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Supplementary Information for

Designing metachronal waves of cilia

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This PDF file includes:

Supplementary text
Fig. S1
Legends for Movies S1 to S12
SI References

Other supplementary materials for this manuscript include the following:

Movies S1 to S12

Supporting Information Text

1. Blake tensor in 2D Fourier space

Blake tensor $\boldsymbol{G}(\boldsymbol{R'};\boldsymbol{R})$ in the real space is (1)

$$G_{ij} \quad = \quad \frac{1}{8\pi\eta}\left\{\frac{\delta_{ij}}{d}+\frac{d_i d_j}{d^3}-\frac{\delta_{ij}}{\bar{d}}-\frac{\bar{d}_i\bar{d}_j}{\bar{d}^3}+2z(\delta_{j\alpha}\delta_{\alpha\xi}-\delta_{j3}\delta_{3\xi})\frac{\partial}{\partial\bar{d}_\xi}\left[\frac{z\bar{d}_i}{\bar{d}^3}-\left(\frac{\delta_{i3}}{\bar{d}}+\frac{\bar{d}_i\bar{d}_3}{\bar{d}^3}\right)\right]\right\}, \qquad [1]$$

with $\boldsymbol{d}=\boldsymbol{R'}-\boldsymbol{R}$, $\bar{\boldsymbol{d}}=\bar{\boldsymbol{R}}'-\boldsymbol{R}$ and Greek letters denoting for the directions in 3D space. The projected positions of $\boldsymbol{R}$ and $\boldsymbol{R'}$ onto the xy plane are denoted as $\boldsymbol{R}_p=(x,y)$ and $\boldsymbol{R}'_p=(x',y')$, respectively.

Blake tensor can be expressed in the 2D Fourier space $\boldsymbol{q}=(q_x,q_y)$. We take $G_{\alpha\beta}$ components as an example, where $\alpha,\beta=x,y$. Explicitly, $G_{\alpha\beta}$ in the real space is:

$$G_{\alpha\beta} \quad = \quad \frac{1}{8\pi\eta}\left\{\underbrace{\left(\frac{\delta_{\alpha\beta}}{d}+\frac{d_\alpha d_\beta}{d^3}\right)}_{\langle 1\rangle}-\underbrace{\left(\frac{\delta_{\alpha\beta}}{\bar{d}}+\frac{\bar{d}_\alpha\bar{d}_\beta}{\bar{d}^3}\right)}_{\langle 2\rangle}+\underbrace{2zz'\frac{\partial^2}{\partial\bar{d}_\alpha\partial\bar{d}_\beta}\frac{1}{\bar{d}}}_{\langle 3\rangle}\right\}, \qquad [2]$$

$$\text{by recalling} \quad \bar{d}_3=z+z', \quad \text{and} \quad \frac{\bar{d}_\alpha}{\bar{d}^3}=-\frac{\partial}{\partial\bar{d}_\alpha}\frac{1}{\bar{d}}. \qquad [3]$$

We now use a Fourier representation as follows:

$$\begin{aligned}\langle 1\rangle &= \frac{1}{\pi^2}\int d^2q dq_z e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}e^{iq_z\cdot(z'-z)}\left[\frac{\delta_{\alpha\beta}}{q^2+q_z^2}-\frac{q_\alpha q_\beta}{(q^2+q_z^2)^2}\right]\\ &= \frac{1}{\pi}\left\{\int d^2q e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}\left[\frac{\delta_{\alpha\beta}}{|\boldsymbol{q}|}e^{-|\boldsymbol{q}||z'-z|}-q_\alpha q_\beta\frac{|z'-z||q|+1}{2|\boldsymbol{q}|^3}e^{-|\boldsymbol{q}||z'-z|}\right]\right\},\end{aligned} \qquad [4]$$

where $q^2=q_x^2+q_y^2$. Similarly, we can have

$$\begin{aligned}\langle 2\rangle &= \frac{1}{\pi^2}\int d^2q dq_z e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}e^{iq_z\cdot(z'+z)}\left[\frac{\delta_{\alpha\beta}}{q^2+q_z^2}-\frac{q_\alpha q_\beta}{(q^2+q_z^2)^2}\right]\\ &= \frac{1}{\pi}\left\{\int d^2q e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}\left[\frac{\delta_{\alpha\beta}}{|\boldsymbol{q}|}e^{-|\boldsymbol{q}|(z'+z)}-q_\alpha q_\beta\frac{(z'+z)|q|+1}{2|\boldsymbol{q}|^3}e^{-|\boldsymbol{q}|(z'+z)}\right]\right\},\end{aligned} \qquad [5]$$

