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Supplementary Information for

The Origin of Cooperation

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Supplementary text
Figs. S1 to S2
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Supporting Information Text

In this SI Appendix, we provide proofs for the main results of the paper and some additional theoretical and computational results.

Evolutionary Optimality

We use the growth rate results Theorems 1 and 2 from the main paper to characterize an evolutionarily optimal random offspring vector. That is, we find under what conditions and in what sense $P_{A,T} + P_{B,T}$ under one random offspring vector dominates $P_{A,T} + P_{B,T}$ under another. To formalize this, we introduce two sets of types with different random offspring vectors.

Let our two sets of types be (A, B) and $(\bar{A}, \bar{B})$. The first set has random offspring vector $(x_{A,t}, x_{B,t})$ at time t, and the second set has random offspring vector $(\bar{x}_{A,t}, \bar{x}_{B,t})$ at time t. We assume that $(x_{A,t}, x_{B,t})$ is IID across t and that $(\bar{x}_{A,t}, \bar{x}_{B,t})$ is IID across t. We don't assume anything about the joint distribution of $(x_{A,t}, x_{B,t})$ and $(\bar{x}_{A,t}, \bar{x}_{B,t})$. In particular, $(x_{A,t}, x_{B,t})$ and $(\bar{x}_{A,t}, \bar{x}_{B,t})$ can be correlated.

We assume additionally that all first and second moments exist of $(x_{A,t}, x_{B,t})$ and $(\bar{x}_{A,t}, \bar{x}_{B,t})$, and are denoted as follows.

$$\begin{array}{rr} \mathbb{E}[\log x_{A,t}] = \mu_A & \mathbb{E}[\log \bar{x}_{A,t}] = \bar{\mu}_A \\ \mathbb{E}[\log x_{B,t}] = \mu_B & \mathbb{E}[\log \bar{x}_{B,t}] = \bar{\mu}_B \\ \mathrm{Var}(\log x_{A,t}) = (\sigma_A)^2 & \mathrm{Var}(\log \bar{x}_{A,t}) = (\bar{\sigma}_A)^2 \\ \mathrm{Var}(\log x_{B,t}) = (\sigma_B)^2 & \mathrm{Var}(\log \bar{x}_{B,t}) = (\bar{\sigma}_B)^2 \\ \mathrm{Corr}(\log x_{A,t}, \log x_{B,t}) = \rho & \mathrm{Corr}(\log \bar{x}_{A,t}, \log \bar{x}_{B,t}) = \bar{\rho} \end{array}$$

The population sizes at generation T are

$$\begin{array}{rr} P_{A,T} = \prod_{t=1}^{T} x_{A,t} & \bar{P}_{A,T} = \prod_{t=1}^{T} \bar{x}_{A,t} \\ P_{B,T} = \prod_{t=1}^{T} x_{B,t} & \bar{P}_{B,T} = \prod_{t=1}^{T} \bar{x}_{B,t} \end{array}$$

For the sake of simplicity, let us introduce names for our random offspring vectors. $V := (x_A, x_B)$ and $\bar{V} := (\bar{x}_A, \bar{x}_B)$. Our first result says that the total population grows exponentially faster under V than under $\bar{V}$ if $\max(\mu_A, \mu_B) > \max(\bar{\mu}_A, \bar{\mu}_B)$.

Proposition 1. *Suppose* $\max(\mu_A, \mu_B) > \max(\bar{\mu}_A, \bar{\mu}_B)$*. Then, almost surely,*

$$\lim_{T\to\infty} \frac{\log\left((P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\right)}{T} = \max(\mu_A, \mu_B) - \max(\bar{\mu}_A, \bar{\mu}_B)\,.$$

This means that if $\max(\mu_A, \mu_B) > \max(\bar{\mu}_A, \bar{\mu}_B)$, then $(P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})$ grows like $e^{\gamma T}$ for some $\gamma > 0$. So regardless of any other aspects of the distributions, if $\max(\mu_A, \mu_B) > \max(\bar{\mu}_A, \bar{\mu}_B)$, the ratio of the populations of the two sets grows exponentially fast. So the evolutionarily optimal random offspring vector is one that maximizes μ for each type separately.

But now suppose we restrict ourselves to random offspring vectors that do maximize μ for each type separately. The next result characterizes the evolutionarily optimal vector among this set.

Proposition 2. *Suppose* $\mu_A = \mu_B = \bar{\mu}_A = \bar{\mu}_B = \mu$*. Let* (N_A, N_B) *be normally distributed with means 0, variances* $(\sigma_A)^2 and (\sigma_B)^2$*, respectively, and correlation* ρ*. Let* $(\bar{N}_A, \bar{N}_B)$ *be normally distributed with means* 0*, variances* $(\bar{\sigma}_A)^2$ *and* $(\bar{\sigma}_B)^2$*, respectively, and correlation* $\bar{\rho}$*. Suppose* $\mathbb{E}[\max(N_A, N_B)] > \mathbb{E}[\max(\bar{N}_A, \bar{N}_B)]$*. Then*

$$\liminf_{T\to\infty} \frac{\log\left(\mathbb{E}\left[(P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\right]\right)}{\sqrt{T}} \geq \mathbb{E}[\max(N_A, N_B)] - \mathbb{E}[\max(\bar{N}_A, \bar{N}_B)]\,.$$

So among the behaviors that achieve the highest possible μ, no matter the other aspects of the distributions, if $\mathbb{E}[\max(N_A, N_B)] > \mathbb{E}[\max(\bar{N}_A, \bar{N}_B)]$, then $P_{A,T} + P_{B,T}$ grows exponentially faster under V than under $\bar{V}$. Note that the correlations ρ and $\bar{\rho}$ affect $\mathbb{E}[\max(N_A, N_B)]$ and $\mathbb{E}[\max(\bar{N}_A, \bar{N}_B)]$. In fact, the correlations are the only thing that matters if we fix the variances. Thus, we next specialize our result to correlation.

Corollary 1. *Suppose* $\mu_A = \mu_B = \bar{\mu}_A = \bar{\mu}_B = \mu$ *and* $\sigma_A = \sigma_B = \bar{\sigma}_A = \bar{\sigma}_B = \sigma$*. Suppose* $\rho < \bar{\rho}$*. Then*

$$\liminf_{T\to\infty} \frac{\log\left(\mathbb{E}\left[(P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\right]\right)}{\sqrt{T}} \geq \frac{1}{\sqrt{\pi}}\left(\sqrt{1-\rho} - \sqrt{1-\bar{\rho}}\right)\,.$$

We come to our most interesting result. If $\rho < \bar{\rho}$, then $\mathbb{E}[(P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})]$ grows like $e^{\gamma\sqrt{T}}$ for some $\gamma > 0$. In other words, decreases in correlation cause exponential increases in population size.

Behavioral Implications: Coordination

Assume that both types each have two possible actions, labeled 0 and 1. The variable $x_{j,i}$ is the random offspring of type j choosing action i, for $j = A, B$ and $i = 0, 1$. In general, the random offspring of type A can be written

$$x_A = (x_{A,0})^{1-I_A}(x_{A,1})^{I_A}$$

and the random offspring of type B can be written

$$x_B = (x_{B,0})^{1-I_B}(x_{B,1})^{I_B}.$$

where the indicator functions I_j denote the action of type j, $j = A, B$.

We assume that the $x_{j,i}$'s and I_j's are both random but also assume that the $x_{j,i}$'s are independent of the I_j's. The assumption of independence in this case implies that the individuals' choice of action is independent of the consequence of the action, which has the interpretation that the individuals have no intelligence. Denote the means, variances, and covariances of the actions x by:

$$\mu_{j,i} \equiv \mathbb{E}[\log x_{j,i}] \quad , \quad (\sigma_{j,i})^2 \equiv \mathrm{Var}(\log x_{j,i}) \quad , \quad \varsigma_{ii'} \equiv \mathrm{Cov}[\log x_{A,i}, \log x_{B,i'}] \qquad [1]$$

Observe the distinction between (for example) μ_A as defined in the main paper and $\mu_{A,0}$ and $\mu_{A,1}$; μ_A is a function of $\mu_{A,0}$, $\mu_{A,1}$ and I_A. Assume that

$$\mu_{A,0} = \mu_{A,1} = \mu_{B,0} = \mu_{B,1} \equiv \mu\,. \qquad [2]$$

so that no action dominates on the basis of μ.

Note that the choice of I_A and I_B completely determines the random offspring vector (x_A, x_B). We now apply Theorem 1 and 2 from the paper (or more precisely, Propositions 1 and 2 from the SI) to characterize the evolutionarily optimal behaviors I_A and I_B. By design, $\max(\mu_A, \mu_B)$ is the same for any I_A and I_B. So the evolutionarily optimal I_A and I_B will be a maximizer of $\mathbb{E}[\max(N_A, N_B)]$.

We can compute

$$\mathbb{E}[\max(N_A, N_B)] = \mu + \frac{1}{\sqrt{2\pi}}\sqrt{\sum_{i,i'\in\{0,1\}} p_{ii'}((\sigma_{A,i})^2 + (\sigma_{B,i'})^2 - 2\varsigma_{ii'})}$$

where $p_{ii'}$ is the probability that $I_A = i$ and $I_B = i'$ are jointly chosen. Thus the evolutionarily optimal I_A and I_B will have all of their probability mass on combinations $(I_A = i, I_B = i')$ that maximize $((\sigma_{A,i})^2 + (\sigma_{B,i'})^2 - 2\varsigma_{ii'})$. In other words, even if there is uncertainty in I_A and I_B marginally, jointly their probabilities are certain.

For example, suppose that $\sigma_{A,0} = \sigma_{A,1} = \sigma_{B,0} = \sigma_{B,1}$ and $\varsigma_{00} = \varsigma_{11} > \varsigma_{01}$. Then $(i = 1, i' = 1)$ and $(i = 0, i' = 0)$ would both maximize $((\sigma_{A,i})^2 + (\sigma_{B,i'})^2 - 2\varsigma_{ii'})$. So at an evolutionarily optimal I_A and I_B, I_A could be uncertain, meaning type A individuals picked action 0 some of the time and 1 some of the time, and I_B could be uncertain, meaning type B individuals picked action 0 some of the time and 1 some of the time. However, at any evolutionarily optimal I_A and I_B, type A individuals would only pick 0 when type B individuals picked 0, and type A individuals would only pick 1 when type B individuals picked 1. This is coordination.

Additional Simulation Results

In all simulations in the paper and the SI Appendix, we compute expectations as an average over 10,000 samples. We generate paths of stochastic processes using the Euler Method with a time increment of 0.001. All populations start with $P_{A,0} = 1$ and $P_{B,0} = 1$.

A. Additional Simulation Results for Basic Model. Table S1 is a reproduction of Table 1 from the paper with a larger range of μ values. The trends mentioned in the paper are consistent over this larger range as well. Notably, decreasing ρ from 0.5 to -0.5 is similar to increasing μ by 0.09. Table S2 halves σ, the standard deviation of log offspring. Table S3 doubles σ. It is no surprise that increasing σ increases the importance of ρ relative to μ, since σ controls how much variation there is to be correlated.

