimal value is $E\{Y(d^{\text{opt}})\} = 21.32$.

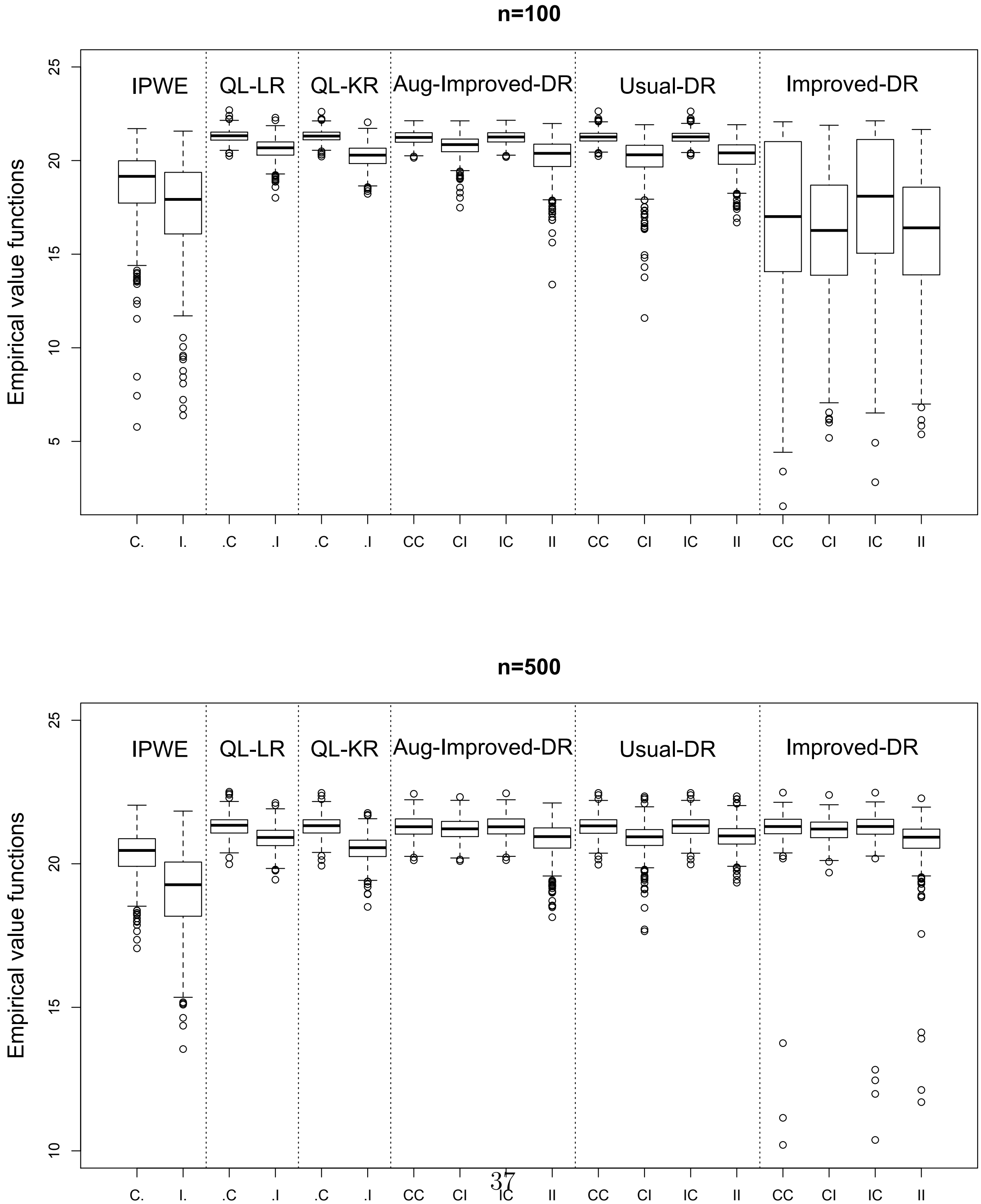

Figure S2: Simulation results for Scenario 2. Value functions over 500 replications. The optimal value is $E\{Y(d^{\rm opt})\} = 21.32$.

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Table S1: Simulation results for Scenario 1. Root MSE for estimating $\boldsymbol{\eta}$. By imposing $||\boldsymbol{\eta}|| = 1$, d^{opt} corresponds to $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4) = (-0.07, -0.71, 0.71, 0, 0)$.