and

$$\langle 3\rangle \quad = \quad -2zz'q_\alpha q_\beta\frac{1}{8\pi^3}\int d^2q dq_z e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}e^{iq_z\cdot(z'-z)}\frac{4\pi}{q^2+q_z^2}=-\frac{zz'}{\pi}q_\alpha q_\beta\int d^2q e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}\frac{1}{|\boldsymbol{q}|}e^{-|\boldsymbol{q}|(z'+z)}. \qquad [6]$$

Then, the α,β components of the Blake tensor can be expressed as

$$\begin{aligned}G_{\alpha\beta} &= \frac{1}{16\pi^2\eta}\int d^2q\frac{1}{|\boldsymbol{q}|}e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}\left\{\left(2\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{|\boldsymbol{q}|^2}\right)\left[e^{-|\boldsymbol{q}||z'-z|}-e^{-|\boldsymbol{q}|(z'+z)}\right]-\right.\\ &\left.\frac{q_\alpha q_\beta}{|\boldsymbol{q}|}\left[(|z'-z|e^{-|\boldsymbol{q}||z'-z|}-(z'+z)e^{-|\boldsymbol{q}|(z'+z)}\right]-2zz'q_\alpha q_\beta e^{-|\boldsymbol{q}|(z'+z)}\right\} \ (\alpha,\beta=x,y).\end{aligned} \qquad [7]$$

By repeating similar calculations, the other components can be represented in Fourier space as follows:

$$G_{\alpha z} \quad = \quad \frac{1}{16\pi^2\eta}\int d^2q\frac{1}{|\boldsymbol{q}|}e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}iq_\alpha\left[-(z'-z)e^{-|\boldsymbol{q}||z'-z|}+(z'-z)e^{-|\boldsymbol{q}|(z'+z)}+2zz'|\boldsymbol{q}|e^{-|\boldsymbol{q}|(z'+z)}\right], \qquad [8]$$

$$G_{z\alpha} \quad = \quad \frac{1}{16\pi^2\eta}\int d^2q\frac{1}{|\boldsymbol{q}|}e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}iq_\alpha\left[-(z'-z)e^{-|\boldsymbol{q}||z'-z|}+(z'-z)e^{-|\boldsymbol{q}|(z'+z)}-2zz'|\boldsymbol{q}|e^{-|\boldsymbol{q}|(z'+z)}\right], \qquad [9]$$

$$\begin{aligned}G_{zz} &= \frac{1}{16\pi^2\eta}\int d^2q\frac{1}{|\boldsymbol{q}|}e^{i\boldsymbol{q}\cdot(\boldsymbol{R}'_p-\boldsymbol{R}_p)}\left[e^{-|\boldsymbol{q}||z'-z|}-e^{-|\boldsymbol{q}|(z'+z)}+|\boldsymbol{q}||z'-z|e^{-|\boldsymbol{q}||z'-z|}\right.\\ &\left.-|\boldsymbol{q}|(z'+z)e^{-|\boldsymbol{q}|(z'+z)}-2zz'|\boldsymbol{q}|^2e^{-|\boldsymbol{q}|(z'+z)}\right].\end{aligned} \qquad [10]$$

2. Tilting trajectories – dynamic equation, dispersion relation and stability condition

A. Dynamic equation. As discussed in the main text, the cilia trajectories are tilted at an angle χ to the z direction, making an angle θ to the x direction ($\theta = \pi/2$ and $\chi = 0$ for trajectories in the yz plane). The position of the bead is: $\boldsymbol{R} - \boldsymbol{r} = a(\cos\phi\cos\theta + \sin\chi\sin\phi\sin\theta, \cos\phi\sin\theta - \sin\chi\sin\phi\cos\theta, \cos\chi\sin\phi)$ with $\boldsymbol{r}$ as lattice coordinates, and the tangential direction along the trajectory is: $\boldsymbol{t} = (-\cos\theta\sin\phi + \sin\chi\sin\theta\cos\phi, -\sin\theta\sin\phi - \sin\chi\cos\theta\cos\phi, \cos\chi\cos\phi)$. Here we introduce two unit vectors for simple notations, $\boldsymbol{b} = (-\sin\theta, \cos\theta, 0)$ and $\boldsymbol{c} = (\cos\theta, \sin\theta, 0)$, and the The unit vector in the normal direction of the rotation plane is $\boldsymbol{n} = -\cos\chi\boldsymbol{b} - \sin\chi\boldsymbol{e}_z$.