B. Additional Simulation Results for Model with Idiosyncratic Risk. In this section, we fix systematic risk at its level in Table 1 (or Table S1) and add a component of idiosyncratic risk. In Table S4 we add a deterministic component of e^{-1} to each individual's offspring. In Table S5, we add a deterministic component of e^0 to each individual's offspring. In Table S6, we add a random component to each individual's offspring that is lognormally distributed with $\mu_{\mathrm{id}} = -1$ and $\sigma_{\mathrm{id}} = \sqrt{2}$. The mean of such a lognormally distributed component is 1, so the results in Table S6 should be very similar to the results in Table S5. What we find in general from this experiment is that idiosyncratic risk, even large amounts, decreases the importance of ρ relative to μ, but only by a small amount. For example, to go from Table 1 (or Table S1) to Table S5 or S6, we add enough idiosyncratic risk to increase finite-time growth rate by over 100 percentage points. Yet decreasing ρ from 0.5 to -0.5 is still comparable to increasing μ by 0.07, as opposed to 0.09 originally.

Table S1. Simulated population growth rate

μ \ ρ	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	20.3%	20.0%	19.1%	18.6%	17.4%	16.2%	15.8%	15.0%	14.1%	13.0%	12.1%	11.2%	10.3%	9.0%	7.8%	7.0%	4.9%	4.3%	1.7%
0.02	22.9%	22.6%	21.6%	21.0%	19.9%	18.9%	18.1%	17.7%	16.3%	15.6%	14.3%	13.6%	12.5%	11.2%	9.9%	8.6%	7.4%	6.2%	4.8%
0.04	25.2%	24.7%	24.1%	23.0%	22.0%	21.5%	20.5%	19.6%	18.4%	18.2%	17.1%	15.7%	14.3%	13.8%	12.8%	10.8%	10.1%	7.8%	6.3%
0.06	28.2%	27.0%	26.3%	25.8%	24.7%	24.3%	22.9%	22.0%	20.8%	20.4%	19.0%	17.7%	16.5%	16.1%	14.4%	13.4%	11.7%	10.0%	8.5%
0.08	30.4%	29.7%	29.1%	28.1%	27.5%	27.1%	25.6%	23.9%	23.9%	22.5%	21.3%	20.5%	19.4%	18.6%	16.8%	14.8%	14.4%	12.8%	11.0%
0.10	33.4%	32.5%	31.0%	30.2%	29.7%	28.9%	27.4%	26.2%	25.9%	24.5%	23.7%	23.4%	21.4%	20.6%	19.2%	17.8%	16.0%	14.7%	12.8%
0.12	36.0%	34.9%	34.1%	33.7%	32.5%	31.5%	30.3%	30.1%	28.4%	27.6%	26.4%	25.9%	24.0%	23.3%	21.8%	20.3%	19.4%	17.3%	15.1%
0.14	38.8%	37.7%	36.7%	35.6%	34.7%	33.7%	33.4%	31.7%	31.1%	30.4%	28.6%	27.3%	27.0%	25.1%	23.9%	23.5%	21.0%	18.9%	17.0%
0.16	41.6%	40.9%	40.0%	39.0%	37.8%	37.1%	35.9%	34.7%	33.8%	32.4%	31.5%	30.1%	28.5%	27.8%	26.2%	25.2%	23.3%	22.1%	20.1%
0.18	44.3%	43.1%	42.4%	41.4%	40.7%	39.5%	38.9%	37.7%	36.2%	35.5%	34.5%	32.6%	31.9%	30.7%	28.8%	27.2%	25.9%	24.7%	22.2%
0.20	47.3%	46.4%	45.6%	44.3%	43.6%	43.0%	42.1%	40.6%	39.2%	37.6%	37.4%	36.3%	34.5%	33.3%	31.9%	30.8%	28.2%	26.9%	25.3%
0.22	50.1%	49.8%	48.6%	46.9%	46.3%	45.2%	44.1%	42.9%	42.2%	41.3%	40.1%	38.6%	37.4%	35.8%	34.7%	32.9%	31.1%	28.7%	27.3%
0.24	53.2%	52.8%	51.0%	50.0%	49.2%	47.9%	47.4%	45.7%	44.8%	43.0%	43.4%	41.1%	40.0%	39.5%	37.3%	35.4%	33.5%	31.9%	29.8%
0.26	56.1%	55.5%	54.0%	53.8%	52.1%	50.5%	50.3%	48.7%	47.8%	47.2%	45.1%	43.6%	42.1%	41.8%	39.8%	38.4%	36.5%	34.8%	32.8%
0.28	60.0%	58.6%	57.4%	56.3%	55.4%	53.9%	53.5%	52.3%	51.7%	50.1%	47.9%	47.3%	45.7%	43.6%	42.8%	41.1%	39.2%	37.0%	35.0%
0.30	62.6%	61.3%	60.8%	59.5%	58.4%	57.3%	56.2%	55.0%	53.9%	52.8%	51.0%	50.1%	48.6%	47.5%	45.8%	44.1%	42.2%	40.6%	37.5%
0.32	66.4%	65.2%	64.3%	63.0%	62.1%	60.6%	59.8%	58.2%	56.9%	55.9%	54.4%	53.2%	51.1%	49.9%	48.9%	46.5%	44.8%	43.3%	41.0%
0.34	69.1%	68.2%	67.5%	66.2%	65.3%	64.3%	62.1%	60.9%	60.6%	58.6%	57.3%	56.3%	54.7%	53.6%	50.8%	49.5%	47.7%	45.8%	43.5%
0.36	72.4%	71.1%	71.3%	69.6%	68.8%	67.4%	65.6%	64.0%	63.1%	62.4%	61.0%	59.3%	58.3%	56.1%	54.9%	52.8%	50.6%	49.2%	47.1%
0.38	75.9%	75.5%	73.9%	72.6%	71.6%	71.0%	69.4%	68.6%	66.8%	65.4%	63.6%	62.6%	60.6%	59.7%	57.6%	55.7%	53.8%	51.0%	49.3%
0.40	79.8%	79.4%	77.2%	76.1%	75.1%	74.0%	72.9%	72.0%	69.7%	68.1%	67.0%	65.4%	63.9%	62.8%	62.2%	59.4%	57.0%	53.9%	52.3%
0.42	83.6%	82.1%	81.7%	80.4%	78.5%	77.6%	76.4%	75.0%	73.6%	72.6%	69.8%	69.4%	67.9%	66.0%	63.9%	62.3%	60.6%	57.4%	55.9%
0.44	87.8%	86.1%	85.7%	84.0%	82.5%	81.1%	79.8%	77.9%	76.2%	75.9%	73.9%	72.0%	71.5%	68.6%	67.4%	65.7%	63.1%	61.0%	58.6%
0.46	90.7%	90.0%	89.0%	87.9%	85.9%	85.0%	82.6%	81.6%	81.0%	78.7%	76.9%	76.0%	73.9%	72.8%	71.4%	67.2%	66.9%	64.6%	61.3%
0.48	94.5%	93.6%	92.5%	90.8%	90.1%	88.8%	87.2%	86.1%	84.6%	82.1%	80.8%	79.7%	77.9%	76.8%	74.4%	72.7%	70.1%	67.3%	65.3%
0.50	98.5%	97.2%	96.1%	94.4%	94.0%	91.7%	90.0%	90.1%	87.2%	86.9%	85.5%	82.6%	81.5%	79.2%	78.0%	75.8%	74.1%	70.9%	68.7%
0.52	103.4%	101.2%	100.1%	99.6%	97.7%	95.1%	95.1%	93.4%	92.2%	90.3%	88.4%	86.7%	85.4%	83.6%	80.5%	79.4%	77.1%	75.1%	72.1%
0.54	106.7%	106.0%	104.4%	103.4%	101.7%	101.1%	99.2%	96.6%	95.5%	94.1%	93.0%	90.1%	89.6%	87.1%	84.9%	83.1%	80.9%	77.3%	75.8%
0.56	110.9%	109.9%	108.2%	107.1%	105.0%	103.9%	103.4%	100.8%	99.9%	97.6%	95.8%	94.9%	92.9%	90.2%	88.8%	87.1%	84.9%	80.4%	78.4%
0.58	116.2%	113.9%	112.3%	111.2%	110.6%	108.2%	107.5%	105.7%	103.5%	101.6%	100.2%	98.3%	96.8%	95.1%	92.7%	90.0%	88.2%	85.5%	82.0%
0.60	120.5%	118.2%	117.1%	115.7%	113.7%	112.6%	111.5%	108.7%	107.7%	105.9%	104.2%	102.4%	100.9%	98.3%	98.0%	94.3%	91.7%	90.0%	86.5%
0.62	124.2%	122.0%	120.9%	120.0%	119.2%	116.3%	115.6%	114.1%	112.0%	109.9%	109.7%	107.0%	104.3%	102.6%	100.4%	98.9%	96.1%	93.1%	90.0%
0.64	128.3%	127.3%	126.1%	124.2%	123.0%	120.7%	119.2%	117.8%	116.6%	114.4%	112.3%	111.4%	108.3%	106.6%	103.7%	101.9%	100.7%	98.6%	93.9%
0.66	133.6%	132.0%	130.5%	128.4%	126.9%	126.3%	124.7%	122.3%	121.6%	118.8%	117.6%	115.5%	112.2%	111.5%	108.6%	106.2%	104.4%	100.3%	98.2%
0.68	137.9%	136.4%	134.7%	133.6%	132.4%	128.8%	127.7%	126.2%	124.4%	122.1%	120.3%	119.9%	117.2%	116.0%	112.1%	110.4%	107.7%	105.6%	102.1%
0.70	143.8%	141.1%	139.9%	137.8%	137.4%	134.8%	133.3%	131.8%	129.1%	127.1%	126.0%	123.4%	122.3%	120.4%	117.1%	114.1%	112.4%	109.6%	106.5%
0.72	147.9%	146.9%	144.4%	143.2%	141.7%	139.6%	138.4%	135.2%	134.5%	131.8%	130.4%	128.6%	126.2%	124.1%	121.3%	119.5%	115.5%	113.5%	109.5%
0.74	152.4%	151.5%	149.9%	147.5%	146.9%	144.1%	142.6%	141.3%	139.5%	136.9%	134.8%	132.6%	130.2%	127.5%	126.5%	123.4%	120.5%	118.5%	113.9%
0.76	157.2%	156.4%	154.4%	153.0%	152.2%	149.4%	147.6%	145.6%	142.7%	142.6%	139.6%	138.6%	135.3%	133.1%	131.9%	128.1%	124.7%	121.6%	119.9%
0.78	164.5%	161.8%	159.8%	158.3%	156.4%	153.5%	153.7%	150.1%	148.7%	148.0%	144.8%	143.1%	140.2%	138.2%	135.7%	131.5%	129.8%	125.8%	123.2%
0.80	168.7%	166.0%	165.6%	162.5%	161.9%	160.3%	157.5%	156.0%	153.8%	152.3%	150.6%	145.8%	145.1%	142.9%	139.9%	137.5%	134.8%	131.5%	127.0%
0.82	172.9%	172.5%	169.9%	168.3%	166.9%	164.5%	162.9%	160.7%	159.1%	156.8%	154.4%	152.6%	149.8%	146.8%	145.5%	142.2%	138.4%	135.8%	132.3%
0.84	179.0%	177.8%	176.5%	174.2%	172.9%	169.4%	167.9%	165.2%	164.5%	161.6%	159.3%	157.0%	154.7%	153.7%	149.9%	146.1%	142.3%	140.8%	137.7%
0.86	185.5%	183.1%	181.6%	180.0%	177.3%	175.4%	173.0%	170.4%	170.3%	166.8%	164.8%	162.8%	159.8%	156.7%	153.6%	150.9%	149.1%	144.6%	140.4%
0.88	191.3%	189.2%	186.4%	184.8%	181.9%	181.6%	180.1%	176.4%	174.4%	172.3%	170.0%	167.1%	164.8%	164.9%	160.9%	156.8%	153.3%	151.2%	145.8%
0.90	196.7%	194.3%	192.3%	190.5%	187.6%	186.2%	185.5%	182.8%	181.9%	179.4%	175.4%	173.7%	170.5%	168.1%	165.1%	163.2%	158.6%	155.1%	150.2%
0.92	202.5%	200.3%	199.0%	195.7%	194.7%	192.2%	191.8%	186.7%	185.4%	184.8%	181.6%	177.3%	175.5%	172.9%	171.6%	167.2%	163.4%	159.6%	155.3%
0.94	208.8%	207.2%	204.6%	203.0%	200.0%	198.8%	196.9%	193.6%	192.0%	188.3%	186.9%	183.9%	183.0%	179.4%	175.3%	171.8%	170.3%	164.2%	161.5%
0.96	215.0%	212.9%	210.0%	209.1%	205.8%	205.0%	202.2%	199.0%	197.4%	195.6%	192.2%	189.8%	186.5%	183.5%	181.3%	178.3%	176.5%	171.7%	167.4%
0.98	222.6%	219.8%	216.4%	214.6%	213.1%	210.6%	207.8%	206.6%	203.6%	203.0%	198.4%	197.5%	193.6%	191.2%	187.0%	184.8%	180.2%	176.2%	171.4%
1.00	227.8%	224.2%	224.8%	221.1%	219.0%	217.9%	214.8%	213.6%	210.2%	207.7%	204.4%	201.9%	199.6%	197.0%	192.4%	189.9%	184.6%	182.5%	176.4%

