Supplementary Materials For “Linkage disequilibrium under polysomic inheritance”

Appendix A. Expansion of Δ_{AB} in triploids

In triploids, the value of Δ_{AB} is given by $\Delta_{AB} = D_s^{AB} + 2D_d^{AB}$, where D_s^{AB} and $2D_d^{AB}$ can be respectively expanded as

$$D_s^{AB} = \frac{1}{3}(D^{A\cdot\cdot}_{B\cdot\cdot} + D^{\cdot A\cdot}_{\cdot B\cdot} + D^{\cdot\cdot A}_{\cdot\cdot B}),$$

$$2D_d^{AB} = \frac{1}{3}(D^{A\cdot\cdot}_{\cdot B\cdot} + D^{A\cdot\cdot}_{\cdot\cdot B} + D^{\cdot A\cdot}_{B\cdot\cdot} + D^{\cdot A\cdot}_{\cdot\cdot B} + D^{\cdot\cdot A}_{B\cdot\cdot} + D^{\cdot\cdot A}_{\cdot B\cdot}),$$

in which the superscripts (or the subscripts) of D on the right side of the equals sign denote the phased genotype at the first (or the second) locus, and the dot $\cdot$ denotes any allele.

Each term on the right side can be further expanded as follows (the terms with the same two-locus unphased genotypes in the expansion are combined):

$$\frac{1}{3}D^{A\cdot\cdot}_{B\cdot\cdot} = \frac{1}{3}(D^{AAA}_{BBB} + D^{AAX}_{BBB} + D^{AXA}_{BBB} + D^{AAA}_{BBX} + D^{AAA}_{BXB} + D^{AAX}_{BBX} + D^{AAX}_{BXB} + D^{AXA}_{BBX} + D^{AXA}_{BXB} + D^{AXX}_{BBB} + D^{AAA}_{BXX} + D^{AXX}_{BXB} + D^{AXX}_{BBX} + D^{AAX}_{BXX} + D^{AXA}_{BXX} + D^{AXX}_{BXX}),$$

$$\frac{1}{3}D^{\cdot A\cdot}_{\cdot B\cdot} = \frac{1}{3}(D^{AAA}_{BBB} + D^{AAX}_{BBB} + D^{XAA}_{BBB} + D^{AAA}_{BBX} + D^{AAA}_{XBB} + D^{AAX}_{BBX} + D^{AAX}_{XBB} + D^{XAA}_{BBX} + D^{XAA}_{XBB} + D^{XAX}_{BBB} + D^{AAA}_{XBX} + D^{XAX}_{XBB} + D^{XAX}_{BBX} + D^{AAX}_{XBX} + D^{XAA}_{XBX} + D^{XAX}_{XBX}),$$

$$\frac{1}{3}D^{\cdot\cdot A}_{\cdot\cdot B} = \frac{1}{3}(D^{AAA}_{BBB} + D^{XAA}_{BBB} + D^{AXA}_{BBB} + D^{AAA}_{XBB} + D^{AAA}_{BXB} + D^{XAA}_{XBB} + D^{XAA}_{BXB} + D^{AXA}_{XBB} + D^{AXA}_{BXB} + D^{XXA}_{BBB} + D^{AAA}_{XXB} + D^{XXA}_{BXB} + D^{XXA}_{XBB} + D^{XAA}_{XXB} + D^{AXA}_{XXB} + D^{XXA}_{XXB}),$$

$$\frac{1}{3}D^{A\cdot\cdot}_{\cdot B\cdot} = \frac{1}{3}(D^{AAA}_{BBB} + D^{AAX}_{BBB} + D^{AXA}_{BBB} + D^{AAA}_{BBX} + D^{AAA}_{XBB} + D^{AAX}_{BBX} + D^{AAX}_{XBB} + D^{AXA}_{BBX} + D^{AXA}_{XBB} + D^{AXX}_{BBB} + D^{AAA}_{XBX} + D^{AXX}_{XBB} + D^{AXX}_{BBX} + D^{AAX}_{XBX} + D^{AXA}_{XBX} + D^{AXX}_{XBX}),$$

$$\frac{1}{3}D^{A\cdot\cdot}_{\cdot\cdot B} = \frac{1}{3}(D^{AAA}_{BBB} + D^{AAX}_{BBB} + D^{AXA}_{BBB} + D^{AAA}_{BXB} + D^{AAA}_{XBB} + D^{AAX}_{BXB} + D^{AAX}_{XBB} + D^{AXA}_{BXB} + D^{AXA}_{XBB} + D^{AXX}_{BBB} + D^{AAA}_{XXB} + D^{AXX}_{XBB} + D^{AXX}_{BXB} + D^{AAX}_{XXB} + D^{AXA}_{XXB} + D^{AXX}_{XXB}),$$

$$\frac{1}{3}D^{\cdot A\cdot}_{B\cdot\cdot} = \frac{1}{3}(D^{AAA}_{BBB} + D^{AAX}_{BBB} + D^{XAA}_{BBB} + D^{AAA}_{BBX} + D^{AAA}_{BXB} + D^{AAX}_{BBX} + D^{AAX}_{BXB} + D^{XAA}_{BBX} + D^{XAA}_{BXB} + D^{XAX}_{BBB} + D^{AAA}_{BXX} + D^{XAX}_{BXB} + D^{XAX}_{BBX} + D^{AAX}_{BXX} + D^{XAA}_{BXX} + D^{XAX}_{BXX}),$$

$$\frac{1}{3}D^{\cdot A\cdot}_{\cdot\cdot B} = \frac{1}{3}(D^{AAA}_{BBB} + D^{AAX}_{BBB} + D^{XAA}_{BBB} + D^{AAA}_{XBB} + D^{AAA}_{BXB} + D^{AAX}_{XBB} + D^{AAX}_{BXB} + D^{XAA}_{XBB} + D^{XAA}_{BXB} + D^{XAX}_{BBB} + D^{AAA}_{XXB} + D^{XAX}_{BXB} + D^{XAX}_{XBB} + D^{AAX}_{XXB} + D^{XAA}_{XXB} + D^{XAX}_{XXB}),$$

$$\frac{1}{3}D^{\cdot\cdot A}_{\cdot B\cdot} = \frac{1}{3}(D^{AAA}_{BBB} + D^{XAA}_{BBB} + D^{AXA}_{BBB} + D^{AAA}_{BBX} + D^{AAA}_{XBB} + D^{XAA}_{BBX} + D^{XAA}_{XBB} + D^{AXA}_{BBX} + D^{AXA}_{XBB} + D^{XXA}_{BBB} + D^{AAA}_{XBX} + D^{XXA}_{XBB} + D^{XXA}_{BBX} + D^{XAA}_{XBX} + D^{AXA}_{XBX} + D^{XXA}_{XBX}),$$

$$\frac{1}{3}D_{B\cdot\cdot}^{\cdot\cdot A} = \frac{1}{3}(D_{BBB}^{AAA} + D_{BBB}^{XAA} + D_{BBB}^{AXA} + D_{BBX}^{AAA} + D_{BXB}^{AAA} + D_{BBX}^{XAA} + D_{BXB}^{XAA} + D_{BBX}^{AXA} + D_{BXB}^{AXA} + D_{BBB}^{XXA}$$
$$+ D_{BXX}^{AAA} + D_{BXB}^{XXA} + D_{BBX}^{XXA} + D_{BXX}^{XAA} + D_{BXX}^{AXA} + D_{BXX}^{XXA}),$$

By summing all terms of the same two-locus unphased genotypes on the right sides of the equals signs, we obtain the following equalities:

$$3D_{BBB}^{AAA} = 3G_{BBB}^{AAA} - 3p_A^3q_B^3,$$

$$2(D_{BBB}^{AAX} + D_{BBB}^{AXA} + D_{BBB}^{XAA}) = 2G_{BBB}^{AAX} - 6p_A^2p_Xq_B^3,$$

$$2(D_{BBX}^{AAA} + D_{BXB}^{AAA} + D_{XBB}^{AAA}) = 2G_{BBX}^{AAA} - 6p_A^3q_B^2q_X,$$

$$\frac{4}{3}(D_{BBX}^{AAX} + D_{BXB}^{AAX} + D_{BBX}^{AXA} + D_{BXB}^{AXA} + D_{XBB}^{AAX}$$
$$+D_{BBX}^{XAA} + D_{XBB}^{XAA} + D_{BXB}^{XAA} + D_{XBB}^{AXA}) = \tfrac{4}{3}G_{BBX}^{AAX} - 12p_A^2p_Xq_B^2q_X,$$
$$D_{BBB}^{AXX} + D_{BBB}^{XAX} + D_{BBB}^{XXA} = G_{BBB}^{AXX} - 3p_Ap_X^2q_B^3,$$
$$D_{BXX}^{AAA} + D_{XBX}^{AAA} + D_{XXB}^{AAA} = G_{BXX}^{AAA} - 3p_A^3q_Bq_X^2,$$

$$\frac{2}{3}(D_{BXB}^{AXX} + D_{BBX}^{AXX} + D_{XBB}^{XAX} + D_{BBX}^{XAX} + D_{BXB}^{XXA}$$
$$+D_{XBB}^{XXA} + D_{XBB}^{AXX} + D_{BXB}^{XAX} + D_{BBX}^{XXA}) = \tfrac{2}{3}G_{BBX}^{AXX} - 6p_Ap_X^2q_B^2q_X,$$

$$\frac{2}{3}(D_{BXX}^{AAX} + D_{BXX}^{AXA} + D_{XBX}^{AAX} + D_{XBX}^{XAA} + D_{XXB}^{XAA}$$
$$+D_{XXB}^{AXA} + D_{XBX}^{AXA} + D_{XXB}^{AAX} + D_{BXX}^{XAA}) = \tfrac{2}{3}G_{BBX}^{AXX} - 6p_Ap_X^2q_B^2q_X,$$