Since the near field effect arising from the trajectory size has been ignored, the resolution for the height difference between cilia is about the trajectory size, i.e., $|z' - z| \sim 2a$, which naturally sets a cutoff for applicable wave length. With Blake tensor in the 2D Fourier space, the dynamic equation of the cilia can be re-written as:

$$
\begin{aligned}
\dot{\phi} \quad \simeq \quad & \frac{f(\phi)}{\zeta(\phi)a} + \sum_{\boldsymbol{r'}} \frac{f(\phi')}{16\pi^2\eta a} \int d^2q \frac{1}{|\boldsymbol{q}|} e^{i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r'})} \times \\
& \Bigg\{ \left[(e^{-2|\boldsymbol{q}|a} - e^{-2|\boldsymbol{q}|h}) \left(2 - \frac{q_x^2}{|\boldsymbol{q}|^2}\right) + 2h\frac{q_x^2}{|\boldsymbol{q}|} e^{-2|\boldsymbol{q}|h} - 2h^2 q_x^2 e^{-2|\boldsymbol{q}|h} \right] \times \\
& \quad (-\cos\theta\sin\phi + \sin\chi\sin\theta\cos\phi)(-\cos\theta\sin\phi' + \sin\chi\sin\theta\cos\phi') - \\
& \frac{q_x q_y}{|\boldsymbol{q}|^2} \left(e^{-2|\boldsymbol{q}|a} - e^{-2|\boldsymbol{q}|h} - 2h|\boldsymbol{q}|e^{-2|\boldsymbol{q}|h} + 2h^2|\boldsymbol{q}|^2 e^{-2|\boldsymbol{q}|h} \right) (\cos\theta\sin\phi - \sin\chi\sin\theta\cos\phi)(\sin\theta\sin\phi' + \sin\chi\cos\theta\cos\phi') - \\
& \frac{q_x q_y}{|\boldsymbol{q}|^2} \left(e^{-2|\boldsymbol{q}|a} - e^{-2|\boldsymbol{q}|h} - 2h|\boldsymbol{q}|e^{-2|\boldsymbol{q}|h} + 2h^2|\boldsymbol{q}|^2 e^{-2|\boldsymbol{q}|h} \right) (\cos\theta\sin\phi' - \sin\chi\sin\theta\cos\phi')(\sin\theta\sin\phi + \sin\chi\cos\theta\cos\phi) + \\
& \left[(e^{-2|\boldsymbol{q}|a} - e^{-2|\boldsymbol{q}|h}) \left(2 - \frac{q_y^2}{|\boldsymbol{q}|^2}\right) + 2h\frac{q_y^2}{|\boldsymbol{q}|} e^{-2|\boldsymbol{q}|h} - 2h^2 q_y^2 e^{-2|\boldsymbol{q}|h} \right] \times \\
& \quad (\sin\theta\sin\phi + \sin\chi\cos\theta\cos\phi)(\sin\theta\sin\phi' + \sin\chi\cos\theta\cos\phi') - \\
& \quad i2q_x|\boldsymbol{q}|h^2 e^{-2|\boldsymbol{q}|h} (\cos\theta\sin\phi - \sin\chi\sin\theta\cos\phi)\cos\chi\cos\phi' - \\
& \quad i2q_y|\boldsymbol{q}|h^2 e^{-2|\boldsymbol{q}|h} (\sin\theta\sin\phi' + \sin\chi\cos\theta\cos\phi')\cos\chi\cos\phi' + \\
& \quad i2q_x|\boldsymbol{q}|h^2 e^{-2|\boldsymbol{q}|h} (\cos\theta\sin\phi' - \sin\chi\sin\theta\cos\phi')\cos\chi\cos\phi + \\
& \quad i2q_y|\boldsymbol{q}|h^2 e^{-2|\boldsymbol{q}|h} (\sin\theta\sin\phi' + \sin\chi\cos\theta\cos\phi')\cos\chi\cos\phi + \\
& \left[(e^{-2|\boldsymbol{q}|a} - e^{-2|\boldsymbol{q}|h} - 2h|\boldsymbol{q}|e^{-2|\boldsymbol{q}|h} - 2h^2|\boldsymbol{q}|^2 e^{-2|\boldsymbol{q}|h} \right] \cos^2\chi\cos\phi\cos\phi' \Bigg\}.
\end{aligned}
\tag{11}
$$