The number of offspring for both types follows a lognormal distribution with the stated μ and ρ, and with $\sigma = 1.0$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

C. Additional Simulation Results for Model with Density Dependence. We reproduce Tables 2-4 from the paper with $r = 1.4$ in Table S7 and $r = 1.5$ in S8 to match the range of μ in Table S1. At first glance, it seems ρ plays a smaller role when r is large. However, this is a bit of a red herring because all growth rates go to zero over time with density dependence, and ρ has actually become much more important compared to r. To get the same effect as increasing r from 1.4 to 1.5, one needs only decrease ρ from 0.5 to about 0.2 when $K = 5$ or to about 0.0 when $K = 80$. As a reminder, to get the same effect as increasing r from 0.5 to 0.6, one needs to decrease ρ from 0.5 to -0.1 when $K = 5$ or 0.5 to -0.3 when $K = 80$. Figures S1 and S2 show the population growth over time when $r = 1.5$ for $K = 10$ and $K = 40$ respectively. The impact of correlation is remarkably consistent over time, even in populations whose growth has plateaued.

Proofs

Lemma 1 (Log-Sum-Exp inequality)**.**

$$\max(x_1, \ldots, x_m) \leq \log(\exp(x_1) + \ldots + \exp(x_m)) \leq \max(x_1, \ldots, x_m) + \log(m)\,.$$

Proof. The left inequality is because

$$\max(x_1, \ldots, x_m) = \log(\max(\exp(x_1), \ldots, \exp(x_m))) \quad [3]$$
$$\leq \log(\exp(x_1) + \ldots + \exp(x_m)) \quad [4]$$

The right inequality is because

$$\exp(x_1) + \ldots + \exp(x_m) \leq m \max(\exp(x_1), \ldots, \exp(x_m)) \quad [5]$$

□

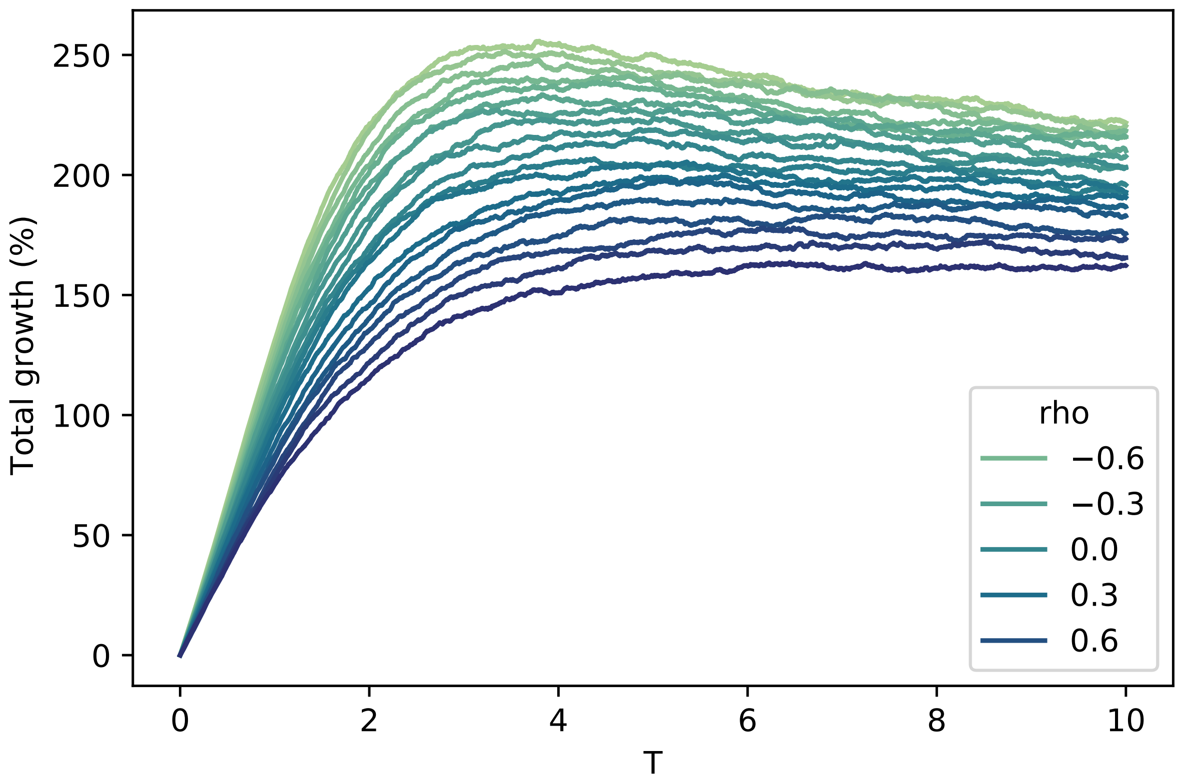

Fig. S1. Total density-dependent population growth vs. time ($K = 10$). Values of ρ range from -0.9 (lightest) to 0.9 (darkest). The number of offspring for both types follows Eq. (25) and Eq. (26) from the main article with $r = 1.5$, $K = 10$, $L = 10$, and $s = 1.0$.

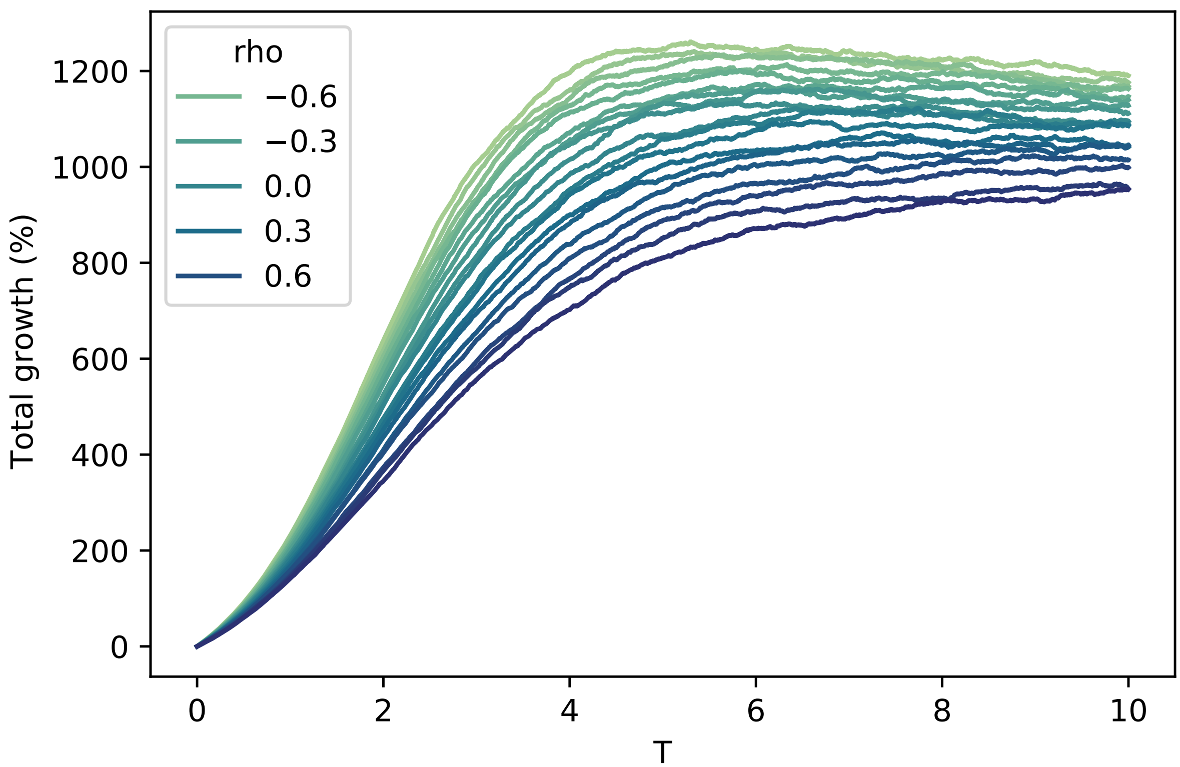

Fig. S2. Total density-dependent population growth vs. time ($K = 40$). Values of ρ range from -0.9 (lightest) to 0.9 (darkest). The number of offspring for both types follows Eq. (25) and Eq. (26) from the main article with $r = 1.5$, $K = 40$, $L = 40$, and $s = 1.0$.