$$\frac{1}{3}(D_{BXX}^{AXX} + D_{XBX}^{XAX} + D_{XXB}^{XXA} + D_{XBX}^{AXX} + D_{XXB}^{AXX}$$
$$+D_{BXX}^{XAX} + D_{XXB}^{XAX} + D_{XBX}^{XXA} + D_{BXX}^{XXA}) = \tfrac{1}{3}G_{BXX}^{AXX} - 3p_Ap_X^2q_Bq_X^2,$$

where each $G_{\cdots}^{\cdots}$ denotes a two-locus unphased genotypic frequency, whose superscript and subscript represent two unphased genotypes. Each expression on the right sides of the above equalities is one of the following:

$$\tfrac{ij}{v}G_{B^jX^{v-j}}^{A^iX^{v-i}} - v\tbinom{v-1}{v-i}\tbinom{v-1}{v-j}p_A^ip_X^{v-i}q_B^jq_X^{v-j},\quad i,j = 1,2,3 \text{ and } v = 3,$$

in which A^iX^{v-i} denotes an unphased genotype containing exactly i copies of A, and the meaning of B^jX^{v-j} is similar. Because Δ_{AB} is the sum of these expressions, it follows

$$\Delta_{AB} = \sum_{i=1}^{v}\sum_{j=1}^{v}\left[\tfrac{ij}{v}G_{B^jX^{v-j}}^{A^iX^{v-i}} - v\binom{v-1}{v-i}\binom{v-1}{v-j}p_A^ip_X^{v-i}q_B^jq_X^{v-j}\right],$$

Note that

$$\sum_{i=1}^{v}\sum_{j=1}^{v}\binom{v-1}{v-i}\binom{v-1}{v-j}p_A^ip_X^{v-i}q_B^jq_X^{v-j} = \left[\sum_{i=1}^{v}\binom{v-1}{v-i}p_A^ip_X^{v-i}\right]\left[\sum_{j=1}^{v}\binom{v-1}{v-j}q_B^jq_X^{v-j}\right]$$

$$= \left[p_A\sum_{k=0}^{v-1}p_X^kp_A^{(v-1)-k}\right]\left[q_B\sum_{l=0}^{v-1}q_X^lq_B^{(v-1)-l}\right]$$

$$= p_Aq_B(p_A+p_X)^{v-1}(q_B+q_X)^{v-1} = p_Aq_B.$$

The next formula is valid for any ploidy level v:

$$\Delta_{AB} = \left(\sum_{i=1}^{v} \sum_{j=1}^{v} \tfrac{ij}{v} G_{B^j X^{v-j}}^{A^i X^{v-i}} \right) - v p_A q_B.$$

Appendix B. Non-identity coefficients

The single non-identity coefficient is defined as the probability that the two alleles of an allele pair are not IBD. There are two configurations for two such alleles: (i) they are sampled from the same individual, or (ii) they are sampled from different individuals. We denote the single non-identity coefficient by P for (i), or by Π for (ii). Then, P and Π can be described by symbols as follows:

$$P \stackrel{\text{def}}{=} 1 - \mathcal{F} \text{ and } \Pi \stackrel{\text{def}}{=} 1 - \bar{\theta},$$

where $\bar{\theta}$ is the average kinship coefficient between all individuals in a population, i.e., the probability that two alleles (each randomly sampled from a separate individual) are IBD.

The double non-identity coefficient is defined as the probability that neither of two allele pairs are IBD. There are multiple configurations for these two allele pairs. Based on Weir & Hill (1980), we established 3 digenic, 6 trigenic and 13 quadgenic two-locus allele configurations for different polysomic inheritances, including four novel allele configurations (9$^{\text{th}}$, 15$^{\text{th}}$, 21$^{\text{st}}$ and 22$^{\text{nd}}$) that have more than two haplotypes within individuals. These allele configurations along with the notations of the corresponding frequencies, double non-identity coefficients, and expectations are presented in Table 1, where the first nine allele configurations do not have corresponding double non-identity coefficients because they share the same alleles.

For example, the 10$^{\text{th}}$ allele configuration $Z_{BB\dots}^{AA\dots}$ in Table 1 means that these two allele pairs are from two haplotypes within the same individual, the first A and first B are in one haplotype, and the second A and second B are in another haplotype. Moreover, the corresponding frequency, double non-identity coefficient and the expectation of frequency are denoted by P_{10}, Θ_1 and E_{10}, respectively.

The expectation E_i of each frequency P_i in Table 1 is derived by assuming no initial LD, which is a linear combination of $p_A q_B$, $p_A q_B(p_X + q_X)$ and $p_A p_X q_B q_X$, whose combination coefficients are listed in the three cells before E_i in Table 1. For example, the combination coefficients of E_{18} are 1, $-\Pi$ and Δ_3. The allele pair AA or BB in the 18$^{\text{th}}$ allele configuration $Z_{\cdot B\dots|\cdot\dots|B\dots}^{A\cdot\dots|A\dots|\cdot\dots}$ consists of the alleles from different individuals, then the single non-identity coefficient of each allele pair is Π and the double non-identity coefficient is Δ_3. Hence the identity states of these two allele pairs can be described by the following three aspects: (i) both pairs are non-IBD with probability Δ_3, (ii) only one pair is IBD with probability $\Pi - \Delta_3$ or (iii) both pairs are IBD with probability $1 - 2\Pi + \Delta_3$. Therefore, the expectation E_{18} is the following linear combination with 1, $-\Pi$ and Δ_3 as the combination coefficients:

$$\begin{aligned} E_{18} &= \Delta_3 p_A^2 q_B^2 + (\Pi - \Delta_3)\left(p_A q_B^2 + p_A^2 q_B\right) + (1 - 2\Pi + \Delta_3) p_A q_B \\ &= p_A q_B - \Pi p_A q_B (p_X + q_X) + \Delta_3 p_A p_X q_B q_X. \end{aligned}$$

Appendix C. Derivation of moments of LD measurements

In the process of deriving the moments of LD measurements, we need to use the frequencies $P_1, P_2, \cdots, P_{22}$ listed in Table 1, and so we first discuss P_1 to P_{22}, and then derive various moments.

P_1 to P_{22}

Denote x_{ij} for the state indicator of allele A related to the j^{th} haplotype within the i^{th} individual at the first locus, and y_{ij} for that of B at the second locus, where $x_{ij} = 1$ if the allele copy at the first locus is A, otherwise $x_{ij} = 0$; the meaning of y_{ij} is similar. Moreover, we let the number of the sampled individuals be n (the sample size), and let the number of haplotypes within each individual be v (the ploidy level). Then P_1 to P_{22} can be expressed as follows.

Digenic:

$$P_1 = \hat{P}^{A\ldots}_{B\ldots} = \frac{1}{nv}\sum_{i}\sum_{j} x_{ij}y_{ij}$$

$$P_2 = \hat{P}^{A\,\cdot\,\ldots}_{\cdot\,B\ldots} = \frac{1}{nv(v-1)}\sum_{i}\sum_{j\neq j'} x_{ij}y_{ij'}$$

$$P_3 = \hat{P}^{A\ldots|\,\cdot\,\ldots}_{\cdot\,\ldots|B\ldots} = \frac{1}{n(n-1)v^2}\sum_{i\neq i'}\sum_{j,j'} x_{ij}y_{i'j'}$$

Trigenic:

$$P_4 = \hat{P}^{AA\ldots}_{B\,\cdot\,\ldots} + \hat{P}^{A\,\cdot\,\ldots}_{BB\ldots} = \frac{1}{nv(v-1)}\sum_{i}\sum_{j\neq j'}\left(x_{ij}y_{ij}x_{ij'} + x_{ij}y_{ij}y_{ij'}\right)$$

$$P_5 = \hat{P}^{AA\ldots|\,\cdot\,\ldots}_{\cdot\,\cdot\,\ldots|B\ldots} + \hat{P}^{\cdot\,\cdot\,\ldots|A\ldots}_{BB\ldots|\,\cdot\,\ldots} = \frac{1}{n(n-1)v^2(v-1)}\sum_{i\neq i'}\sum_{j\neq j'}\sum_{j''}\left(x_{ij}x_{ij'}y_{i'j''} + y_{ij}y_{ij'}x_{i'j''}\right)$$

$$P_6 = \hat{P}^{A\ldots|A\ldots}_{B\ldots|\,\cdot\,\ldots} + \hat{P}^{A\ldots|\,\cdot\,\ldots}_{B\ldots|B\ldots} = \frac{1}{n(n-1)v^2}\sum_{i\neq i'}\sum_{j,j'}\left(x_{ij}y_{ij}x_{i'j'} + x_{ij}y_{ij}y_{i'j'}\right)$$

$$P_7 = \hat{P}^{A\,\cdot\,\ldots|A\ldots}_{\cdot\,B\ldots|\,\cdot\,\ldots} + \hat{P}^{A\,\cdot\,\ldots|\,\cdot\,\ldots}_{\cdot\,B\ldots|B\ldots} = \frac{1}{n(n-1)v^2(v-1)}\sum_{i\neq i'}\sum_{j\neq j'}\sum_{j''}\left(x_{ij}y_{ij'}x_{i'j''} + x_{ij}y_{ij'}y_{i'j''}\right)$$

$$P_8 = \hat{P}^{A\ldots|A\ldots|\,\cdot\,\ldots}_{\cdot\,\ldots|\,\cdot\,\ldots|B\ldots} + \hat{P}^{A\ldots|\,\cdot\,\ldots|\,\cdot\,\ldots}_{\cdot\,\ldots|B\ldots|B\ldots}$$

$$= \frac{1}{n(n-1)(n-2)v^3}\sum_{\substack{i,i',i'' \\ \text{are distinct}}}\sum_{j,j',j''}\left(x_{ij}x_{i'j'}y_{i''j''} + x_{ij}y_{i'j'}y_{i''j''}\right)$$

$$P_9 = \hat{P}^{AA\cdot\ldots}_{\cdot\cdot B\ldots} + \hat{P}^{A\cdot\cdot\ldots}_{\cdot BB\ldots} = \frac{1}{nv(v-1)(v-2)} \sum_i \sum_{\substack{j,j',j'' \\ \text{are distinct}}} \left(x_{ij} x_{ij'} y_{ij''} + x_{ij} y_{ij'} y_{ij''} \right)$$