B. Separation of timescales. In the system, the phase difference between cilia $\phi - \phi'$ is a slow variable, compared with the sum of the two phases $\phi + \phi'$. The dynamical equation Eq. (11) can be re-expressed in terms $\phi - \phi'$ and $\phi + \phi'$, then we can time-average the fast variable over a beat period of the cilia to obtain the dynamical equation expressed in the slow variable.

Equation (7) in the main text gives us the dynamical equation for the new coordinate, $\partial_t \bar{\phi} = \dot{\phi}\Omega_0 a\zeta(\phi)/f(\phi)$, and we substitute Eq. (11) in for $\dot{\phi}$. This gives us $\zeta(\phi)f(\phi')/f(\phi)$ multiplied by a linear combination of $\sin\phi\sin\phi'$, $\sin\phi\cos\phi'$, $\cos\phi\sin\phi'$ and $\cos\phi\cos\phi'$, with different coefficients. We will use the term $\zeta(\phi)\frac{f(\phi')}{f(\phi)}\sin\phi\sin\phi'$ as an example to show how we can approximate the dynamical equation, Eq. (11), in terms of only the slow variable, using the following steps:

$$
\begin{aligned}
F_{\text{e.g.}} \quad &:= \quad \zeta(\phi)\frac{f(\phi')}{f(\phi)}\sin\phi\sin\phi' \\
&= \quad (1 + \sum_{n=1} C_n\cos n\phi + D_n\sin n\phi)(1 + \sum_{n=1} A_n\cos n\phi' + B_n\sin n\phi')(1 - \sum_{n=1} A_n\cos n\phi + B_n\sin n\phi)\sin\phi\sin\phi'
\end{aligned}
\tag{12}
$$

Expanding this up to first order in the harmonics we have

$$
F_{\text{e.g.}} = \left\{ 1 + \sum_{n=1} \left[(C_n - A_n)\cos n\phi + (D_n - B_n)\sin n\phi + A_n\cos n\phi' + B_n\sin n\phi' \right] \right\} \frac{1}{2} \left[\cos(\phi - \phi') - \cos(\phi + \phi') \right] \tag{13}
$$

Now we express this in the new coordinate, defined by Eq. (8) in the main text:

$$
\begin{aligned}
F_{\text{e.g.}} =& \Bigg\{ 1 + \sum_{n=1} \Bigg[(C_n - A_n) \cos\left(n\bar{\phi} + n \sum_{m=1} (\frac{A_m - C_m}{m} \sin m\bar{\phi} - \frac{B_m - D_m}{m} \cos m\bar{\phi}) \right) \\
&+ (D_n - B_n) \sin\left(n\bar{\phi} + n \sum_{m=1} (\frac{A_m - C_m}{m} \sin m\bar{\phi} - \frac{B_m - D_m}{m} \cos m\bar{\phi}) \right) \\
&+ A_n \cos\left(n\bar{\phi}' + n \sum_{m=1} (\frac{A_m - C_m}{m} \sin m\bar{\phi}' - \frac{B_m - D_m}{m} \cos m\bar{\phi}') \right) \\
&+ B_n \sin\left(n\bar{\phi}' + n \sum_{m=1} (\frac{A_m - C_m}{m} \sin m\bar{\phi}' - \frac{B_m - D_m}{m} \cos m\bar{\phi}') \right) \Bigg] \Bigg\} \\
&\times \frac{1}{2} \Bigg[\cos\left(\bar{\phi} - \bar{\phi}' + \sum_{m=1} (\frac{A_m - C_m}{m} (\sin m\bar{\phi} - \sin m\bar{\phi}') - \frac{B_m - D_m}{m} (\cos m\bar{\phi} - \cos m\bar{\phi}')) \right) \\
&- \cos\left(\bar{\phi} + \bar{\phi}' + \sum_{m=1} (\frac{A_m - C_m}{m} (\sin m\bar{\phi} + \sin m\bar{\phi}') - \frac{B_m - D_m}{m} (\cos m\bar{\phi} + \cos m\bar{\phi}')) \right) \Bigg]
\end{aligned}
\quad [14]
$$