Table S2. Simulated population growth rate ($\sigma = 0.5$)

μ \ ρ	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	7.7%	7.2%	6.8%	6.5%	6.4%	6.0%	5.8%	5.2%	5.1%	4.5%	4.2%	3.8%	3.5%	3.3%	2.4%	2.0%	1.7%	1.3%	0.3%
0.02	9.8%	9.3%	9.1%	8.7%	8.5%	8.0%	7.8%	7.4%	7.2%	6.7%	6.2%	6.0%	5.3%	4.9%	4.5%	4.2%	3.7%	3.0%	2.7%
0.04	11.9%	11.5%	11.2%	11.0%	10.6%	10.3%	9.9%	9.5%	9.1%	8.9%	8.3%	8.0%	7.7%	7.2%	6.7%	6.4%	5.9%	5.0%	4.7%
0.06	14.3%	13.8%	13.4%	13.2%	12.9%	12.3%	12.2%	12.0%	11.4%	11.1%	10.6%	10.1%	9.9%	9.4%	8.9%	8.1%	8.1%	7.6%	6.9%
0.08	16.4%	16.3%	15.8%	15.4%	15.0%	15.0%	14.4%	14.1%	13.6%	13.3%	12.9%	12.5%	12.1%	11.7%	11.2%	10.9%	10.1%	9.3%	9.1%
0.10	19.1%	18.6%	18.3%	17.9%	17.5%	17.2%	16.8%	16.3%	16.1%	15.5%	15.0%	14.7%	14.2%	13.6%	13.0%	12.9%	12.6%	11.9%	11.4%
0.12	21.4%	20.9%	20.6%	20.2%	20.1%	19.6%	19.1%	18.6%	18.4%	17.9%	17.2%	17.1%	16.5%	16.2%	15.4%	14.9%	14.7%	14.0%	13.1%
0.14	24.0%	23.3%	22.8%	22.6%	22.1%	22.2%	21.3%	21.1%	20.7%	20.4%	19.9%	19.5%	19.0%	18.6%	17.7%	17.5%	17.1%	16.0%	15.9%
0.16	26.2%	25.7%	25.6%	25.3%	24.9%	24.4%	23.9%	23.7%	23.1%	22.9%	22.5%	21.8%	21.1%	20.5%	20.5%	19.9%	19.2%	18.8%	18.0%
0.18	28.7%	28.4%	28.0%	27.7%	27.1%	27.0%	26.4%	26.1%	25.6%	25.3%	24.5%	24.3%	23.9%	23.1%	22.9%	22.0%	21.8%	20.9%	20.4%
0.20	31.4%	31.1%	30.9%	30.3%	30.1%	29.6%	29.5%	28.7%	27.9%	27.5%	27.4%	27.0%	26.4%	26.1%	25.3%	24.7%	24.3%	23.5%	22.8%
0.22	33.8%	33.5%	33.2%	32.9%	32.5%	32.0%	31.5%	31.2%	31.0%	30.4%	29.8%	29.5%	29.0%	28.6%	27.6%	27.4%	26.7%	26.2%	25.2%
0.24	36.8%	36.3%	35.6%	35.6%	35.2%	34.5%	34.1%	33.7%	33.6%	33.0%	32.3%	32.1%	31.6%	30.8%	30.6%	30.0%	29.3%	28.6%	27.7%
0.26	39.7%	39.1%	38.5%	38.3%	38.1%	37.4%	37.0%	36.6%	36.3%	35.8%	35.5%	34.5%	33.9%	33.5%	33.0%	32.3%	32.0%	31.4%	30.4%
0.28	42.4%	42.0%	41.4%	41.0%	40.7%	40.3%	39.8%	39.4%	38.9%	38.6%	37.8%	37.1%	36.7%	36.2%	35.8%	35.3%	34.6%	33.7%	33.0%
0.30	45.2%	44.7%	44.4%	43.9%	43.8%	42.9%	42.7%	42.4%	41.5%	41.0%	40.5%	39.7%	39.6%	38.8%	38.4%	38.0%	37.1%	36.6%	36.0%
0.32	48.0%	47.5%	47.2%	46.7%	46.4%	45.5%	45.2%	44.8%	44.8%	44.2%	43.7%	43.0%	42.4%	41.9%	40.9%	40.9%	40.5%	39.5%	38.3%
0.34	51.0%	50.6%	50.4%	50.0%	49.1%	48.8%	48.2%	47.9%	47.6%	46.9%	46.6%	46.0%	45.3%	44.7%	44.2%	43.8%	42.9%	42.2%	41.1%
0.36	54.3%	53.7%	53.4%	53.0%	52.4%	51.8%	51.5%	51.3%	50.3%	50.3%	49.2%	48.8%	48.2%	47.8%	46.9%	46.1%	45.8%	45.5%	44.3%
0.38	57.2%	56.9%	56.4%	56.0%	55.5%	54.6%	54.3%	54.0%	53.3%	53.1%	52.5%	51.5%	51.4%	50.7%	50.3%	49.5%	48.9%	47.8%	47.1%
0.40	60.4%	60.0%	59.7%	59.1%	58.6%	58.2%	57.3%	57.1%	56.3%	56.1%	55.4%	54.9%	54.1%	53.8%	53.2%	52.3%	51.9%	50.9%	50.5%
0.42	63.9%	63.3%	62.5%	62.5%	61.6%	61.5%	60.7%	60.4%	59.5%	59.1%	58.5%	58.3%	57.7%	56.7%	56.1%	55.5%	54.4%	54.0%	53.3%
0.44	66.9%	66.5%	65.8%	65.4%	64.8%	64.7%	63.9%	63.6%	62.9%	62.0%	61.8%	61.3%	60.8%	59.9%	59.2%	58.6%	57.7%	57.0%	56.3%
0.46	70.6%	70.0%	69.9%	68.9%	68.6%	68.1%	67.3%	66.6%	66.3%	65.6%	65.0%	64.5%	63.9%	63.0%	62.4%	61.7%	60.6%	60.5%	59.0%
0.48	73.9%	73.3%	73.0%	72.4%	71.8%	71.5%	70.8%	70.0%	69.6%	69.1%	68.6%	67.6%	67.1%	66.4%	65.9%	64.9%	64.3%	63.2%	62.3%
0.50	77.6%	76.9%	76.1%	75.8%	75.0%	74.8%	73.9%	73.7%	73.2%	72.2%	71.8%	71.4%	70.4%	69.5%	69.2%	68.3%	68.0%	66.2%	66.0%
0.52	80.8%	80.3%	80.0%	79.4%	78.5%	78.3%	78.1%	77.2%	76.8%	76.2%	75.3%	74.8%	74.0%	73.1%	72.9%	72.1%	71.0%	70.2%	68.7%
0.54	84.8%	84.1%	83.7%	82.9%	82.4%	82.1%	81.0%	80.7%	80.2%	79.2%	78.8%	78.2%	77.3%	76.4%	76.4%	75.2%	74.4%	73.5%	73.3%
0.56	88.4%	87.8%	87.3%	86.8%	86.0%	85.5%	84.7%	84.6%	83.6%	82.8%	82.5%	81.8%	81.2%	80.1%	79.5%	79.1%	77.6%	77.3%	76.3%
0.58	92.0%	91.7%	91.0%	90.4%	90.0%	89.4%	88.7%	88.2%	87.4%	86.8%	85.7%	85.2%	84.2%	83.8%	83.0%	82.3%	81.9%	80.7%	79.4%
0.60	96.1%	95.2%	94.9%	94.4%	93.9%	93.3%	92.5%	91.7%	91.2%	90.6%	89.7%	88.8%	88.5%	87.7%	87.1%	86.2%	85.0%	84.6%	83.3%
0.62	99.8%	99.6%	99.1%	98.3%	97.7%	96.8%	96.5%	95.7%	95.3%	94.8%	93.8%	92.7%	92.3%	91.7%	90.9%	90.1%	88.8%	88.1%	86.3%
0.64	103.9%	103.5%	102.6%	102.1%	101.7%	101.2%	100.3%	99.6%	99.4%	98.0%	97.2%	96.7%	95.9%	95.2%	94.5%	93.7%	92.6%	92.1%	91.3%
0.66	107.8%	107.5%	107.2%	106.0%	105.4%	105.0%	104.4%	103.7%	102.7%	102.3%	101.4%	101.1%	100.0%	99.1%	98.3%	97.6%	96.6%	95.8%	94.9%
0.68	112.5%	111.7%	111.6%	110.3%	110.0%	109.2%	108.2%	108.1%	106.7%	106.3%	105.9%	105.1%	103.8%	103.1%	102.4%	101.8%	100.7%	100.1%	98.3%
0.70	116.6%	116.4%	115.6%	114.6%	114.1%	113.1%	112.7%	112.2%	111.2%	110.7%	109.8%	109.4%	108.3%	107.5%	106.7%	105.3%	104.4%	104.1%	102.1%
0.72	120.8%	120.3%	119.7%	118.9%	118.3%	118.1%	117.1%	116.3%	115.5%	114.8%	114.0%	113.5%	112.7%	111.9%	110.7%	109.9%	108.6%	107.7%	106.2%
0.74	125.2%	124.5%	124.2%	123.5%	122.9%	122.2%	121.3%	120.4%	119.7%	119.1%	118.3%	117.4%	116.7%	116.1%	115.0%	114.3%	113.0%	112.1%	111.0%
0.76	129.9%	129.6%	129.0%	127.7%	127.8%	126.5%	125.7%	125.5%	124.1%	123.5%	122.4%	122.3%	121.4%	120.5%	119.3%	118.1%	117.8%	115.5%	115.1%
0.78	134.5%	133.8%	133.2%	132.6%	132.1%	131.2%	130.3%	129.7%	128.8%	128.5%	127.1%	126.4%	125.6%	124.5%	123.9%	122.4%	122.4%	120.6%	119.1%
0.80	139.5%	138.7%	138.2%	137.4%	136.8%	136.3%	135.0%	134.5%	134.0%	132.8%	131.5%	131.0%	130.3%	129.3%	127.8%	127.4%	126.4%	125.7%	123.6%
0.82	144.7%	143.2%	143.1%	142.4%	141.3%	140.7%	140.0%	138.5%	138.6%	137.1%	136.6%	136.3%	134.7%	134.2%	133.1%	131.9%	130.8%	129.1%	128.3%
0.84	149.2%	148.2%	147.7%	146.9%	145.7%	145.2%	144.8%	144.0%	142.7%	142.4%	141.3%	140.4%	140.2%	138.7%	137.4%	136.2%	135.5%	134.6%	133.2%
0.86	154.3%	153.6%	152.6%	152.0%	151.4%	150.4%	150.0%	148.6%	147.9%	146.8%	146.7%	145.2%	144.6%	144.0%	142.2%	141.4%	140.3%	139.5%	138.0%
0.88	159.5%	158.5%	158.3%	157.8%	156.4%	155.4%	154.4%	153.6%	153.4%	152.2%	151.1%	150.0%	149.9%	148.7%	147.1%	146.6%	144.8%	143.7%	142.5%
0.90	164.2%	163.9%	163.1%	162.2%	161.4%	160.8%	159.7%	159.1%	158.6%	157.0%	156.6%	155.4%	154.3%	153.3%	152.4%	150.6%	149.2%	148.7%	147.5%
0.92	170.0%	168.9%	168.5%	167.2%	167.2%	165.6%	165.4%	164.1%	163.0%	162.6%	161.4%	160.7%	160.1%	158.4%	157.2%	156.5%	154.9%	154.3%	152.4%
0.94	175.5%	174.7%	173.8%	173.0%	172.4%	171.4%	171.2%	169.9%	168.9%	167.5%	167.0%	165.5%	164.7%	163.4%	162.3%	161.3%	160.1%	158.8%	157.4%
0.96	180.8%	180.6%	179.6%	179.0%	177.3%	176.8%	175.4%	175.0%	174.0%	173.0%	172.1%	171.6%	170.6%	168.8%	167.8%	166.4%	165.2%	164.4%	162.4%
0.98	186.8%	186.2%	184.7%	183.9%	183.1%	182.1%	181.7%	180.2%	179.8%	178.6%	177.6%	176.4%	175.1%	174.0%	173.3%	171.7%	170.2%	170.3%	168.0%
1.00	192.3%	191.6%	190.5%	189.9%	189.3%	187.6%	187.0%	186.6%	185.3%	184.7%	183.7%	181.7%	180.7%	179.8%	179.7%	177.5%	176.7%	175.3%	173.6%