Quadgenic:

Dihaplotypic:

$$P_{10} = \hat{P}^{AA\ldots}_{BB\ldots} = \frac{1}{nv(v-1)} \sum_i \sum_{j \neq j'} x_{ij} x_{ij'} y_{ij} y_{ij'}$$

$$P_{11} = \hat{P}^{A\ldots|A\ldots}_{B\ldots|B\ldots} = \frac{1}{n(n-1)v^2} \sum_{i \neq i'} \sum_{j,j'} x_{ij} y_{ij} x_{i'j'} y_{i'j'}$$

Trihaplotypic:

$$P_{12} = \hat{P}^{A\ldots|A\cdot\ldots}_{B\ldots|\cdot B\ldots} = \frac{1}{n(n-1)v^2(v-1)} \sum_{i \neq i'} \sum_{j,j' \neq j''} x_{ij} y_{ij} x_{i'j'} y_{i'j''}$$

$$P_{13} = \hat{P}^{AA\ldots|\cdot\ldots}_{B\cdot\ldots|B\ldots} + \hat{P}^{A\cdot\ldots|A\ldots}_{BB\ldots|\cdot\ldots} = \frac{1}{n(n-1)v^2(v-1)} \sum_{i \neq i'} \sum_{j \neq j'} \sum_{j''} \left(x_{ij} y_{ij} x_{ij'} y_{i'j''} + x_{ij} y_{ij} y_{ij'} x_{i'j''} \right)$$

$$P_{14} = \hat{P}^{A\ldots|A\ldots|\cdot\ldots}_{B\ldots|\cdot\ldots|B\ldots} = \frac{1}{n(n-1)(n-2)v^3} \sum_{\substack{i,i',i'' \\ \text{are distinct}}} \sum_{j,j',j''} x_{ij} y_{ij} x_{i'j'} y_{i''j''}$$

$$P_{15} = \hat{P}^{AA\cdot\ldots}_{B\cdot B\ldots} = \frac{1}{nv(v-1)(v-2)} \sum_i \sum_{\substack{j,j',j'' \\ \text{are distinct}}} x_{ij} y_{ij} x_{ij'} y_{ij''}$$

Quadhaplotypic:

$$P_{16} = \hat{P}^{A\cdot\ldots|A\cdot\ldots}_{\cdot B\ldots|\cdot B\ldots} = \frac{1}{n(n-1)v^2(v-1)^2} \sum_{i \neq i'} \sum_{\substack{j \neq j' \\ j'' \neq j'''}} x_{ij} y_{ij'} x_{i'j''} y_{i'j'''}$$

$$P_{17} = \hat{P}^{AA\ldots|\cdot\cdot\ldots}_{\cdot\cdot\ldots|BB\ldots} = \frac{1}{n(n-1)v^2(v-1)^2} \sum_{i \neq i'} \sum_{\substack{j \neq j' \\ j'' \neq j'''}} x_{ij} x_{ij'} y_{i'j''} y_{i'j'''}$$

$$P_{18} = \hat{P}^{A\cdot\ldots|A\ldots|\cdot\ldots}_{\cdot B\ldots|\cdot\ldots|B\ldots} = \frac{1}{n(n-1)(n-2)v^3(v-1)} \sum_{\substack{i,i',i'' \\ \text{are distinct}}} \sum_{j \neq j'} \sum_{j'',j'''} x_{ij} y_{ij'} x_{i'j''} y_{i''j'''}$$

$$P_{19} = \hat{P}^{AA\ldots|\cdot\ldots|\cdot\ldots}_{\cdot\cdot\ldots|B\ldots|B\ldots} + \hat{P}^{\cdot\cdot\ldots|A\ldots|A\ldots}_{BB\ldots|\cdot\ldots|\cdot\ldots} = \frac{1}{n(n-1)(n-2)v^3(v-1)}$$

$$\sum_{\substack{i,i',i'' \\ \text{are distinct}}} \sum_{j \neq j'} \sum_{j'',j'''} \left(x_{ij} x_{ij'} y_{i'j''} y_{i''j'''} + y_{ij} y_{ij'} x_{i'j''} x_{i''j'''} \right)$$

$$P_{20} = \hat{P}^{A\ldots|A\ldots|\cdot\ldots|\cdot\ldots}_{\cdot\ldots|\cdot\ldots|B\ldots|B\ldots} = \frac{1}{n(n-1)(n-2)(n-3)v^4} \sum_{\substack{i,i',i'',i''' \\ \text{are distinct}}} \sum_{j,j',j'',j'''} x_{ij} x_{i'j'} y_{i''j''} y_{i'''j'''}$$

$$P_{21} = \hat{P}^{AA\cdot\cdot\ldots}_{\cdot\cdot BB\ldots} = \frac{1}{nv(v-1)(v-2)(v-3)} \sum_i \sum_{\substack{j,j',j''j''' \\ \text{are distinct}}} x_{ij} x_{ij'} y_{ij''} y_{ij'''}$$

$$P_{22} = \hat{P}^{AA\cdot\ldots|\cdot\ldots}_{\cdot\cdot B\ldots|B\ldots} + \hat{P}^{\cdot\cdot A\ldots|A\ldots}_{BB\cdot\ldots|\cdot\ldots} = \frac{1}{n(n-1)v^2(v-1)(v-2)}$$

$$\sum_{i\neq i'} \sum_{\substack{j,j',j'' \\ \text{are distinct}}} \sum_{j'''} (x_{ij}x_{ij'}y_{ij''}y_{i'j'''} + y_{ij}y_{ij'}x_{ij''}x_{i'j'''})$$

$\mathbf{E}\left(\hat{\boldsymbol{D}}_{\boldsymbol{w}}\right)$ and $\mathbf{E}\left(\hat{\boldsymbol{D}}_{\boldsymbol{w}}^{2}\right)$

$\hat{D}_w = \hat{P}^{A\ldots}_{B\ldots} - \hat{P}^{A\cdot\ldots}_{\cdot B\ldots} = P_1 - P_2$ by the definition of D_w, in which

$$P_1 = \frac{1}{nv}\sum_i \sum_j x_{ij}y_{ij} \text{ and } P_2 = \frac{1}{nv(v-1)}\sum_i \sum_{j\neq j'} x_{ij}y_{ij'}.$$

$\mathrm{E}\left(\hat{D}_w\right) = \mathrm{E}(P_1) - \mathrm{E}(P_2) = E_1 - E_2$

$\hat{D}_w^2 = (P_1 - P_2)^2$

$$= \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} x_{ij}y_{ij}x_{i'j'}y_{i'j'} - \frac{2}{n^2v^2(v-1)}\sum_{i,i'}\sum_{j,j'\neq j''} x_{ij}y_{ij}x_{i'j'}y_{i'j''}$$

$$+ \frac{1}{n^2v^2(v-1)^2}\sum_{i,i'}\sum_{j\neq j',j''\neq j'''} x_{ij}y_{ij'}x_{i'j''}y_{i'j'''}$$

$$= \frac{1}{n^2v^2}[C_1P_1 + C_{10}P_{10} + C_{11}P_{11}] - \frac{2}{n^2v^2(v-1)}[C_4P_4 + C_{12}P_{12} + C_{15}P_{15}]$$

$$+ \frac{1}{n^2v^2(v-1)^2}[C_2P_2 + C_9P_9 + C_{10}P_{10} + 2C_{15}P_{15} + C_{16}P_{16} + C_{21}P_{21}]$$

$$\mathrm{E}\left(\hat{D}_w^2\right) = \frac{1}{n^2v^2}[C_1E_1 + C_{10}E_{10} + C_{11}E_{11}] - \frac{2}{n^2v^2(v-1)}[C_4E_4 + C_{12}E_{12} + C_{15}E_{15}]$$

$$+ \frac{1}{n^2v^2(v-1)^2}[C_2E_2 + C_9E_9 + C_{10}E_{10} + 2C_{15}E_{15} + C_{16}E_{16} + C_{21}E_{21}]$$

where the coefficient C_i is the reciprocal of coefficient before the summation sign in the expression of P_i, e.g., the final coefficient C_{21} is $nv(v-1)(v-2)(v-3)$.