Expand out the harmonic terms from the trig functions to first order, using

$$
\cos\left(\bar{\phi} + \sum \text{harmonics} \right) = \cos\bar{\phi}(1 + O(\sum \text{harmonics})^2) - \sin\bar{\phi}(\sum \text{harmonics} + O(\sum \text{harmonics})^3), \quad [15]
$$

$$
\sin\left(\bar{\phi} + \sum \text{harmonics} \right) = \sin\bar{\phi}(1 + O(\sum \text{harmonics})^2) + \cos\bar{\phi}(\sum \text{harmonics} + O(\sum \text{harmonics})^3). \quad [16]
$$

This gives us (to first order in the harmonics)

$$
\begin{aligned}
F_{\text{e.g.}} =& \Bigg\{ 1 + \sum_{n=1} \Bigg[(C_n - A_n) \cos\left(n\bar{\phi} \right) + (D_n - B_n) \sin\left(n\bar{\phi} \right) + A_n \cos\left(n\bar{\phi}' \right) + B_n \sin\left(n\bar{\phi}' \right) \Bigg] \Bigg\} \frac{1}{2} \Bigg[\cos\left(\bar{\phi} - \bar{\phi}' \right) - \cos\left(\bar{\phi} + \bar{\phi}' \right) \Bigg] \\
&- \frac{1}{2} \Bigg[\sin(\bar{\phi} - \bar{\phi}') \sum_{m=1} \left(\frac{A_m - C_m}{m} (\sin m\bar{\phi} - \sin m\bar{\phi}') - \frac{B_m - D_m}{m} (\cos m\bar{\phi} - \cos m\bar{\phi}') \right) \\
&- \sin(\bar{\phi} + \bar{\phi}') \sum_{m=1} \left(\frac{A_m - C_m}{m} (\sin m\bar{\phi} + \sin m\bar{\phi}') - \frac{B_m - D_m}{m} (\cos m\bar{\phi} + \cos m\bar{\phi}') \right) \Bigg]
\end{aligned}
\quad [17]
$$

We now rewrite the equation in terms of phase differences and sum of phases:

$$
\Delta = \bar{\phi} - \bar{\phi}', \quad \varphi = \bar{\phi} + \bar{\phi}', \quad \bar{\phi} = \frac{\varphi + \Delta}{2}, \quad \bar{\phi}' = \frac{\varphi - \Delta}{2}. \quad [18]
$$

Now we have

$$
\begin{aligned}
F_{\text{e.g.}} =& \Bigg\{ 1 + \sum_{n=1} \Bigg[(C_n - A_n) \cos\left(\frac{n}{2}(\varphi + \Delta) \right) + (D_n - B_n) \sin\left(\frac{n}{2}(\varphi + \Delta) \right) + A_n \cos\left(\frac{n}{2}(\varphi - \Delta) \right) \\
&+ B_n \sin\left(\frac{n}{2}(\varphi - \Delta) \right) \Bigg] \Bigg\} \frac{1}{2} \Bigg[\cos\left(\Delta \right) - \cos\left(\varphi \right) \Bigg] \\
&- \frac{1}{2} \Bigg[\sin(\Delta) \sum_{m=1} \left(\frac{A_m - C_m}{m} 2 \cos\left(\frac{m\varphi}{2} \right) \sin\left(\frac{m\Delta}{2} \right) + \frac{B_m - D_m}{m} 2 \sin\left(\frac{m\varphi}{2} \right) \sin\left(\frac{m\Delta}{2} \right) \right) \\
&- \sin(\varphi) \sum_{m=1} \left(\frac{A_m - C_m}{m} 2 \sin\left(\frac{m\varphi}{2} \right) \cos\left(\frac{m\Delta}{2} \right) - \frac{B_m - D_m}{m} 2 \cos\left(\frac{m\varphi}{2} \right) \cos\left(\frac{m\Delta}{2} \right) \right) \Bigg].
\end{aligned}
\quad [19]
$$