The number of offspring for both types follows a lognormal distribution with the stated μ and ρ, and with $\sigma = 0.5$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

Proof of Theorem 1.

$$\log(P_{A,T} + P_{B,T}) = \log\left(\exp\left(\sum_{t=1}^{T} \log x_{A,t}\right) + \exp\left(\sum_{t=1}^{T} \log x_{B,t}\right)\right). \quad [6]$$

Using Lemma 1 with $m = 2$,

$$\max\left(\sum_{t=1}^{T} \log x_{A,t}, \sum_{t=1}^{T} \log x_{B,t}\right) \leq \log(P_{A,T} + P_{B,T}) \leq \max\left(\sum_{t=1}^{T} \log x_{A,t}, \sum_{t=1}^{T} \log x_{B,t}\right) + \log(2). \quad [7]$$

Dividing by T on all sides and using the strong law of large numbers, we have the desired result. □

Proof of Theorem 2. Again,

$$\log(P_{A,T} + P_{B,T}) - T\mu = \log\left(\exp\left(\sum_{t=1}^{T} \log x_{A,t}\right) + \exp\left(\sum_{t=1}^{T} \log x_{B,t}\right)\right) - T\mu. \quad [8]$$

Using the left hand inequality of Lemma 1 with $m = 2$,

$$\max\left(\sum_{t=1}^{T} \log x_{A,t}, \sum_{t=1}^{T} \log x_{B,t}\right) - T\mu \leq \log(P_{A,T} + P_{B,T}) - T\mu, \quad [9]$$

Table S3. Simulated population growth rate ($\sigma = 2.0$)

μ \ ρ	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	53.1%	51.9%	48.8%	46.9%	45.9%	42.1%	41.5%	38.8%	36.8%	34.0%	31.8%	31.1%	27.0%	23.7%	21.8%	18.7%	16.2%	11.1%	7.2%
0.02	57.0%	53.9%	52.8%	50.4%	46.9%	45.5%	44.2%	42.2%	39.1%	36.9%	34.7%	32.4%	28.0%	26.2%	25.0%	20.1%	16.0%	13.7%	8.8%
0.04	60.0%	57.2%	55.7%	53.2%	51.1%	50.4%	46.7%	44.4%	42.2%	39.9%	36.4%	35.3%	32.8%	28.3%	26.5%	22.8%	19.2%	14.8%	11.3%
0.06	63.9%	60.5%	58.6%	57.2%	52.8%	51.9%	51.3%	48.6%	45.2%	41.4%	39.9%	37.4%	34.7%	33.0%	28.8%	25.1%	22.3%	18.5%	14.6%
0.08	67.4%	63.6%	60.1%	58.7%	58.3%	55.2%	54.6%	51.6%	48.5%	45.8%	42.6%	40.7%	38.5%	34.2%	31.2%	27.1%	25.4%	21.4%	15.2%
0.10	69.7%	66.9%	64.7%	62.8%	60.8%	58.3%	56.0%	53.0%	52.2%	49.0%	45.9%	42.1%	40.4%	39.4%	35.5%	31.3%	27.0%	22.9%	17.1%
0.12	73.3%	71.3%	67.8%	66.2%	63.5%	60.9%	58.2%	55.8%	53.8%	52.4%	47.2%	46.0%	42.1%	40.2%	36.3%	31.7%	28.8%	25.4%	21.4%
0.14	77.0%	74.1%	71.9%	71.1%	65.2%	66.0%	61.1%	59.4%	56.4%	54.5%	53.2%	49.0%	46.9%	42.2%	40.0%	35.8%	31.8%	26.7%	22.1%
0.16	79.8%	77.4%	74.9%	72.7%	71.4%	68.0%	65.3%	63.4%	57.7%	58.1%	53.1%	51.9%	49.6%	43.5%	43.3%	37.4%	34.6%	31.2%	25.4%
0.18	82.7%	81.9%	79.0%	76.9%	73.9%	71.7%	67.9%	66.4%	63.1%	61.3%	58.4%	55.8%	51.8%	48.0%	45.5%	40.4%	38.1%	32.9%	28.4%
0.20	86.2%	84.9%	81.1%	79.0%	77.5%	76.0%	72.4%	69.9%	67.0%	64.0%	60.9%	59.2%	53.8%	51.2%	47.7%	44.8%	40.0%	35.8%	30.0%
0.22	90.7%	88.7%	85.4%	84.5%	81.6%	79.3%	75.3%	74.4%	70.6%	67.8%	64.2%	60.7%	58.4%	54.4%	51.6%	46.6%	43.6%	38.2%	33.8%
0.24	95.5%	93.2%	89.7%	86.4%	84.6%	81.2%	79.2%	74.9%	73.5%	71.1%	66.1%	64.7%	61.5%	57.9%	55.1%	50.6%	46.6%	40.5%	34.8%
0.26	99.3%	95.9%	93.7%	90.9%	88.1%	85.5%	83.4%	79.5%	77.1%	74.7%	69.6%	65.9%	65.3%	59.4%	56.5%	50.8%	50.0%	44.6%	38.0%
0.28	103.0%	99.5%	98.7%	92.7%	93.4%	89.6%	88.2%	83.8%	80.9%	77.5%	74.0%	70.4%	68.5%	64.2%	59.3%	55.3%	50.1%	47.0%	43.3%
0.30	106.0%	104.0%	102.3%	100.3%	96.2%	94.5%	89.4%	88.8%	84.5%	82.0%	78.8%	74.3%	71.5%	66.9%	64.1%	59.5%	54.5%	49.6%	43.3%
0.32	111.5%	109.1%	106.9%	102.8%	98.6%	97.6%	94.8%	90.8%	87.4%	85.2%	82.4%	78.8%	74.6%	70.5%	66.9%	62.9%	57.1%	51.9%	47.8%
0.34	115.2%	111.9%	110.2%	107.1%	104.9%	101.6%	98.5%	95.4%	92.8%	88.7%	85.4%	83.1%	80.1%	74.3%	70.4%	66.4%	61.9%	54.8%	50.1%
0.36	120.9%	118.3%	114.3%	110.9%	108.0%	106.3%	100.1%	98.6%	96.1%	92.8%	88.0%	85.0%	81.1%	77.5%	74.3%	68.7%	64.9%	59.2%	51.0%
0.38	125.1%	122.0%	118.7%	114.9%	113.4%	109.5%	107.5%	103.3%	99.3%	94.3%	94.5%	88.9%	84.8%	81.0%	80.1%	73.2%	66.2%	63.1%	56.1%
0.40	128.8%	126.2%	123.2%	119.4%	117.2%	113.0%	110.9%	108.3%	104.5%	101.2%	97.7%	93.8%	89.8%	85.1%	82.7%	78.1%	72.4%	67.5%	59.1%
0.42	133.7%	131.7%	126.9%	123.1%	121.9%	119.2%	114.6%	110.0%	109.7%	104.1%	100.3%	95.6%	92.6%	89.0%	85.9%	79.4%	75.6%	70.2%	60.7%
0.44	137.7%	135.4%	132.7%	127.7%	126.6%	122.1%	121.3%	114.9%	112.1%	107.8%	104.9%	99.8%	96.5%	92.2%	89.1%	84.8%	76.8%	72.1%	67.5%
0.46	142.1%	140.6%	137.7%	133.6%	130.1%	125.8%	124.3%	117.5%	118.0%	111.2%	107.9%	104.4%	102.1%	96.4%	92.7%	87.3%	82.6%	76.3%	69.5%
0.48	147.8%	144.3%	140.5%	136.7%	136.5%	130.5%	127.6%	124.0%	121.6%	116.5%	113.2%	110.1%	104.2%	100.0%	97.0%	90.0%	84.9%	77.5%	72.2%
0.50	151.8%	151.0%	145.4%	142.7%	140.1%	137.3%	132.3%	128.7%	125.4%	121.4%	118.0%	114.2%	107.1%	105.1%	99.9%	94.7%	89.0%	83.2%	76.2%
0.52	158.1%	154.4%	150.7%	146.5%	144.8%	143.2%	135.4%	133.3%	130.7%	125.1%	122.4%	117.5%	112.9%	110.6%	103.3%	98.2%	93.4%	87.3%	79.6%
0.54	162.1%	160.6%	157.1%	153.9%	150.9%	147.1%	142.5%	137.8%	134.9%	130.3%	125.7%	121.4%	118.5%	111.5%	111.0%	102.6%	96.8%	90.5%	83.6%
0.56	169.5%	165.3%	161.8%	158.2%	152.7%	150.1%	146.6%	142.2%	140.6%	132.9%	133.5%	125.8%	122.5%	116.4%	112.8%	105.3%	101.9%	94.9%	87.3%
0.58	174.7%	171.9%	166.6%	164.1%	158.3%	155.1%	150.5%	148.7%	145.2%	137.4%	135.6%	130.6%	128.7%	121.0%	116.2%	111.6%	104.9%	96.7%	89.2%
0.60	182.0%	177.1%	172.7%	169.8%	165.2%	160.7%	155.1%	155.1%	146.7%	145.1%	140.6%	135.6%	132.8%	125.5%	121.9%	115.9%	113.7%	102.0%	93.1%
0.62	186.0%	183.0%	178.7%	173.3%	172.2%	165.3%	161.6%	159.0%	153.4%	147.7%	142.4%	140.4%	134.3%	129.4%	125.4%	122.1%	114.2%	107.7%	96.3%
0.64	191.8%	189.3%	181.9%	179.7%	175.8%	172.1%	167.7%	161.9%	159.2%	155.0%	148.5%	146.1%	140.5%	136.5%	131.1%	124.9%	118.9%	110.9%	103.4%
0.66	198.8%	194.2%	187.3%	183.9%	177.9%	178.5%	173.3%	167.4%	162.1%	158.0%	156.4%	150.1%	143.7%	139.8%	132.6%	127.2%	121.6%	114.9%	106.7%
0.68	204.0%	197.9%	195.9%	191.4%	187.7%	182.3%	176.8%	176.3%	170.9%	164.3%	162.0%	154.7%	152.1%	144.0%	141.2%	133.6%	128.9%	120.0%	110.3%
0.70	207.6%	204.3%	202.5%	195.3%	194.3%	187.8%	185.8%	178.8%	174.8%	172.6%	167.4%	160.5%	155.8%	149.9%	144.2%	136.5%	132.6%	123.5%	117.0%
0.72	218.4%	211.0%	207.4%	202.1%	197.9%	194.8%	189.8%	184.7%	178.6%	174.4%	170.6%	166.0%	159.1%	155.4%	147.6%	143.2%	136.6%	130.1%	116.6%
0.74	222.9%	219.4%	213.5%	207.8%	206.4%	197.5%	195.3%	188.6%	183.3%	181.0%	175.5%	169.8%	167.3%	161.2%	156.0%	145.6%	138.7%	133.9%	123.3%
0.76	230.2%	225.1%	221.3%	213.2%	211.5%	207.6%	201.9%	195.8%	193.5%	188.2%	180.5%	174.1%	172.9%	163.5%	158.7%	152.1%	145.8%	139.7%	128.8%
0.78	234.2%	229.6%	224.6%	218.4%	218.8%	213.4%	207.0%	202.8%	199.1%	194.1%	189.1%	182.1%	177.7%	168.7%	164.8%	158.4%	147.9%	143.9%	133.1%
0.80	241.8%	238.1%	231.8%	229.3%	226.3%	220.3%	211.3%	212.4%	200.5%	199.6%	195.9%	189.6%	184.0%	176.4%	169.6%	161.0%	154.0%	147.9%	136.2%
0.82	246.9%	243.0%	237.6%	234.8%	231.1%	224.7%	220.9%	215.1%	211.9%	206.0%	197.8%	195.1%	188.1%	179.5%	178.2%	169.6%	162.8%	152.6%	143.1%
0.84	255.3%	249.0%	246.5%	239.8%	235.6%	230.5%	228.0%	221.2%	214.9%	209.9%	204.7%	198.0%	193.8%	188.0%	181.3%	172.6%	167.3%	157.4%	145.4%
0.86	262.7%	258.1%	253.4%	246.3%	242.6%	237.6%	233.9%	229.7%	221.9%	216.6%	213.9%	204.4%	201.1%	195.9%	185.9%	179.2%	171.0%	164.8%	152.6%
0.88	270.3%	265.8%	259.8%	255.2%	251.0%	242.4%	240.2%	232.7%	227.1%	224.2%	217.3%	211.9%	209.7%	200.0%	189.8%	188.7%	178.8%	168.3%	156.8%
0.90	275.8%	272.0%	265.9%	263.2%	254.9%	254.2%	245.9%	244.4%	238.4%	230.7%	220.9%	220.9%	215.1%	209.2%	199.3%	191.2%	179.0%	169.6%	164.6%
0.92	286.1%	281.5%	270.1%	268.6%	266.8%	255.9%	255.0%	249.1%	244.8%	236.7%	228.3%	227.2%	219.2%	209.1%	201.9%	200.7%	190.8%	178.8%	167.3%
0.94	292.9%	289.8%	282.5%	280.0%	270.2%	264.9%	260.2%	257.9%	248.6%	246.1%	240.1%	229.4%	224.6%	217.7%	211.2%	202.3%	193.3%	184.9%	174.6%
0.96	303.6%	298.2%	291.9%	286.3%	278.7%	272.9%	268.3%	263.1%	258.2%	250.1%	243.7%	234.9%	229.9%	220.1%	221.1%	211.0%	198.9%	187.7%	179.6%
0.98	308.1%	302.3%	297.6%	293.7%	286.5%	279.6%	276.5%	271.4%	263.3%	256.5%	245.4%	244.5%	240.0%	229.0%	225.0%	214.4%	209.0%	198.9%	185.8%
1.00	319.3%	311.2%	306.1%	300.3%	295.0%	289.3%	283.3%	276.5%	272.5%	263.4%	255.4%	250.5%	245.3%	234.1%	230.5%	221.3%	213.3%	205.0%	187.7%