$\mathbf{E}\left(\hat{\boldsymbol{D}}_{\boldsymbol{b}}\right)$ and $\mathbf{E}\left(\hat{\boldsymbol{D}}_{\boldsymbol{b}}^{2}\right)$

$\hat{D}_b = \hat{P}^{A\cdot\ldots}_{\cdot B\ldots} - \hat{p}\hat{q} = P_2 - \hat{p}\hat{q}$ by the definition of D_b, in which

$$P_2 = \frac{1}{nv(v-1)}\sum_i \sum_{j\neq j'} x_{ij}y_{ij'} \quad \hat{p}_A = \frac{1}{nv}\sum_i \sum_j x_{ij} \text{ and } \hat{q}_B = \frac{1}{nv}\sum_i \sum_j y_{ij}.$$

$$\hat{D}_b = \frac{1}{nv(v-1)}\sum_i \sum_{j\neq j'} x_{ij}y_{ij'} - \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} x_{ij}y_{i'j'}$$

$$= P_2 - \frac{1}{n^2v^2}[nvP_1 + nv(v-1)P_2 + n(n-1)v^2P_3]$$

$$\mathrm{E}\left(\hat{D}_b\right) = E_2 - \frac{1}{n^2v^2}[C_1E_1 + C_2E_2 + C_3E_3]$$

$$\hat{D}_b^2 = \frac{1}{n^2v^2(v-1)^2}\sum_{i,i'}\sum_{\substack{j\neq j'\\ j''\neq j'''}} x_{ij}y_{ij'}x_{i'j''}y_{i'j'''}$$
$$-\frac{2}{n^3v^3(v-1)}\sum_{i,i',i''}\sum_{j\neq j'}\sum_{j'',j'''} x_{ij}y_{ij'}x_{i'j''}y_{i''j'''}$$
$$+\frac{1}{n^4v^4}\sum_{i,i',i'',i'''}\sum_{j,j',j'',j'''} x_{ij}y_{i'j'}x_{i''j''}y_{i'''j'''}$$

$$\mathrm{E}\left(\hat{D}_b^2\right) = \frac{1}{n^2v^2(v-1)^2}\left[C_2E_2 + C_9E_9 + C_{10}E_{10} + 2C_{15}E_{15} + C_{16}E_{16} + C_{21}E_{21}\right]$$
$$-\frac{2}{n^3v^3(v-1)}\left[C_2E_2 + C_4E_4 + C_7E_7 + C_9E_9 + C_{10}E_{10} + C_{12}E_{12} + C_{13}E_{13}\right.$$
$$\left.+ 3C_{15}E_{15} + C_{16}E_{16} + C_{18}E_{18} + C_{21}E_{21} + C_{22}E_{22}\right]$$
$$+\frac{1}{n^4v^4}\left[C_1E_1 + C_2E_2 + C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8\right.$$
$$+ C_9E_9 + 2C_{10}E_{10} + 2C_{11}E_{11} + 4C_{12}E_{12} + 4C_{13}E_{13} + 4C_{14}E_{14} + 4C_{15}E_{15}$$
$$\left.+ 2C_{16}E_{16} + C_{17}E_{17} + 4C_{18}E_{18} + C_{19}E_{19} + C_{20}E_{20} + C_{21}E_{21} + 2C_{22}E_{22}\right]$$

$\mathbf{E}\left(\hat{D}_w\hat{D}_b\right)$

$$\hat{D}_w\hat{D}_b = \left(\frac{1}{nv}\sum_i\sum_j x_{ij}y_{ij} - \frac{1}{nv(v-1)}\sum_i\sum_{j\neq j'} x_{ij}y_{ij'}\right)\left(\frac{1}{nv(v-1)}\sum_i\sum_{j\neq j'} x_{ij}y_{ij'} - \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} x_{ij}y_{i'j'}\right)$$

$$= \frac{1}{n^2v^2(v-1)}\sum_{i,i'}\sum_j\sum_{j'\neq j''} x_{ij}y_{ij}x_{i'j'}y_{i'j''} - \frac{1}{n^3v^3}\sum_{i,i',i''}\sum_{j,j',j''} x_{ij}y_{ij}x_{i'j'}y_{i''j''}$$

$$-\frac{1}{n^2v^2(v-1)^2}\sum_{i,i'}\sum_{\substack{j\neq j'\\ j''\neq j'''}} x_{ij}y_{ij'}x_{i'j''}y_{i'j'''} + \frac{1}{n^3v^3(v-1)}\sum_{i,i',i''}\sum_{j\neq j'}\sum_{j'',j'''} x_{ij}y_{ij'}x_{i'j''}y_{i''j'''}$$

$$\mathrm{E}\left(\hat{D}_w\hat{D}_b\right) = \frac{1}{n^2v^2(v-1)}\left[C_4E_4 + C_{12}E_{12} + C_{15}E_{15}\right]$$
$$-\frac{1}{n^3v^3}\left[C_1E_1 + C_4E_4 + C_6E_6 + C_{10}E_{10} + C_{11}E_{11} + C_{12}E_{12} + C_{13}E_{13} + C_{14}E_{14}\right.$$
$$\left.+ C_{15}E_{15}\right]$$
$$-\frac{1}{n^2v^2(v-1)^2}\left[C_2E_2 + C_9E_9 + C_{10}E_{10} + 2C_{15}E_{15} + C_{16}E_{16} + C_{21}E_{21}\right]$$
$$+\frac{1}{n^3v^3(v-1)}\left[C_2E_2 + C_4E_4 + C_7E_7 + C_9E_9 + C_{10}E_{10} + C_{12}E_{12} + C_{13}E_{13}\right.$$
$$\left.+ 3C_{15}E_{15} + C_{16}E_{16} + C_{18}E_{18} + C_{21}E_{21} + C_{22}E_{22}\right]$$

$\mathbf{E}\left(\hat{D}\right)$ and $\mathbf{E}\left(\hat{D}^2\right)$

$\hat{D} = \hat{D}_w + \hat{D}_b$ by the definition of D, then

$$\mathrm{E}\left(\hat{D}\right) = \mathrm{E}\left(\hat{D}_w\right) + \mathrm{E}\left(\hat{D}_b\right) = E_1 - \frac{1}{n^2v^2}\left[C_1E_1 + C_2E_2 + C_3E_3\right]$$

$$\hat{D}^2 = \left(\hat{D}_w + \hat{D}_b\right)^2 = \hat{D}_w^2 + 2\hat{D}_w\hat{D}_b + \hat{D}_b^2$$

$$\mathrm{E}\left(\hat{D}^2\right) = \mathrm{E}(\hat{D}_w^2) + 2\mathrm{E}(\hat{D}_w\hat{D}_b) + \mathrm{E}(\hat{D}_b^2)$$

$$= \frac{1}{n^2v^2}[C_1E_1 + C_{10}E_{10} + C_{11}E_{11}] - \frac{2}{n^2v^2(v-1)}[C_4E_4 + C_{12}E_{12} + C_{15}E_{15}]$$
$$+ \frac{1}{n^4v^4}[C_1E_1 + C_2E_2 + C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8$$
$$+ C_9E_9 + 2C_{10}E_{10} + 2C_{11}E_{11} + 4C_{12}E_{12} + 4C_{13}E_{13} + 4C_{14}E_{14} + 4C_{15}E_{15}$$
$$+ 2C_{16}E_{16} + C_{17}E_{17} + 4C_{18}E_{18} + C_{19}E_{19} + C_{20}E_{20} + C_{21}E_{21} + 2C_{22}E_{22}]$$
$$+ \frac{2}{n^2v^2(v-1)}[C_4E_4 + C_{12}E_{12} + C_{15}E_{15}]$$
$$- \frac{2}{n^3v^3}[C_1E_1 + C_4E_4 + C_6E_6 + C_{10}E_{10} + C_{11}E_{11} + C_{12}E_{12} + C_{13}E_{13} + C_{14}E_{14}$$
$$+ C_{15}E_{15}]$$

$\mathbf{E}(\hat{\Delta})$ and $\mathbf{E}(\hat{\Delta}^2)$

$\hat{\Delta} = \hat{D}_w + v\hat{D}_b$ by the definition of Δ, then

$$\mathrm{E}(\hat{\Delta}) = \mathrm{E}(\hat{D}_w) + v\mathrm{E}(\hat{D}_b) = E_1 - \frac{1}{n^2v}[C_1E_1 + C_2E_2 + C_3E_3]$$

$$\hat{\Delta}^2 = \left(\hat{D}_w + v\hat{D}_b\right)^2 = \hat{D}_w^2 + 2v\hat{D}_w\hat{D}_b + v^2\hat{D}_b^2$$

$$\mathrm{E}(\hat{\Delta}^2) = \mathrm{E}(\hat{D}_w^2) + 2v\mathrm{E}(\hat{D}_w\hat{D}_b) + v^2\mathrm{E}(\hat{D}_b^2)$$
$$= \frac{1}{n^2v^2}[C_1E_1 + C_{10}E_{10} + C_{11}E_{11}] - \frac{2}{n^2v^2(v-1)}[C_4E_4 + C_{12}E_{12} + C_{15}E_{15}]$$
$$+ \frac{1}{n^2v^2}[C_2E_2 + C_9E_9 + C_{10}E_{10} + 2C_{15}E_{15} + C_{16}E_{16} + C_{21}E_{21}]$$
$$- \frac{2}{n^3v^2}[C_2E_2 + C_4E_4 + C_7E_7 + C_9E_9 + C_{10}E_{10} + C_{12}E_{12} + C_{13}E_{13} + 3C_{15}E_{15}$$
$$+ C_{16}E_{16} + C_{18}E_{18} + C_{21}E_{21} + C_{22}E_{22}]$$
$$+ \frac{1}{n^4v^2}[C_1E_1 + C_2E_2 + C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8$$
$$+ C_9E_9 + 2C_{10}E_{10} + 2C_{11}E_{11} + 4C_{12}E_{12} + 4C_{13}E_{13} + 4C_{14}E_{14} + 4C_{15}E_{15}$$
$$+ 2C_{16}E_{16} + C_{17}E_{17} + 4C_{18}E_{18} + C_{19}E_{19} + C_{20}E_{20} + C_{21}E_{21} + 2C_{22}E_{22}]$$
$$+ \frac{2}{n^2v(v-1)}[C_4E_4 + C_{12}E_{12} + C_{15}E_{15}]$$
$$- \frac{2}{n^3v^2}[C_1E_1 + C_4E_4 + C_6E_6 + C_{10}E_{10} + C_{11}E_{11} + C_{12}E_{12} + C_{13}E_{13} + C_{14}E_{14}$$
$$+ C_{15}E_{15}]$$