Expand out the trig functions with argument $\varphi \pm \Delta$:

$$\begin{aligned} F_{\text{e.g.}} = &\Bigg\{1+\sum_{n=1}\Bigg[(C_n-A_n)(\cos\left(\frac{n\varphi}{2}\right)\cos\left(\frac{n\Delta}{2}\right)-\sin\left(\frac{n\varphi}{2}\right)\sin\left(\frac{n\Delta}{2}\right)) \\ &+(D_n-B_n)(\sin\left(\frac{n\varphi}{2}\right)\cos\left(\frac{n\Delta}{2}\right)+\cos\left(\frac{n\varphi}{2}\right)\sin\left(\frac{n\Delta}{2}\right))+A_n(\cos\left(\frac{n\varphi}{2}\right)\cos\left(\frac{n\Delta}{2}\right)+\sin\left(\frac{n\varphi}{2}\right)\sin\left(\frac{n\Delta}{2}\right)) \\ &+B_n(\sin\left(\frac{n\varphi}{2}\right)\cos\left(\frac{n\Delta}{2}\right)-\cos\left(\frac{n\varphi}{2}\right)\sin\left(\frac{n\Delta}{2}\right))\Bigg]\Bigg\}\frac{1}{2}\Bigg[\cos\Big(\Delta\Big)-\cos\Big(\varphi\Big)\Bigg] \\ &-\frac{1}{2}\Bigg[\sin(\Delta)\sum_{m=1}\left(\frac{A_m-C_m}{m}2\cos\left(\frac{m\varphi}{2}\right)\sin\left(\frac{m\Delta}{2}\right)+\frac{B_m-D_m}{m}2\sin\left(\frac{m\varphi}{2}\right)\sin\left(\frac{m\Delta}{2}\right)\right) \\ &-\sin(\varphi)\sum_{m=1}\left(\frac{A_m-C_m}{m}2\sin\left(\frac{m\varphi}{2}\right)\cos\left(\frac{m\Delta}{2}\right)-\frac{B_m-D_m}{m}2\cos\left(\frac{m\varphi}{2}\right)\cos\left(\frac{m\Delta}{2}\right)\right)\Bigg]. \end{aligned} \quad [20]$$

We now use separation of timescales. φ is fast compared to Δ, so we time average φ terms over a beat cycle, while holding Δ terms constant. The time average is obtained by integrating over a beat cycle:

$$\begin{aligned} < F_{\text{e.g.}} > &= \frac{1}{T}\int_0^T dt F_{\text{e.g.}}(\varphi(t)), \\ &= \frac{1}{T}\int_0^{4\pi}\frac{d\varphi}{2\Omega_0}F_{\text{e.g.}}(\varphi) \end{aligned} \quad [21]$$

where

$$T=\frac{2\pi}{\Omega_0}, \quad [22]$$

and we have made use of the constant intrinsic velocity in the new coordinate system. When we do the time average, many of the terms in Eq. (20) average to zero. Only terms with $\cos^2(n\varphi)$ or $\sin^2(n\varphi)$ survive. Time averaging leaves the following terms:

$$\begin{aligned} < F_{\text{e.g.}} > &= \frac{1}{2}\Bigg\{\cos\Delta-\left(\frac{(C_2-A_2)}{2}\cos\Delta+\frac{(D_2-B_2)}{2}\sin\Delta+\frac{A_2}{2}\cos\Delta-\frac{B_2}{2}\sin\Delta\right)+\frac{(A_2-C_2)}{2}\cos\Delta\Bigg\} \\ &= \frac{1}{2}\left[1+\frac{A_2}{2}-C_2\right]\cos\Delta+\frac{1}{2}\left[B_2-\frac{D_2}{2}\right]\sin\Delta+O(\text{harmonics})^2. \end{aligned} \quad [23]$$