The number of offspring for both types follows a lognormal distribution with the stated μ and ρ, and with $\sigma = 2.0$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

so

$$\max\left(\frac{\sum_{t=1}^{T}\log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^{T}\log x_{B,t} - T\mu}{\sqrt{T}}\right) \leq \frac{\log(P_{A,T} + P_{B,T}) - T\mu}{\sqrt{T}}\,. \tag{10}$$

Similarly, using the right hand inequality of Lemma 1 with $m = 2$,

$$\log(P_{A,T} + P_{B,T}) - T\mu \leq \max\left(\sum_{t=1}^{T}\log x_{A,t}, \sum_{t=1}^{T}\log x_{B,t}\right) + \log(2) - T\mu\,, \tag{11}$$

and

$$\frac{\log(P_{A,T} + P_{B,T}) - T\mu}{\sqrt{T}} \leq \max\left(\frac{\sum_{t=1}^{T}\log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^{T}\log x_{B,t} - T\mu}{\sqrt{T}}\right) + \frac{\log(2)}{\sqrt{T}}\,. \tag{12}$$

The central limit theorem states that the joint vector $\left(\frac{\sum_{t=1}^{T}\log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^{T}\log x_{B,t} - T\mu}{\sqrt{T}}\right)$ converges in distribution to (N_A, N_B), where N_A and N_B are normally distributed vector with means 0, variances σ_A^2 and σ_B^2, and correlation ρ. Since max is continuous, the continuous mapping theorem theorem implies that $\max\left(\frac{\sum_{t=1}^{T}\log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^{T}\log x_{B,t} - T\mu}{\sqrt{T}}\right)$ converges to $\max(N_A, N_B)$. Since $\left|\frac{\log(P_{A,T}+P_{B,T}) - T\mu}{\sqrt{T}} - \max\left(\frac{\sum_{t=1}^{T}\log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^{T}\log x_{B,t} - T\mu}{\sqrt{T}}\right)\right|$ converges almost surely to 0, $\frac{\log(P_{A,T}+P_{B,T}) - T\mu}{\sqrt{T}}$ converges in distribution to $\max(N_A, N_B)$. □

Table S4. Simulated population growth rate with idiosyncratic risk ($\mu_{\text{id}} = -1.0$, $\sigma_{\text{id}} = 0.0$)

μ \ ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0.1	0.3	0.5	0.7	0.9
0.00	69.1%	67.6%	66.0%	65.0%	63.3%	61.3%	59.3%	57.0%	54.5%	52.7%
0.01	69.5%	69.2%	67.0%	65.9%	63.6%	62.5%	59.6%	57.7%	56.5%	53.0%
0.02	71.2%	69.7%	68.4%	67.0%	65.6%	63.7%	61.6%	59.3%	57.2%	54.6%
0.03	72.3%	71.7%	69.4%	67.6%	65.9%	64.5%	63.0%	60.3%	58.2%	55.1%
0.04	73.7%	72.7%	71.3%	69.3%	67.2%	65.7%	63.9%	61.7%	59.1%	56.2%
0.05	75.3%	74.0%	72.1%	70.5%	69.5%	66.6%	65.1%	62.5%	60.4%	57.8%
0.06	76.8%	75.0%	73.4%	72.2%	70.4%	68.8%	66.1%	64.4%	61.8%	58.3%
0.07	77.5%	76.9%	75.0%	73.4%	71.9%	68.9%	66.8%	65.3%	62.8%	59.1%
0.08	78.8%	77.5%	76.2%	74.9%	72.5%	71.3%	68.5%	66.8%	64.6%	60.0%
0.09	80.1%	79.0%	77.1%	76.3%	74.0%	72.8%	70.2%	68.4%	65.2%	61.8%
0.10	82.2%	80.0%	78.5%	77.1%	75.3%	73.3%	71.6%	68.5%	66.2%	62.7%
0.11	83.4%	81.6%	79.4%	78.6%	75.9%	75.5%	72.4%	70.3%	67.8%	63.4%
0.12	85.2%	83.2%	81.5%	80.0%	77.8%	75.8%	73.8%	71.1%	68.4%	66.1%
0.13	85.8%	84.8%	82.7%	81.1%	79.5%	77.6%	74.5%	72.7%	70.5%	67.6%
0.14	87.0%	85.9%	84.3%	82.7%	80.5%	78.8%	76.7%	74.2%	71.1%	68.4%
0.15	89.0%	87.2%	85.8%	83.5%	82.2%	79.9%	77.6%	75.6%	72.9%	68.7%
0.16	90.6%	88.8%	86.7%	85.2%	83.3%	81.4%	78.9%	77.1%	73.0%	70.5%
0.17	92.3%	90.1%	88.4%	86.5%	84.3%	82.9%	80.3%	78.7%	75.3%	71.2%
0.18	93.1%	91.5%	89.5%	87.7%	86.1%	83.9%	81.1%	78.9%	76.8%	72.8%
0.19	94.5%	92.8%	91.1%	89.3%	87.9%	85.3%	82.7%	80.9%	77.6%	74.0%
0.20	96.1%	94.4%	92.7%	91.3%	88.9%	86.6%	84.7%	82.3%	78.9%	74.9%

The systematic component of number of offspring for both types follows a lognormal distribution with the stated μ and ρ, and with $\sigma = 1.0$. The idiosyncratic component follows a lognormal distribution with $\mu_{\text{id}} = -1.0$ and $\sigma_{\text{id}} = 0.0$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

Proof of Proposition 1.