	η_0	η_1	η_2	η_3	η_4	η_0	η_1	η_2	η_3	η_4
					$n = 250$					
	CC: both models correct					CI: only propensity correct				
IPWE	0.22	0.31	0.17	0.29	0.30					
Usual-DR	0.03	0.02	0.02	0.03	0.03	0.15	0.17	0.11	0.22	0.18
Improved-DR	0.15	0.22	0.18	0.06	0.07	0.14	0.18	0.15	0.14	0.16
Aug-Improved-DR	0.03	0.02	0.02	0.03	0.03	**0.10**	**0.07**	**0.07**	**0.11**	**0.11**
	IC: only outcome correct					II: both models incorrect				
IPWE	0.30	0.52	0.14	0.27	0.28					
Usual-DR	0.03	0.03	0.03	0.03	0.03	0.23	0.17	0.10	0.18	0.15
Improved-DR	0.16	0.21	0.16	0.07	0.06	0.16	0.21	0.16	0.16	0.19
Aug-Improved-DR	0.03	0.03	0.03	0.03	0.03	0.11	0.08	0.08	0.12	0.13
					$n = 1000$					
	CC: both models correct					CI: only propensity correct				
IPWE	0.14	0.18	0.11	0.20	0.20					
Usual-DR	0.02	0.01	0.01	0.02	0.02	0.08	0.08	0.07	0.12	0.09
Improved-DR	0.04	0.06	0.03	0.02	0.02	0.06	0.05	0.05	0.07	0.07
Aug-Improved-DR	0.02	0.01	0.01	0.02	0.02	**0.06**	**0.04**	**0.04**	**0.07**	**0.06**
	IC: only outcome correct					II: both models incorrect				
IPWE	0.23	0.53	0.17	0.21	0.22					
Usual-DR	0.02	0.03	0.02	0.02	0.02	0.19	0.11	0.07	0.12	0.09
Improved-DR	0.02	0.01	0.01	0.02	0.02	0.08	0.07	0.06	0.09	0.09
Aug-Improved-DR	0.03	0.03	0.02	0.03	0.03	0.08	0.08	0.06	0.09	0.09

Table S2: Simulation results for Scenario 2. Root MSE for estimating $\boldsymbol{\eta}$. By imposing $||\boldsymbol{\eta}|| = 1$, d^{opt} corresponds to $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4) = (-0.07, -0.71, 0.71, 0, 0)$.

	η_0	η_1	η_2	η_3	η_4	η_0	η_1	η_2	η_3	η_4
					$n = 250$					
	CC: both models correct					CI: only propensity correct				
IPWE	0.21	0.22	0.18	0.26	0.28					
Usual-DR	0.03	0.02	0.02	0.03	0.03	0.16	0.14	0.11	0.19	0.17
Improved-DR	0.16	0.09	0.15	0.06	0.06	0.24	0.12	0.21	0.15	0.13
Aug-Improved-DR	0.03	0.02	0.02	0.03	0.03	**0.09**	**0.07**	**0.06**	**0.11**	**0.10**
	IC: only outcome correct					II: both models incorrect				
IPWE	0.43	0.30	0.16	0.27	0.28					
Usual-DR	0.03	0.02	0.02	0.03	0.03	0.16	0.08	0.11	0.18	0.14
Improved-DR	0.15	0.09	0.15	0.07	0.05	0.29	0.18	0.24	0.20	0.19
Aug-Improved-DR	0.03	0.02	0.02	0.03	0.03	0.16	0.15	0.10	0.18	0.17
					$n = 1000$					
	CC: both models correct					CI: only propensity correct				
IPWE	0.14	0.14	0.12	0.17	0.16					
Usual-DR	0.02	0.01	0.01	0.02	0.02	0.09	0.08	0.06	0.12	0.10
Improved-DR	0.02	0.01	0.01	0.02	0.02	0.04	0.03	0.03	0.05	0.04
Aug-Improved-DR	0.02	0.01	0.01	0.02	0.02	**0.04**	**0.03**	**0.03**	**0.05**	**0.04**
	IC: only outcome correct					II: both models incorrect				
IPWE	0.45	0.26	0.09	0.18	0.18					
Usual-DR	0.02	0.01	0.01	0.02	0.02	0.10	0.05	0.07	0.11	0.09
Improved-DR	0.04	0.02	0.03	0.02	0.02	0.09	0.10	0.07	0.12	0.09
Aug-Improved-DR	0.02	0.01	0.01	0.02	0.02	0.09	0.12	0.08	0.11	0.10