Then the dynamic equation Eq. (11) can be re-expressed as:

$$\partial_t\bar{\phi}(\boldsymbol{r},t)=\Omega_0+\frac{\Omega_0 hb}{16\pi}\sum_{\boldsymbol{r'}}\int d^2\boldsymbol{q}\,e^{i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r'})}\left[\mathcal{M}(\boldsymbol{q})\cos\left(\bar{\phi}(\boldsymbol{r})-\bar{\phi}(\boldsymbol{r'})\right)+\mathcal{S}(\boldsymbol{q})\sin\left(\bar{\phi}(\boldsymbol{r})-\bar{\phi}(\boldsymbol{r'})\right)\right],$$

where

$$\mathcal{M} = g_0(\boldsymbol{q}h)+g_1(\boldsymbol{q}h)(A_2-2C_2)+g_2(\boldsymbol{q}h)(B_2-2D_2), \qquad \mathcal{S}=g_1(\boldsymbol{q}h)(2B_2-D_2)-g_2(\boldsymbol{q}h)(2A_2-C_2), \quad [24]$$

with

$$\begin{cases} g_0(\boldsymbol{p})=2p^{-1}(e^{-2p(a/h)}-e^{-2p})[3-\cos^2\chi(\hat{\boldsymbol{p}}\cdot\boldsymbol{c})^2]+4(1-p)e^{-2p}[1-\cos^2\chi(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})^2]-4(1+p)e^{-2p}\cos^2\chi \\ g_1(\boldsymbol{p})=p^{-1}(e^{-2p(a/h)}-e^{-2p})[(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})^2-\sin^2\chi(\hat{\boldsymbol{p}}\cdot\boldsymbol{c})^2]+2(1-p)e^{-2p}[(\hat{\boldsymbol{p}}\cdot\boldsymbol{c})^2-\sin^2\chi(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})^2]+2(1+p)e^{-2p}\cos^2\chi \\ g_2(\boldsymbol{p})=2[p^{-1}(1-e^{-2p})-2(1-p)e^{-2p}]\sin\chi(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})(\hat{\boldsymbol{p}}\cdot\boldsymbol{c}) \end{cases} \quad [25]$$

and this gives Equation ~~(3)~~(9) in the main text.

C. Dispersion relation and stability condition. By assuming the cilia are beating in the form of metachronal wave with the wave frequency ω and the wave vector $\boldsymbol{k}$, i.e., $\bar{\phi}=\omega t-\boldsymbol{k}\cdot\boldsymbol{r}+\delta\bar{\phi}$, then the phase difference between cilia will become,

$$\bar{\Delta} = \boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})+\delta\bar{\phi}-\delta\bar{\phi}'. \quad [26]$$

$\sin\bar{\Delta}$ and $\cos\bar{\Delta}$ in Equation Eq. (11) can be approximated as,

$$\begin{aligned} \sin\bar{\Delta} &\simeq \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}-e^{-i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}}{2i}+\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}+e^{-i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}}{2}(\delta\bar{\phi}-\delta\bar{\phi}'), \\ \cos\bar{\Delta} &\simeq \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}+e^{-i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}}{2}+\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}-e^{-i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}}{2}i(\delta\bar{\phi}-\delta\bar{\phi}'). \end{aligned} \quad [27]$$

By taking the above equations and the Fourier transformation in a discrete lattice: $\sum_{\boldsymbol{r'}} e^{i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r'})} = \sum_{\boldsymbol{G}} \frac{4\pi^2}{\ell^2}\delta^2(\boldsymbol{q}+\boldsymbol{G})$, where $\boldsymbol{G}$ denote the vectors of the reciprocal lattice, then the dynamic equation can be simply calculated as:

$$\dot{\bar{\phi}} = \omega(\boldsymbol{k}) + \partial_t \delta\bar{\phi}. \quad [28]$$

where $\omega(\boldsymbol{k})$ represents the dispersion relation of the system, and $\partial_t \delta\bar{\phi}$ represents the evolution of the perturbation.