$$\frac{\log\left((P_{A,T}+P_{B,T})/(\bar{P}_{A,T}+\bar{P}_{B,T})\right)}{T} = \left(\frac{\log(P_{A,T}+P_{B,T})}{T} - \frac{\log(\bar{P}_{A,T}+\bar{P}_{B,T})}{T}\right), \qquad [13]$$

so by Theorem 1,

$$\frac{\log\left((P_{A,T}+P_{B,T})/(\bar{P}_{A,T}+\bar{P}_{B,T})\right)}{T} \to \max(\mu_A, \mu_B) - \max(\bar{\mu}_A, \bar{\mu}_B) \qquad [14]$$

almost surely. □

Proof of Proposition 2. The central limit theorem implies that the joint vector

$$\left(\frac{\sum_{t=1}^T \log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log x_{B,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log \bar{x}_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log \bar{x}_{B,t} - T\mu}{\sqrt{T}}\right)$$

converges in distribution to

$$(N_A, N_B, \bar{N}_A, \bar{N}_B)$$

where $(N_A, N_B, \bar{N}_A, \bar{N}_B)$ has the same correlation structure as $(x_{A,t}, x_{B,t}, \bar{x}_{A,t}, \bar{x}_{B,t})$. By the continuous mapping theorem,

$$\max\left(\frac{\sum_{t=1}^T \log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log x_{B,t} - T\mu}{\sqrt{T}}\right) - \max\left(\frac{\sum_{t=1}^T \log \bar{x}_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log \bar{x}_{B,t} - T\mu}{\sqrt{T}}\right)$$

converges in distribution to

$$\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)$$

As in the proof of Theorem 2, using Lemma 1,

$$\left|\frac{\log(P_{A,T}+P_{B,T}) - T\mu}{\sqrt{T}} - \max\left(\frac{\sum_{t=1}^T \log x_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log x_{B,t} - T\mu}{\sqrt{T}}\right)\right| \to^{\text{a.s.}} 0, \qquad [15]$$

and

$$\left|\frac{\log(\bar{P}_{A,T}+\bar{P}_{B,T}) - T\mu}{\sqrt{T}} - \max\left(\frac{\sum_{t=1}^T \log \bar{x}_{A,t} - T\mu}{\sqrt{T}}, \frac{\sum_{t=1}^T \log \bar{x}_{B,t} - T\mu}{\sqrt{T}}\right)\right| \to^{\text{a.s.}} 0, \qquad [16]$$

Table S5. Simulated population growth rate with idiosyncratic risk ($\mu_{\text{id}} = 0.0$, $\sigma_{\text{id}} = 0.0$)

μ \ ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0.1	0.3	0.5	0.7	0.9
0.00	140.9%	139.2%	138.4%	137.0%	135.9%	134.2%	131.9%	130.5%	127.7%	126.2%
0.01	142.4%	141.3%	139.8%	138.4%	137.5%	135.5%	133.6%	131.6%	128.9%	126.7%
0.02	143.8%	142.6%	141.4%	139.7%	139.0%	136.4%	134.5%	133.4%	130.7%	127.6%
0.03	145.3%	143.8%	142.8%	140.9%	139.5%	137.5%	135.5%	134.0%	131.3%	128.7%
0.04	146.0%	145.3%	143.5%	142.5%	140.9%	138.4%	136.8%	135.5%	132.6%	130.5%
0.05	147.8%	146.8%	145.3%	143.7%	142.4%	140.0%	138.4%	136.0%	133.9%	131.9%
0.06	148.8%	147.2%	147.0%	144.4%	143.4%	141.6%	139.5%	138.0%	135.1%	132.7%
0.07	150.5%	149.3%	148.1%	146.1%	145.0%	143.2%	140.6%	138.3%	136.4%	134.2%
0.08	151.9%	150.8%	149.7%	147.6%	146.2%	144.2%	142.3%	139.8%	137.4%	135.4%
0.09	153.5%	151.8%	150.2%	149.4%	147.5%	145.3%	143.6%	141.4%	139.3%	135.7%
0.10	154.8%	153.4%	152.4%	150.7%	148.3%	146.9%	145.1%	142.3%	140.0%	137.7%
0.11	155.9%	154.9%	153.3%	151.8%	150.2%	148.4%	146.1%	144.6%	141.7%	139.3%
0.12	157.6%	156.8%	154.9%	153.7%	151.2%	149.7%	147.8%	144.6%	142.2%	139.0%
0.13	159.0%	157.7%	155.9%	155.2%	152.4%	151.2%	149.8%	147.1%	145.0%	141.7%
0.14	160.6%	159.1%	157.3%	156.0%	154.3%	152.2%	150.5%	148.1%	145.6%	142.4%
0.15	161.7%	160.7%	158.9%	157.8%	155.5%	153.5%	152.2%	149.6%	146.7%	144.7%
0.16	163.7%	162.6%	160.8%	158.5%	157.5%	154.8%	154.1%	151.7%	147.8%	145.0%
0.17	164.8%	164.2%	162.4%	160.7%	158.8%	156.3%	154.2%	151.8%	149.9%	146.2%
0.18	166.3%	165.0%	164.1%	162.4%	160.6%	158.3%	155.9%	153.6%	151.3%	147.5%
0.19	168.6%	166.8%	165.5%	163.0%	161.3%	159.7%	157.1%	155.6%	152.6%	148.8%
0.20	170.1%	168.3%	166.6%	164.3%	163.2%	161.2%	158.8%	155.9%	153.2%	150.3%

The systematic component of number of offspring for both types follows a lognormal distribution with the stated μ and ρ, and with $\sigma = 1.0$. The idiosyncratic component follows a lognormal distribution with $\mu_{\text{id}} = 0.0$ and $\sigma_{\text{id}} = 0.0$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

If $X_n \to Z$ in distribution and $|X_n - Y_n| \to 0$ almost surely, then $Y_n \to Z$ in distribution. So

$$\frac{\log(P_{A,T} + P_{B,T}) - T\mu}{\sqrt{T}} - \frac{\log(\bar{P}_{A,T} + \bar{P}_{B,T}) - T\mu}{\sqrt{T}} \to^{\text{d}} \max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)\,, \qquad [17]$$

and rearranging the left hand side,

$$\frac{\log\big((P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\big)}{\sqrt{T}} \to^{\text{d}} \max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)\,. \qquad [18]$$

By the continuous mapping theorem,

$$\big((P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\big)^{1/\sqrt{T}} \to^{\text{d}} \exp\big(\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)\big)\,. \qquad [19]$$

Passing to a probability space where the above convergence happens almost surely and using Fatou's lemma,

$$\liminf_{T\to\infty} \mathbb{E}\left[\big((P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\big)^{1/\sqrt{T}}\right] \geq \mathbb{E}\left[\exp\big(\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)\big)\right]\,. \qquad [20]$$

Since $\rho < \bar{\rho}$,

$$\mathbb{E}\left[\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)\right] > 0\,. \qquad [21]$$

By Jensen's inequality,

$$\mathbb{E}\left[\exp\big(\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)\big)\right] \geq \exp\big(\mathbb{E}[\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)]\big) \qquad [22]$$

where $c > 1$. Also by Jensen's inequality,

$$\mathbb{E}\left[(P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\right]^{1/\sqrt{T}} \geq \mathbb{E}\left[\big((P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\big)^{1/\sqrt{T}}\right]\,, \qquad [23]$$

so

$$\liminf_{T\to\infty} \mathbb{E}\left[(P_{A,T} + P_{B,T})/(\bar{P}_{A,T} + \bar{P}_{B,T})\right]^{1/\sqrt{T}} \geq \exp\big(\mathbb{E}[\max(N_A, N_B) - \max(\bar{N}_A, \bar{N}_B)]\big)\,. \qquad [24]$$

Taking the log of both sides, we have the desired result. □

Table S6. Simulated population growth rate with idiosyncratic risk ($\mu_{\text{id}} = -1.0$, $\sigma_{\text{id}} = \sqrt{2}$)

μ \ ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0.1	0.3	0.5	0.7	0.9
0.00	131.7%	129.7%	128.2%	127.3%	125.5%	123.1%	121.6%	118.4%	118.0%	114.4%
0.01	132.9%	131.5%	129.5%	127.7%	126.8%	125.0%	123.7%	121.3%	118.6%	116.1%
0.02	133.8%	133.5%	131.3%	129.3%	127.6%	125.8%	124.5%	122.3%	119.7%	117.9%
0.03	135.5%	134.2%	132.5%	132.1%	128.3%	127.4%	125.6%	122.9%	121.1%	117.9%
0.04	137.1%	135.8%	134.1%	132.4%	131.1%	129.2%	127.2%	124.9%	122.6%	119.0%
0.05	138.6%	136.5%	135.6%	133.3%	132.4%	130.2%	128.1%	125.8%	123.3%	121.6%
0.06	139.8%	138.6%	137.5%	135.1%	133.1%	131.3%	130.3%	127.0%	125.6%	121.6%
0.07	141.4%	139.8%	138.1%	137.1%	134.9%	133.0%	131.8%	129.0%	126.2%	123.4%
0.08	142.7%	141.7%	139.4%	138.0%	136.4%	134.5%	131.7%	131.1%	127.6%	125.0%
0.09	143.8%	143.0%	141.5%	139.4%	137.8%	135.9%	133.4%	131.7%	129.1%	126.1%
0.10	146.1%	143.9%	142.4%	141.5%	139.0%	137.7%	134.7%	133.2%	130.3%	128.2%
0.11	147.9%	146.2%	143.7%	142.9%	140.5%	138.3%	136.3%	134.7%	132.3%	128.0%
0.12	149.0%	147.8%	145.6%	144.2%	141.4%	140.1%	137.3%	135.6%	132.6%	130.4%
0.13	150.1%	148.3%	146.8%	145.4%	143.2%	141.9%	139.1%	137.0%	134.1%	131.4%
0.14	151.9%	150.3%	148.8%	147.3%	144.5%	143.0%	141.5%	138.5%	136.3%	133.0%
0.15	153.6%	152.1%	150.2%	148.3%	146.2%	145.6%	142.2%	139.4%	137.0%	133.8%
0.16	154.8%	153.3%	151.5%	149.9%	147.8%	146.1%	144.0%	141.9%	138.2%	135.8%
0.17	156.3%	155.2%	152.9%	151.4%	149.3%	147.5%	145.4%	142.4%	140.3%	136.3%
0.18	157.7%	156.3%	154.8%	152.5%	151.0%	148.5%	146.4%	144.9%	141.0%	138.1%
0.19	159.6%	158.4%	156.2%	154.7%	152.5%	149.9%	148.0%	145.4%	143.1%	139.6%
0.20	161.1%	159.7%	157.7%	154.8%	154.2%	152.3%	149.4%	146.4%	143.5%	141.4%

The systematic component of number of offspring for both types follows a lognormal distribution with the stated μ and ρ, and with $\sigma = 1.0$. The idiosyncratic component follows a lognormal distribution with $\mu_{\text{id}} = -1.0$ and $\sigma_{\text{id}} = \sqrt{2}$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

Table S7. Simulated density-dependent population growth rate ($r = 1.4$)

K \ ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0.1	0.3	0.5	0.7	0.9
5.0	4.6%	4.3%	4.2%	4.1%	3.8%	3.6%	3.4%	3.0%	2.8%	2.1%
10.0	12.0%	11.7%	11.6%	11.5%	11.3%	11.0%	10.7%	10.3%	9.8%	9.4%
20.0	20.0%	19.7%	19.6%	19.5%	19.2%	19.0%	18.5%	18.3%	17.9%	17.4%
40.0	28.5%	28.5%	28.2%	28.0%	27.6%	27.3%	27.0%	26.8%	26.2%	25.4%
80.0	38.0%	37.5%	37.4%	37.2%	37.0%	36.5%	36.3%	35.7%	35.1%	34.5%