$\mathbf{E}(\hat{Q})$

$\hat{Q} = \hat{p}_A\hat{p}_X\hat{q}_B\hat{q}_X = (\hat{p}_A - \hat{p}_A^2)(\hat{q}_B - \hat{q}_B^2)$ by $Q = p_Ap_Xq_Bq_X$, in which

$$\hat{p}_A = \frac{1}{nv}\sum_i\sum_j x_{ij} \text{ and } \hat{q}_B = \frac{1}{nv}\sum_i\sum_j y_{ij}.$$

$$\hat{Q} = \left(\frac{1}{nv}\sum_i\sum_j x_{ij} - \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} x_{ij}\,x_{i'j'}\right)\left(\frac{1}{nv}\sum_i\sum_j y_{ij} - \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} y_{ij}\,y_{i'j'}\right)$$
$$= \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} x_{ij}\,y_{i'j'} - \frac{1}{n^3v^3}\sum_{i,i',i''}\sum_{j,j',j''}\left(x_{ij}y_{i'j'}y_{i''j''} + x_{ij}x_{i'j'}y_{i''j''}\right)$$
$$+ \frac{1}{n^4v^4}\sum_{i,i',i'',i'''}\sum_{j,j',j'',j'''} x_{ij}x_{i'j'}y_{i''j''}y_{i'''j'''}$$

$$\mathrm{E}(\hat{Q}) = \frac{1}{n^2v^2}[C_1E_1 + C_2E_2 + C_3E_3]$$
$$- \frac{1}{n^3v^3}[2C_1E_1 + 2C_2E_2 + 2C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8 + C_9E_9]$$
$$+ \frac{1}{n^4v^4}[C_1E_1 + C_2E_2 + C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8 + C_9E_9 + 2C_{10}E_{10} + 2C_{11}E_{11} + 4C_{12}E_{12} + 4C_{13}E_{13} + 4C_{14}E_{14} + 4C_{15}E_{15} + 2C_{16}E_{16} + C_{17}E_{17} + 4C_{18}E_{18} + C_{19}E_{19} + C_{20}E_{20} + C_{21}E_{21} + 2C_{22}E_{22}]$$

$\mathbf{E}(\hat{\boldsymbol{R}})$

$\hat{P}_{AA} = \frac{1}{nv(v-1)}\sum_i \sum_{j\neq j'} x_{ij}x_{ij'}$ and $\hat{P}_{BB} = \frac{1}{nv(v-1)}\sum_i \sum_{j\neq j'} y_{ij}y_{ij'}$ by the definition of P_{AA}.

$\hat{R} = [\hat{p}_A - v\hat{p}_A^2 + (v-1)\hat{P}_{AA}][\hat{q}_B - v\hat{q}_B^2 + (v-1)\hat{P}_{BB})]$ by Equation (2).

$$\hat{R} = \left(\frac{1}{nv}\sum_i\sum_j x_{ij} - \frac{1}{n^2v}\sum_{i,i'}\sum_{j,j'} x_{ij}\, x_{i'j'} + \frac{1}{nv}\sum_i\sum_{j\neq j'} x_{ij}x_{ij'}\right)\left(\frac{1}{nv}\sum_i\sum_j y_{ij} - \frac{1}{n^2v}\sum_{i,i'}\sum_{j,j'} y_{ij}\, y_{i'j'} + \frac{1}{nv}\sum_i\sum_{j\neq j'} y_{ij}y_{ij'}\right)$$
$$= \frac{1}{n^2v^2}\sum_{i,i'}\sum_{j,j'} x_{ij}\, y_{i'j'} - \frac{1}{n^3v^2}\sum_{i,i',i''}\sum_{j,j',j''}\left(x_{ij}y_{i'j'}y_{i''j''} + x_{ij}x_{i'j'}y_{i''j''}\right)$$
$$+ \frac{1}{n^4v^2}\sum_{i,i',i'',i'''}\sum_{j,j',j'',j'''} x_{ij}x_{i'j'}y_{i''j''}y_{i'''j'''}$$
$$+ \frac{1}{n^2v^2}\sum_{i,i'}\sum_j\sum_{j'\neq j''}\left(x_{ij}y_{i'j'}y_{i'j''} + y_{ij}x_{i'j'}x_{i'j''}\right)$$
$$- \frac{1}{n^3v^2}\sum_{i,i',i''}\sum_{j,j'}\sum_{j''\neq j'''}\left(x_{ij}x_{i'j'}y_{i''j''}y_{i''j'''} + y_{ij}y_{i'j'}x_{i''j''}x_{i''j'''}\right)$$
$$+ \frac{1}{n^2v^2}\sum_{i,i'}\sum_{\substack{j\neq j'\\ j''\neq j'''}} x_{ij}x_{ij'}y_{i'j''}y_{i'j'''}$$

$$\mathrm{E}(\hat{R}) = \frac{1}{n^2v^2}[C_1E_1 + C_2E_2 + C_3E_3]$$
$$- \frac{1}{n^3v^2}[2C_1E_1 + 2C_2E_2 + 2C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8 + C_9E_9]$$
$$+ \frac{1}{n^4v^2}[C_1E_1 + C_2E_2 + C_3E_3 + 2C_4E_4 + C_5E_5 + 2C_6E_6 + 2C_7E_7 + C_8E_8 + C_9E_9 + 2C_{10}E_{10} + 2C_{11}E_{11} + 4C_{12}E_{12} + 4C_{13}E_{13} + 4C_{14}E_{14} + 4C_{15}E_{15} + 2C_{16}E_{16} + C_{17}E_{17} + 4C_{18}E_{18} + C_{19}E_{19} + C_{20}E_{20} + C_{21}E_{21} + 2C_{22}E_{22}]$$
$$+ \frac{1}{n^2v^2}[2C_4E_4 + C_5E_5 + C_9E_9] - \frac{1}{n^3v^2}[2C_4E_4 + C_5E_5 + C_9E_9 + 4C_{10}E_{10} + 4C_{13}E_{13} + 8C_{15}E_{15} + 2C_{17}E_{17} + C_{19}E_{19} + 2C_{21}E_{21} + 2C_{22}E_{22}]$$
$$+ \frac{1}{n^2v^2}[2C_{10}E_{10} + 4C_{15}E_{15} + C_{17}E_{17} + C_{21}E_{21}]$$

Appendix D. HS mating system

The double non-identity coefficients are closely related to the haplotypes that are used to detect the specific alleles. In this appendix, we consider such relationships in the haplotype sampling (HS) mating system. The effective population size N_e in this system is assumed to be the same as the population size N. We will adopt N_e instead of N in our discussion in order to accommodate other mating systems, and we will also write the double non-identity coefficient as the dni-coefficient for brevity.

For the case of dni-coefficients Θ_1 and Θ_2, it can be seen from Table 1 that only two haplotypes (say H_1 and H_2) are sampled, and both alleles in each haplotype need to be detected. These two haplotypes can be copied from either the same haplotype (written as $H_1 \equiv H_2$), or different haplotypes in the same individual (written as $H_1 \asymp H_2$), or different haplotypes in different individuals (written as $H_1 \sim H_2$). For this case, we will divide into three situations (named HSΘ1, HSΘ2 and HSΘ3) to carry out our discussion.

HSΘ1 $H_1 \equiv H_2$, weight 1, dni-coefficient 0;
HSΘ2 $H_1 \asymp H_2$, weight $v - 1$,
(a) none recombined, probability $(1-c)^2$, dni-coefficient Θ_1;
(b) one recombined, probability $2c(1-c)$, dni-coefficient $\frac{v-2}{v-1}\Gamma_4$;
(c) both recombined, prob. c^2, dni-coefficient $\frac{1}{(v-1)^2}\Theta_1 + \frac{2(v-2)}{(v-1)^2}\Gamma_4 + \frac{(v-2)(v-3)}{(v-1)^2}\Delta_6$;
HSΘ3 $H_1 \sim H_2$, weight $(N_e - 1)v$,
(a) none recombined, probability $(1-c)^2$, dni-coefficient Θ_2;
(b) one recombined, probability $2c(1-c)$, dni-coefficient Γ_1;
(c) both recombined, probability c^2, dni-coefficient Δ_1.

Now, the expression of dni-coefficients Θ_1' or Θ_2' in the next generation can be written out. In fact, let $\mathbf{W}_\theta = [w_{1\theta}, w_{2\theta}, w_{3\theta}]$ be the vector consisting of those weights, i.e. $\mathbf{W}_\theta = [1, v-1, (N_e-1)v]$, and let $\boldsymbol{\Theta} = [\theta_1, \theta_2, \theta_3]$, where each θ_i is the weighted sum of dni-coefficients in HSΘi, with the corresponding recombination probabilities as their weights (if the recombination probability does not occur, θ_i is set as the dni-coefficient in HSΘi), $i = 1, 2, 3$, that is

$$\theta_1 = 0,$$
$$\theta_2 = (1-c)^2\Theta_1 + 2c(1-c)\frac{v-2}{v-1}\Gamma_4 + c^2\left[\frac{1}{(v-1)^2}\Theta_1 + \frac{2(v-2)}{(v-1)^2}\Gamma_4 + \frac{(v-2)(v-3)}{(v-1)^2}\Delta_6\right],$$
$$\theta_3 = (1-c)^2\Theta_2 + 2c(1-c)\Gamma_1 + c^2\Delta_1.$$

Then $\Theta_1' = \Theta_2' = \mathbf{W}_\theta\boldsymbol{\Theta}^T/\mathbf{W}_\theta\mathbf{1} = \frac{w_{1\theta}\theta_1 + w_{2\theta}\theta_2 + w_{3\theta}\theta_3}{w_{1\theta} + w_{2\theta} + w_{3\theta}}$, where $\mathbf{1}$ is the column vector $[1,1,1]^T$. This is a linear combination of dni-coefficients in the current generation, and the products of combination coefficients times $N_e v(v-1)$ are listed in the second column of Table S3.

For the case of Γ_1 to Γ_4, it can be seen from Table 1 that three haplotypes are sampled, in which one is the haplotype that both alleles need to be detected, denoted by H_1, another is that only the allele at the first locus needs to be detected, denoted by H_2, and the third is that only the allele at the second locus needs to be detected, denoted by H_3. Because H_2 and H_3 are only detected the allele at a single locus, it is unnecessary to model their recombination. For this case, the combinations among three relations $\equiv$, $\asymp$ and $\sim$ can be divided into nine situations (named HSΓ1 to HSΓ9).