Dispersion relation. The explicit form of $\omega(\boldsymbol{k})$ and $\partial_t \delta\bar{\phi}$ is:

$$\omega = \Omega_0 \left[1 + \frac{\pi}{4}\frac{hb}{\ell^2}\sum_{\boldsymbol{G}} \mathcal{M}(\boldsymbol{k}+\boldsymbol{G})\right]. \quad [29]$$

Evolution of $\delta\bar{\phi}$. The explicit form of $\partial_t \delta\bar{\phi}$ is:

$$\begin{aligned} \partial_t \delta\bar{\phi}_{\boldsymbol{k}}(\boldsymbol{q}) &= -\Big(\Gamma(\boldsymbol{q},\boldsymbol{k}) - \Gamma(\boldsymbol{0},\boldsymbol{k})\Big)\delta\bar{\phi}_{\boldsymbol{k}}(\boldsymbol{q}), \quad [30] \\ \text{with} \quad \Gamma(\boldsymbol{q},\boldsymbol{k}) &= \frac{\pi b \Omega_0}{8\ell^2}\sum_{\boldsymbol{G}} \big[\mathcal{S}(\boldsymbol{q}+\boldsymbol{k}-\boldsymbol{G}) + \mathcal{S}(\boldsymbol{q}-\boldsymbol{k}-\boldsymbol{G})\big], \end{aligned}$$

respectively.

References

1. Blake JR (1971) A note on the image system for a stokeslet in a no-slip boundary. *Mathematical Proceedings of the Cambridge Philosophical Society* 70(02):303.

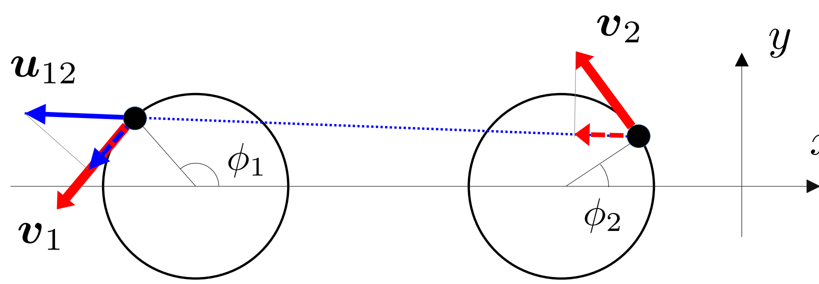

Fig. S1. Schematic picture of two interacting cilia making circular orbits in the $xy-$plane. Their velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are tangential to each orbit (red solid arrows). The motion of cilium $\#2$ causes a flow along the line connecting the two cilia (blue dashed line), which is identical to a streamline and is almost parallel to the x-axis due to large distance. The fluid velocity $\boldsymbol{u}_{12}$ induced by the cilium $\#2$ at the position of $\#1$ (blue arrow) has the magnitude proportional to the projection of $\boldsymbol{v}_2$ onto the streamline ($v_2 \sin\phi_2$, red dashed arrow). The tangential component of $\boldsymbol{u}_{12}$ ($u_{12}\sin\phi_1 \propto v_2 \sin\phi_2 \sin\phi_1$, blue dashed line) adds to the velocity of cilium $\#1$. Since v_2 is proportional to the driving force $f(\phi_2)$, the velocity shift contains the factor $\sin\phi_1 \sin\phi_2\, f(\phi_2)$.

Movie S1. Wave development of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\theta = 0$ and $\chi = 0$.

Movie S2. Stable wave of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\theta = 0$ and $\chi = 0$.

Movie S3. Wave development of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = 0$ and $\theta = \pi/6$.

Movie S4. Stable wave of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = 0$ and $\theta = \pi/6$.

Movie S5. Wave development of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = 0$ and $\theta = \pi/4$.

Movie S6. Stable wave of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = 0$ and $\theta = \pi/4$.

Movie S7. Wave development of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = \pi/36$ and $\theta = \pi/4$.

Movie S8. Stable wave of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = \pi/36$ and $\theta = \pi/4$.

Movie S9. Wave development of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = \pi/6$ and $\theta = \pi/4$.

Movie S10. Stable wave of the cilia placed on a square lattice, with the tilting angles of the trajectory as $\chi = \pi/6$ and $\theta = \pi/4$.

Movie S11. Wave development of the cilia placed on a triangular lattice, with the tilting angles of the trajectory as $\theta = 0$ and $\chi = 0$.

Movie S12. Stable wave of the cilia placed on a triangular lattice, with the tilting angles of the trajectory as $\theta = 0$ and $\chi = 0$.