The number of offspring for both types follows Eq. (25) and Eq. (26) from the main article with the stated K and ρ, and with $r = 1.4$, $s = 1.0$ and $L = K$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

Proof of Theorem 3. We have that

$$\frac{\log(P_{nT_0})}{nT_0} = \frac{\sum_{i=1}^{n} \log P_{iT_0} - \log P_{(i-1)T_0}}{nT_0} \qquad [25]$$

Since

$$\log P_{iT_0} - \log P_{(i-1)T_0} = \log\left(\frac{1}{2}\prod_{t=(i-1)T_0+1}^{iT_0} x_{A,t} + \frac{1}{2}\prod_{t=(i-1)T_0+1}^{iT_0} x_{B,t}\right), \qquad [26]$$

$\log P_{iT_0} - \log P_{(i-1)T_0}$ is IID over i, and the strong law of large numbers gives the result. $\square$

Proof of Theorem 4. Using Lemma 1 with $m = 2$,

$$\max\left(\log P_{A,T}, \log P_{B,T}\right) \leq \log(P_{A,T} + P_{B,T}) \leq \max\left(\log P_{A,T}, \log P_{B,T}\right) + \log(2)\,. \qquad [27]$$

Therefore, if we show separately that $\frac{\log P_{A,T}}{T}$ converges almost surely to

$$\mathbb{E}[\log(\mathbb{E}[\bar{x}_{A,i,t}] + x_{A,t})]\,,$$

and $\frac{\log P_{B,T}}{T}$ converges almost surely to

$$\mathbb{E}[\log(\mathbb{E}[\bar{x}_{B,i,t}] + x_{B,t})]\,,$$

Table S8. Simulated density-dependent population growth rate ($r = 1.5$)

K \ ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0.1	0.3	0.5	0.7	0.9
5.0	4.9%	4.7%	4.5%	4.4%	4.2%	3.9%	3.8%	3.5%	3.2%	2.8%
10.0	12.3%	12.1%	12.1%	11.8%	11.8%	11.5%	11.2%	10.8%	10.7%	10.2%
20.0	20.5%	20.2%	20.2%	20.0%	19.7%	19.5%	19.3%	18.8%	18.5%	18.2%
40.0	29.1%	28.9%	28.7%	28.4%	28.3%	28.1%	27.6%	27.5%	27.2%	26.4%
80.0	38.6%	38.1%	37.9%	37.8%	37.5%	37.4%	37.1%	36.8%	36.1%	35.4%

The number of offspring for both types follows Eq. (25) and Eq. (26) from the main article with the stated K and ρ, and with $r = 1.5$, $s = 1.0$ and $L = K$. Quantities shown represent population growth rate per generation after 10 generations, computed as $e^{\mathbb{E}[\log(P_{10}/P_0)]/10}$, where $P_T = P_{A,T} + P_{B,T}$.

then the result is proved. We will show the claim for $\frac{\log P_{A,T}}{T}$, and the claim for $\frac{\log P_{B,T}}{T}$ follows by an identical argument replacing A with B.

First, decompose

$$\frac{\log P_{A,T}}{T} = \frac{\log P_{A,0} + \log \frac{P_{A,1}}{P_{A,0}} + \ldots + \log \frac{P_{A,T}}{P_{A,T-1}}}{T} . \qquad [28]$$

Since

$$P_{A,T} = \sum_{i=1}^{P_{A,T-1}} x_{A,T} + \bar{x}_{A,i,T} , \qquad [29]$$

$\log \frac{P_{A,T}}{P_{A,T-1}}$ can be written as

$$\log \frac{P_{A,T}}{P_{A,T-1}} = \log \left(x_{A,T} + \frac{\sum_{i=1}^{P_{A,T-1}} \bar{x}_{A,i,T}}{P_{A,T-1}} \right) . \qquad [30]$$

Since $\mathbb{E}[\log(x_{A,t})] > 0$, $P_{A,T} \to \infty$ almost surely. Using the strong law of large numbers, and since $x_{A,t}$ is bounded,

$$\left| \log \left(x_{A,T} + \frac{\sum_{i=1}^{P_{A,T-1}} \bar{x}_{A,i,T}}{P_{A,T-1}} \right) - \log \left(x_{A,T} + \mathbb{E}[\log(\bar{x}_{A,i,t})] \right) \right| \overset{\text{a.s.}}{\to} 0 . \qquad [31]$$

Using the strong law of large numbers once more in Eq. (28), the claim is proved. □

Proof of Theorem 5. We will show separately that $\frac{\log P_{2,T}}{T}$ converges almost surely to

$$\mathbb{E}\left[\log \left(\frac{1}{2} x_{A,2,T} + \frac{1}{2} x_{B,2,T} \right) \right]$$

and that $\frac{\log P_{1,T}}{T}$ converges almost surely to

$$\mathbb{E}\left[\log \left(\frac{1}{2} x_{A,1,T} + \frac{1}{2} x_{B,1,T} \right) \right] .$$

First, observe that

$$\frac{\log P_{2,T}}{T} = \frac{\log P_{2,0} + \log \frac{P_{2,1}}{P_{2,0}} + \ldots + \log \frac{P_{2,T}}{P_{2,T-1}}}{T} . \qquad [32]$$

Since

$$P_{2,T} = \sum_{i=1}^{P_{2,T-1}} 1_{\{q_{2,i,T-1}=A2\}} x_{A,2,T} + 1_{\{q_{2,i,T-1}=B2\}} x_{B,2,T} , \qquad [33]$$

$P_{2,T} \overset{\text{a.s.}}{\to} \infty$ almost surely (implicit in the assumption of the law of large numbers for random matching), and $x_{A,2,t}$ and $x_{B,2,t}$ are bounded, by the strong law of large numbers,

$$\left| \log \left(\frac{P_{2,T}}{P_{2,T-1}} \right) - \log \left(\frac{1}{2} x_{A,2,T} + \frac{1}{2} x_{B,2,T} \right) \right| \overset{\text{a.s.}}{\to} 0 . \qquad [34]$$

Again by the strong law of large numbers, as T increases without bound,

$$\frac{\log P_{2,0} + \log \frac{P_{2,1}}{P_{2,0}} + \ldots + \log \frac{P_{2,T}}{P_{2,T-1}}}{T} \overset{\text{a.s.}}{\to} \mathbb{E}\left[\log \left(\frac{1}{2} x_{A,2,T} + \frac{1}{2} x_{B,2,T} \right) \right] , \qquad [35]$$

and hence as T increases without bound,

$$\frac{\log P_{2,T}}{T} \overset{\text{a.s.}}{\to} \mathbb{E}\left[\log\left(\frac{1}{2}x_{A,2,T} + \frac{1}{2}x_{B,2,T}\right)\right] . \qquad [36]$$

Similarly,

$$\frac{\log P_{1,T}}{T} = \frac{\log P_{1,0} + \log \frac{P_{1,1}}{P_{1,0}} + \ldots + \log \frac{P_{1,T}}{P_{1,T-1}}}{T} . \qquad [37]$$

From the relation

$$P_{1,T} = \sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A1\}} x_{A,1,T} + 1_{\{q_{1,i,T-1}=B1\}} x_{B,1,T} + 1_{\{q_{1,i,T-1}=A2\}} x_{A,2,T} + 1_{\{q_{1,i,T-1}=B2\}} x_{B,2,T} , \qquad [38]$$

we can write

$$\begin{aligned} P_{1,T} = &\left(\sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A1\}} + 1_{\{q_{1,i,T-1}=B1\}}\right)\left(\frac{1}{2}x_{A,1,T} + \frac{1}{2}x_{B,1,T}\right) \\ &+ \left(\sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A2\}} + 1_{\{q_{1,i,T-1}=B2\}}\right)\left(\frac{1_{\{q_{1,i,T-1}=A2\}}}{\sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A2\}} + 1_{\{q_{1,i,T-1}=B2\}}} x_{A,2,T}\right. \\ &\left. + \frac{1_{\{q_{1,i,T-1}=B2\}}}{\sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A2\}} + 1_{\{q_{1,i,T-1}=B2\}}} x_{B,2,T}\right) \end{aligned} \qquad [39]$$

Let $n_{k,T-1}$ be the number of type 1 individuals matched with type k for $k = 1, 2$ in generation $T - 1$:

$$n_{1,T-1} = \sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A1\}} + 1_{\{q_{1,i,T-1}=B1\}}, \quad n_{2,T-1} = \sum_{i=1}^{P_{1,T-1}} 1_{\{q_{1,i,T-1}=A2\}} + 1_{\{q_{1,i,T-1}=B2\}} . \qquad [40]$$

By the assumption of the law of large numbers for random matching, and since $x_{j,k,t}$ and $n_{k,t-1}/P_{1,t-1}$ are bounded for $j = A, B$ and $k = 1, 2$, by the strong law of large numbers,

$$\left|\log\left(\frac{P_{1,T}}{P_{1,T-1}}\right) - \log\left(\frac{n_{1,T-1}}{P_{1,T-1}}\left(\frac{1}{2}x_{A,1,T} + \frac{1}{2}x_{B,1,T}\right) + \frac{n_{2,T-1}}{P_{1,T-1}}\left(\frac{1}{2}x_{A,2,T} + \frac{1}{2}x_{B,2,T}\right)\right)\right| \overset{\text{a.s.}}{\to} 0 \qquad [41]$$

Fix an outcome ω. Consider any subsequence of the sequence $\frac{\log P_{1,T}(\omega)}{T}$. From the decomposition Eq. (37), the fact that $\frac{n_{1,T-1}}{P_{1,T-1}}$ and $\frac{n_{2,T-1}}{P_{1,T-1}}$ are bounded, and the strong law of large numbers, there exists a further subsequence such that

$$\frac{\log P_{1,T}(\omega)}{T} \overset{\text{a.s.}}{\to} \mathbb{E}\left[\log\left(p\left(\frac{1}{2}x_{A,1,T} + \frac{1}{2}x_{B,1,T}\right) + (1-p)\left(\frac{1}{2}x_{A,2,T} + \frac{1}{2}x_{B,2,T}\right)\right)\right] \qquad [42]$$

for some p. If $p = 0$, it would imply that $P_{1,T}$ and $P_{2,T}$ grow at the same rates, which would make it impossible for p, a subsequential limit of proportions of type 1 individuals matched to type 1, to be 0. Similarly, if $0 < p < 1$, by Jensen's inequality, $P_{1,T}$ would grow exponentially faster than $P_{2,T}$, making it impossible that $p < 1$. Thus the only possibility is that $p = 1$. Therefore, any subsequence of $\frac{\log P_{1,T}(\omega)}{T}$ has a further subsequence converging to $\mathbb{E}\left[\log\left(\frac{1}{2}x_{A,1,T} + \frac{1}{2}x_{B,1,T}\right)\right]$, proving that $\frac{\log P_{1,T}}{T}$ converges to $\mathbb{E}\left[\log\left(\frac{1}{2}x_{A,1,T} + \frac{1}{2}x_{B,1,T}\right)\right]$ almost surely.

□