HSΓ1 $H_1 \equiv H_2 \equiv H_3$, weight 1, dni-coefficient 0;

HSΓ2 $H_1 \equiv H_2 \asymp H_3$ or $H_1 \equiv H_3 \asymp H_2$, weight $2(v-1)$,

(a) not recombined, probability $1-c$, dni-coefficient 0;

(b) recombined, probability c, dni-coefficient $\frac{1}{2(v-1)}\Theta_1 + \frac{v-2}{2(v-1)}\Gamma_4$;

HSΓ3 $H_1 \asymp H_2 \equiv H_3$, weight $v-1$,

(a) not recombined, probability $1-c$, dni-coefficient Θ_1;

(b) recombined, probability c, dni-coefficient $\frac{v-2}{v-1}\Gamma_4$;

HSΓ4 $H_1 \asymp H_2 \asymp H_3$, weight $(v-1)(v-2)$,

(a) not recombined, probability $1-c$, dni-coefficient Γ_4;

(b) recombined, probability c, dni-coefficient $\frac{1}{v-1}\Gamma_4 + \frac{v-3}{v-1}\Delta_6$;

HSΓ5 $H_1 \equiv H_2 \sim H_3$ or $H_1 \equiv H_3 \sim H_2$, weight $2(N_e-1)v$,

(a) not recombined, probability $1-c$, dni-coefficient 0;

(b) recombined, probability c, dni-coefficient $\Gamma_2/2$;

HSΓ6 $H_2 \equiv H_3 \sim H_1$, weight $(N_e-1)v$,

(a) not recombined, probability $1-c$, dni-coefficient Θ_2;

(b) recombined, probability c, dni-coefficient Γ_1;

HSΓ7 $H_1 \asymp H_2 \sim H_3$ or $H_1 \asymp H_3 \sim H_2$, weight $2(N_e-1)v(v-1)$,

(a) not recombined, probability $1-c$, dni-coefficient Γ_2;

(b) recombined, probability c, dni-coefficient $\frac{1}{2(v-1)}\Gamma_2 + \frac{v-2}{v-1}\Delta_7$;

HSΓ8 $H_1 \sim H_2 \asymp H_3$, weight $(N_e-1)v(v-1)$,

(a) not recombined, probability $1-c$, dni-coefficient Γ_1;

(b) recombined, probability c, dni- coefficient Δ_1;

HSΓ9 $H_1 \sim H_2 \sim H_3$, weight $(N_e-1)(NN_e-2)v^2$,

(a) not recombined, probability $1-c$, dni-coefficient Γ_3;

(b) recombined, probability c, dni-coefficient Δ_3.

Now, let $\mathbf{W}_\gamma = [w_{1\gamma}, w_{2\gamma}, \cdots, w_{9\gamma}]$ and $\mathbf{\Gamma} = [\gamma_1, \gamma_2, \cdots, \gamma_9]$, where the definitions of $\mathbf{W}_\gamma$ and $\mathbf{\Gamma}$ are similar to those of $\mathbf{W}_\theta$ and $\mathbf{\Theta}$. Then

$$\Gamma_1' = \Gamma_2' = \Gamma_3' = \Gamma_4' = \frac{\mathbf{W}_\gamma \mathbf{\Gamma}^T}{\mathbf{W}_\gamma \mathbf{1}} = \frac{w_{1\gamma}\gamma_1 + w_{2\gamma}\gamma_2 + \cdots + w_{9\gamma}\gamma_9}{w_{1\gamma} + w_{2\gamma} + \cdots + w_{9\gamma}}.$$

This is also a linear combination, and the products of combination coefficients times $N_e^2 v^2$ are listed in the third column of Table S3.

For the case of Δ_1 to Δ_7, we see from Table 1 that four haplotypes are sampled, in which two are the haplotypes that the allele at the first locus is detected, denoted by H_1 and H_2, and the other two are that the allele at the second locus is detected, denoted by H_3 and H_4. Because there is only the allele at a single locus to be detected, the recombination of these four haplotypes need not be modelled. For this case, the combinations of three relations can be divided into 22 situations (named HSΔ1 to HSΔ22).

HSΔ1 $H_1 \equiv H_2 \equiv H_3 \equiv H_4$, weight 1, dni-coefficient 0;

HSΔ2 $H_1 \equiv H_2 \asymp H_3 \equiv H_4$, weight $v-1$, dni-coefficient 0;

HSΔ3 $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $(N_e-1)v$, dni-coefficient 0;

HSΔ4 $H_1 \equiv H_2 \equiv H_3 \asymp H_4$ or $H_1 \equiv H_2 \equiv H_4 \asymp H_3$ or $H_1 \asymp H_2 \equiv H_3 \equiv H_4$ or $H_2 \asymp H_1 \equiv H_3 \equiv H_4$, weight $4(v-1)$, dni-coefficient 0;

HSΔ5 $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $2(N_e-1)v(v-1)$, dni-coefficient 0;

HSΔ6 $H_1 \equiv H_3 \equiv H_4 \sim H_2$ or $H_2 \equiv H_3 \equiv H_4 \sim H_1$ or $H_1 \equiv H_2 \equiv H_3 \sim H_4$ or $H_1 \equiv H_2 \equiv H_4 \sim H_3$, weight $4(N_e-1)v$, dni-coefficient 0;

HSΔ7 $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$ or $H_1 \equiv H_2 \asymp H_4 \sim H_3$, weight $4(N_e - 1)v(v - 1)$, dni-coefficient 0;

HSΔ8 $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $2(N_e - 1)(N_e - 2)v^2$, dni-coefficient 0;

HSΔ9 $H_1 \asymp H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \asymp H_4$, weight $2(v - 1)(v - 2)$, dni-coefficient 0;

HSΔ10 $H_1 \equiv H_3 \asymp H_2 \equiv H_4$ or $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $2(v - 1)$, dni-coefficient Θ_1;

HSΔ11 $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $2(N_e - 1)v$, dni-coefficient Θ_2;

HSΔ12 $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$ or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $4(N_e - 1)v(v - 1)$, dni-coefficient Γ_1;

HSΔ13 $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$ or $H_1 \equiv H_3 \asymp H_4 \sim H_2$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $8(N_e - 1)v(v - 1)$, dni-coefficient Γ_2;

HSΔ14 $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $4(N_e - 1)(N_e - 2)v^2$, dni-coefficient Γ_3;

HSΔ15 $H_1 \equiv H_3 \asymp H_2 \asymp H_4$ or $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $4(v - 1)(v - 2)$, dni-coefficient Γ_4;

HSΔ16 $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $2(N_e - 1)v(v - 1)^2$, dni-coefficient Δ_1;

HSΔ17 $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $(N_e - 1)v(v - 1)^2$, dni-coefficient Δ_2;

HSΔ18 $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $4(N_e - 1)(N_e - 2)v^2(v - 1)$, dni-coefficient Δ_3;

HSΔ19 $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $2(N_e - 1)(N_e - 2)v^2(v - 1)$, dni-coefficient Δ_4;

HSΔ20 $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $(N_e - 1)(N_e - 2)(N_e - 3)v^3$, dni-coefficient Δ_5;

HSΔ21 $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $(v - 1)(v - 2)(v - 3)$, dni-coefficient Δ_6;

HSΔ22 $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$ weight $4(N_e - 1)v(v - 1)(v - 2)$, dni-coefficient Δ_7.

Now, let $\mathbf{W}_\delta$ and $\boldsymbol{\Delta}$ be the row vectors consisting of 22 weights and 22 dni-coefficients in HSΔ1 to HSΔ22, respectively. Then

$$\Delta_1' = \Delta_2' = \cdots = \Delta_7' = \mathbf{W}_\delta \boldsymbol{\Delta}^T / \mathbf{W}_\delta \mathbf{1}.$$

This is still a linear combination, and the products of combination coefficients times $N_e^3 v^3$ are listed in the final column of Table S3.

The expressions in Table S3 are the essential factors to form $\boldsymbol{\Omega}^T$ of the transition matrix $\boldsymbol{\Omega}$ for the HS mating system. Moreover, the matrices $\mathbf{T}$ and $\mathbf{S}$ in the principal part of $\boldsymbol{\Omega}$ are listed in Appendix I.

Appendix E. Corrections for finite sample size

Matrix $\mathbf{A}$ can be decomposed as the following combination:

$$\mathbf{A} = \mathbf{A}_1 + n^{-1}\mathbf{A}_2 + n^{-2}\mathbf{A}_3 + n^{-3}\mathbf{A}_4 + \boldsymbol{O}(n^{-4}),$$

where n is the sample size, and the principal parts can be calculated by:

$$\mathbf{A}_1 = \lim_{n\to\infty} \mathbf{A},$$

$$\mathbf{A}_2 = \lim_{n\to\infty} n(\mathbf{A} - \mathbf{A}_1),$$

$$\mathbf{A}_3 = \lim_{n\to\infty} n^2(\mathbf{A} - \mathbf{A}_1 - \mathbf{A}_2/n),$$

$$\mathbf{A}_4 = \lim_{n\to\infty} n^3(\mathbf{A} - \mathbf{A}_1 - \mathbf{A}_2/n - \mathbf{A}_3/n^2).$$

The elements in $\mathbf{A}_1$ and $\mathbf{A}_2$ are listed in Tables S1 and S2, respectively.

It is clear from $\mathbf{M}_{\boldsymbol{\omega}} = \boldsymbol{\omega}^T\mathbf{A}$ that the next formula is valid:

$$\mathbf{M}_{\boldsymbol{\omega}} = \boldsymbol{\omega}^T\mathbf{A}_1 + n^{-1}\boldsymbol{\omega}^T\mathbf{A}_2 + n^{-2}\boldsymbol{\omega}^T\mathbf{A}_3 + n^{-3}\boldsymbol{\omega}^T\mathbf{A}_4 + \boldsymbol{\omega}^T\boldsymbol{O}(n^{-4}). \tag{S1}$$

When the sample size n is large enough, the principal part of $\mathbf{M}_{\boldsymbol{\omega}}$ is $\boldsymbol{\omega}^T\mathbf{A}_1$, and the remainder can be neglected, then $\mathbf{M}_{\boldsymbol{\omega}} \approx \boldsymbol{\omega}^T\mathbf{A}_1 = \mathbf{M}_{\boldsymbol{\omega}1}$, indicating that the matrix $\mathbf{A}_1$ can be used to approximate the moments of LD measurements. We will use HS mating system as an example to illustrate, using the fourth and the sixth column elements in $\mathbf{A}_1$ (Table S1) and the approximated $\boldsymbol{\omega}$ in Section 'Approximations',

$$\boldsymbol{\omega} \approx \left[1 + \frac{2c + c_1^2 v - 1}{c_2 c v_1 v N_e}, 1 + \frac{2c + c_1^2 v - 1}{c_2 c v_1 v N_e}, 1, 1, \cdots, 1\right]^T,$$

we have

$$d_{\mathrm{HS}}^2 = \frac{\mathrm{E}(\widehat{D}^2)}{\mathrm{E}(\widehat{Q})} \approx \frac{\boldsymbol{\omega}^T\mathbf{A}_1^{(4)}}{\boldsymbol{\omega}^T\mathbf{A}_1^{(6)}} = \frac{c^2 v + (1-2c)v_1}{(2-c)cN_e v_1 v} = d_{\mathrm{HS}1}^2,$$

where $\mathbf{A}_j^{(i)}$ denotes the i^{th} column of $\mathbf{A}_j$. Similarly, δ^2 can be approximately expressed as

$$\delta_{\mathrm{HS}}^2 = \frac{\mathrm{E}(\widehat{\Delta}^2)}{\mathrm{E}(\widehat{R})} \approx \frac{\boldsymbol{\omega}^T\mathbf{A}_1^{(5)}}{\boldsymbol{\omega}^T\mathbf{A}_1^{(7)}} = \frac{c^2 v + (1-2c)v_1}{(2-c)cN_e v_1 v} = \delta_{\mathrm{HS}1}^2.$$

In real studies, the finite sample size n will influence the estimation of $\hat{r}^2$ and $\hat{r}_\Delta^2$. For example, if the two loci are unlinked, $r^2 = r_\Delta^2 = 0$ while $\hat{r}^2$ and $\hat{r}_\Delta^2$ are greater than zero. To avoid such an error, higher-order terms in Equation (S1) should be considered. To accommodate this effect, we use the following approximates to include more higher-order terms:

$$\mathbf{M}_{\boldsymbol{\omega}} \approx \boldsymbol{\omega}^T\mathbf{A}_1 + n^{-1}\boldsymbol{\omega}^T\mathbf{A}_2 = \mathbf{M}_{\boldsymbol{\omega}2},$$

$$\mathbf{M}_{\boldsymbol{\omega}} \approx \boldsymbol{\omega}^T\mathbf{A}_1 + n^{-1}\boldsymbol{\omega}^T\mathbf{A}_2 + n^{-2}\boldsymbol{\omega}^T\mathbf{A}_3 = \mathbf{M}_{\boldsymbol{\omega}3},$$

$$\mathbf{M}_{\boldsymbol{\omega}} \approx \boldsymbol{\omega}^T\mathbf{A}_1 + n^{-1}\boldsymbol{\omega}^T\mathbf{A}_2 + n^{-2}\boldsymbol{\omega}^T\mathbf{A}_3 + n^{-3}\boldsymbol{\omega}^T\mathbf{A}_4 = \mathbf{M}_{\boldsymbol{\omega}4}.$$

The resulting approximations of d_{HS}^2 and δ_{HS}^2 are respectively $d_{\mathrm{HS}2}^2$, $d_{\mathrm{HS}3}^2$, $d_{\mathrm{HS}4}^2$, $\delta_{\mathrm{HS}2}^2$, $\delta_{\mathrm{HS}3}^2$ and $\delta_{\mathrm{HS}4}^2$. For example,

$$d_{\mathrm{HS}2}^2 = \frac{c^2 v(N_e v_1 - nv + 3) - v_1(nv - 3) + 2cv_1(N_e v - nv + 3)}{c_2 c N_e v_1 v(nv - 2)}.$$

The difference $d_2^2 - d_1^2$ can be expanded as

$$d_{\mathrm{HS}2}^2 - d_{\mathrm{HS}1}^2 = \frac{1}{nv - 2} + \frac{1}{c_2 v_1 N_e(nv - 2)}\left(\frac{1}{c} + c_2 + \frac{2}{v} - \frac{1}{cv}\right).$$

The rightmost term is tiny and can be ignored. The net effect for d_{HS}^2 caused by including $\mathbf{A}_2$ is approximately $\frac{1}{nv-2}$. This can be written as

$$\lim_{N_e \to \infty}(d_{\text{HS2}}^2 - d_{\text{HS1}}^2) = \frac{1}{nv-2}.$$

We use the same way and derived the following differences

$$\lim_{N_e \to \infty}(d_{\text{HS3}}^2 - d_{\text{HS1}}^2) = \frac{1}{nv-1},$$

$$\lim_{N_e \to \infty}(d_{\text{HS4}}^2 - d_{\text{HS1}}^2) = \frac{1}{nv-1}.$$

It can be found the sequence of differences is converged to $\frac{1}{nv-1}$. Similarly, for δ^2, the sequence is

$$\lim_{N_e \to \infty}(\delta_{\text{HS2}}^2 - \delta_{\text{HS1}}^2) = \frac{1}{n-2},$$

$$\lim_{N_e \to \infty}(\delta_{\text{HS3}}^2 - \delta_{\text{HS1}}^2) = \frac{1}{n-1},$$

$$\lim_{N_e \to \infty}(\delta_{\text{HS4}}^2 - \delta_{\text{HS1}}^2) = \frac{1}{n-1}.$$

Thus, the approximate expressions of d^2 and δ^2 considered the effect of sample size n on sampling are

$$d_{\text{HS}}^2 \approx \frac{c^2 v + (1-2c)v_1}{(2-c)cN_e v_1 v} + \frac{1}{vn-1},$$

$$\delta_{\text{HS}}^2 \approx \frac{c^2 v + (1-2c)v_1}{(2-c)cN_e v_1 v} + \frac{1}{n-1}.$$

For the remaining mating systems, the same method can be used, and the compensate term is the same.

Appendix F. MS and ME mating systems

The method to derive the expression of each element in $\mathbf{\Omega}$ for the monoecious mating systems is the same as that for the HS mating system. It is noteworthy that unlike the HS mating system, two haplotypes sampled within the same individual need to be detected whether they are from the same gamete: (i) if they are from the same gamete, the probability is $\frac{v/2-1}{v-1}$; (ii) otherwise, the probability is $\frac{v/2}{v-1}$. We will denote $(H, H', \cdots)$ for which those haplotypes within brackets are from the same gamete.

For (i), it is assumed that the chromosomes form bivalents during meiosis, and the double-reduction will never happen, then the paired chromosomes will segregate into different oocytes, which means that two haplotypes H and H' within the same gamete are copied from different haplotypes. However, this is not strictly equivalent to $H \asymp H'$. This is because the paired chromosomes will segregate into different oocytes. In order to avoid repetition, we first discuss nine situations similar to HSΓ1 to HSΓ9 in Appendix D, named MOΓ1 to MOΓ9 in turn.

MOΓ1 $H_1 \equiv H_2 \equiv H_3$, weight 1, dni-coefficient 0;

MOΓ2 $H_1 \equiv H_2 \asymp H_3$ or $H_1 \equiv H_3 \asymp H_2$, weight $2(v-1)$,
- (a) not recombined, probability $1-c$, dni-coefficient 0;
- (b) recombined, probability c, dni-coefficient $\Gamma_4/2$;

MOΓ3 $H_1 \asymp H_2 \equiv H_3$, weight $v-1$,
- (a) not recombined, probability $1-c$, dni-coefficient Θ_1;
- (b) recombined, probability c, dni-coefficient Γ_4;

MOΓ4 $H_1 \asymp H_2 \asymp H_3$, weight $(v-1)(v-2)$,
   (a) not recombined, probability $1-c$, dni-coefficient Γ_4;
   (b) recombined, probability c, dni-coefficient $\frac{1}{2v-4}\Gamma_4 + \frac{v-3}{v-2}\Delta_6$;

MOΓ5 $H_1 \equiv H_2 \sim H_3$ or $H_1 \equiv H_3 \sim H_2$, weight $2(N_e-1)v$,
   (a) not recombined, probability $1-c$, dni-coefficient 0;
   (b) recombined, probability c, dni-coefficient $\Gamma_2/2$;

MOΓ6 $H_2 \equiv H_3 \sim H_1$, weight $(N_e-1)v$,
   (a) not recombined, probability $1-c$, dni-coefficient Θ_2;
   (b) recombined, probability c, dni-coefficient Γ_1;

MOΓ7 $H_1 \asymp H_2 \sim H_3$ or $H_1 \asymp H_3 \sim H_2$, weight $2(N_e-1)v(v-1)$,
   (a) not recombined, probability $1-c$, dni-coefficient Γ_2;
   (b) recombined, probability c, dni-coefficient Δ_7;

MOΓ8 $H_1 \sim H_2 \asymp H_3$, weight $(N_e-1)v(v-1)$,
   (a) not recombined, probability $1-c$, dni-coefficient Γ_1;
   (b) recombined, probability c, dni-coefficient Δ_1;

MOΓ9 $H_1 \sim H_2 \sim H_3$, weight $(N_e-1)(N_e-2)v^2$,
   (a) not recombined, probability $1-c$, dni-coefficient Γ_3;
   (b) recombined, probability c, dni-coefficient Δ_3.

For (ii), if the mating system is MS, because selfing is allowed, two haplotypes H and H' from different gametes can be copied from either the same haplotype ($H \equiv H'$), or different haplotypes in the same individual ($H \asymp H'$), or different haplotypes in different individuals ($H \sim H'$). These relationships are obviously equivalent to those in the HS mating system. If the mating system is ME, because selfing is excluded, two haplotypes from different gametes must be from different individuals ($H \sim H'$).

There may be more than one situation of HSΘ (HSΓ, HSΔ or MOΓ) appearing in an item, and we will use some symbols to denote this phenomenon. For example, the symbol (MOΓ2/2,3,4,7/2) in the item (1) of Γ_2' below denotes that there are four situations (i.e., MOΓ2/2, MOΓ3, MOΓ4 and MOΓ7/2) appearing in this item, in which MOΓ2/2 represents that the number of expressions describing the relations among the haplotypes is half of that for MOΓ2. This is because H_1 and H_2 are in the same gamete in the item (1) of Γ_2', thus only the second expression in the two expressions in MOΓ2 ($H_1 \equiv H_2 \asymp H_3$ and $H_1 \equiv H_3 \asymp H_2$) can hold. The weight for MOΓ2/2 is also half of the weight for MOΓ2, and the meaning of MOΓ7/2 is analogous. It is noteworthy that MOΓ2 and MOΓ2/2 are the same except their number of expressions and weights, as are MOΓ7 and MOΓ7/2.

We next discuss the dni-coefficients in the next generation one by one.

Θ_1':

(1) (H_1, H_2), probability $\frac{v/2-1}{v-1}$,
   (a) none recombined, probability $(1-c)^2$, dni-coefficient Θ_1;
   (b) one recombined, probability $2c(1-c)$, dni-coefficient Γ_4;
   (c) both recombined, probability c^2, dni-coefficient Δ_6;

(2) $(H_1), (H_2)$, probability $\frac{v/2}{v-1}$,
   others identical to (HSΘ1-3) for MS;
   others identical to (HSΘ3) for ME;

then $\Theta_1' = m_\theta + \frac{v/2}{v-1}\mathbf{W}_\theta\mathbf{\Theta}^T/\mathbf{W}_\theta\mathbf{1}$ for MS, or $\Theta_1' = m_\theta + \frac{v/2}{v-1}(w_{3\theta}\theta_3/\mathbf{W}_\theta\mathbf{1})$ for ME, where

$$m_\theta = \frac{v/2-1}{v-1}[(1-c)^2\Theta_1 + 2c(1-c)\Gamma_4 + c^2\Delta_6].$$

The expression of Θ_1' is a linear combination of dni-coefficients in the current generation, whose combination coefficients are the first row of $\mathbf{\Omega}$ for the MS or the ME mating system.

Θ_2': identical to (HSΘ1-3), and thus $\Theta_2' = \mathbf{W}_\theta \mathbf{\Theta}^T / \mathbf{W}_\theta \mathbf{1}$ for MS or ME.

Γ_1':

(1) $H_1, (H_2, H_3)$, probability $\frac{v/2-1}{v-1}$, others identical to (HSΓ2,4,8);

(2) $H_1, (H_2), (H_3)$, probability $\frac{v/2}{v-1}$, others identical to (HSΓ1-9) for MS; others identical to (HSΓ5,7,9) for ME;

then $\Gamma_1' = m_\gamma + \frac{v/2}{v-1} \mathbf{W}_\gamma \mathbf{\Gamma}^T / \mathbf{W}_\gamma \mathbf{1}$ for MS or $\Gamma_1' = m_\gamma + \frac{v/2}{v-1} \frac{w_{5\gamma}\gamma_5 + w_{7\gamma}\gamma_7 + w_{9\gamma}\gamma_9}{\mathbf{W}_\gamma \mathbf{1}}$ for ME, where

$$m_\gamma = \frac{v/2-1}{v-1} \frac{w_{2\gamma}\gamma_2 + w_{4\gamma}\gamma_4 + w_{8\gamma}\gamma_8}{\mathbf{W}_\gamma \mathbf{1}}.$$

In this way, the expressions of other double non-identity coefficients can be written down, so can the elements in the corresponding rows of $\mathbf{\Omega}$. We will omit these lengthy explanations from the following discussion.

Γ_2':

(1) $(H_1, H_2), H_3$, probability $\frac{v/2-1}{v-1}$, others identical to (MOΓ2/2,3,4,7/2);

(2) $(H_1), (H_2), H_3$, probability $\frac{v/2}{v-1}$, MS: identical to (HSΓ1-9); ME: identical to (HSΓ5/2,6,7/2,8,9).

Here *`others'* is omitted for brevity, the same below.

Γ_3': identical to (HSΓ1-9).

Γ_4':

(1) (H_1, H_2, H_3), probability $\frac{(v/2-1)(v/2-2)}{(v-1)(v-2)}$,
(a) not recombined, probability $1-c$, dni-coefficient Γ_4;
(b) recombined, probability c, dni-coefficient Δ_6;

(2) $(H_1, H_2), (H_3)$ or $(H_1, H_3), (H_2)$, probability $\frac{2(v/2-1)(v/2)}{(v-1)(v-2)}$, MS: identical to (MOΓ2/2,3,4,7/2); ME: identical to (MOΓ7/2);

(3) $(H_1), (H_2, H_3)$, probability $\frac{(v/2-1)(v/2)}{(v-1)(v-2)}$, MS: identical to (HSΓ2,4,8); ME: identical to (HSΓ8).

Δ_1':

(1) $(H_1, H_3), (H_2, H_4)$, probability $\frac{(v/2-1)^2}{(v-1)^2}$, identical to (HSΔ2,9,10/2,15/2,16/2,21);

(2) $(H_1, H_3), (H_2)(H_4)$ or $(H_1), (H_3), (H_2, H_4)$, probability $\frac{2(v/2-1)(v/2)}{(v-1)^2}$, MS: identical to (HSΔ2,4/2,7/2,9,10/2,12/4,13/4,15*3/4,16/2,18/4,21,22/2); ME: identical to (HSΔ7/2,13/4,18/4,22/2);

(3) $(H_1), (H_3), (H_2), (H_4)$, probability $\frac{(v/2)^2}{(v-1)^2}$, MS: identical to (HSΔ1-22); ME: identical to (HSΔ3,5,8,11/2,12/2,14/2,16/2,17,18/2,19,20).

Δ_2':

(1) $(H_1,H_2),(H_3,H_4)$, probability $\frac{(v/2-1)^2}{(v-1)^2}$, identical to (HSΔ10,15,17,21);

(2) $(H_1,H_2),(H_3),(H_4)$ or $(H_1),(H_2),(H_3,H_4)$, probability $\frac{2(v/2-1)(v/2)}{(v-1)^2}$, MS: identical to (HS Δ 4/2,5/2,9/2,10,13/2,15,17,19/2,21,22/2); ME: identical to (HSΔ13/2,19/2,22/2);

(3) $(H_1),(H_2),(H_3),(H_4)$, probability $\frac{(v/2)^2}{(v-1)^2}$, MS: identical to (HS Δ 1-22); ME: identical to (HSΔ11,12,14,16,18,20).

Δ_3':

(1) (H_1,H_3), probability $\frac{v/2-1}{v-1}$, identical to (HSΔ2,4/2,7/2,9,10/2,12/4,13/4,15*3/4,16/2,18/4,21,22/2);

(2) $(H_1),(H_3)$, probability $\frac{v/2}{v-1}$, MS: identical to (HS Δ 1-22); ME: identical to (HS Δ 3,5,6/2,7/2,8,11/2,12/2,13/2,14*3/4,16/2,17,18*3/4,19,20, 22/2).

Δ_4':

(1) (H_1,H_2), probability $\frac{v/2-1}{v-1}$, identical to (HSΔ4/2,5/2,9/2,10,13/2,15,17,19/2,21,22/2);

(2) $(H_1),(H_2)$, probability $\frac{v/2}{v-1}$, MS: identical to (HS Δ 1-22); ME: identical to (HSΔ6/2,7/2,8/2,11,12,13/2,14,16,18,19/2,20,22/2).

Δ_5': identical to (HSΔ1-22).

Δ_6':

(1) (H_1,H_2,H_3,H_4), probability $\frac{(v-4)(v-6)}{8(v-1)(v-3)}$, identical to (HSΔ21);

(2) $(H_1),(H_2,H_3,H_4)$ or $(H_2),(H_1,H_3,H_4)$ or $(H_3),(H_1,H_2,H_4)$ or $(H_4),(H_1,H_2,H_3)$, probability $\frac{v(v-4)}{2(v-1)(v-3)}$, MS: identical to (HS Δ 9/2,15/2,21,22/4); ME: identical to (HSΔ22/4);

(3) $(H_1,H_2),(H_3,H_4)$, probability $\frac{v(v-2)}{8(v-1)(v-3)}$, MS: identical to (HS Δ 10,15,17,21); ME: identical to (HSΔ17);

(4) $(H_1,H_3),(H_2,H_4)$ or $(H_1,H_4),(H_2,H_3)$ probability $\frac{v(v-2)}{4(v-1)(v-3)}$, MS: identical to (HS Δ 2,9,10/2,15/2,16/2,21); ME: identical to (HSΔ16/2).

Δ_7':

(1) (H_1,H_2,H_3), probability $\frac{v-4}{4(v-1)}$, identical to (HSΔ9/2,15/2,21,22/4);

(2) $(H_1,H_3),(H_2)$ or $(H_1),(H_2,H_3)$, probability $\frac{2v}{4(v-1)}$, MS: identical to (HS Δ 2,4/2,7/2,9,10/2,12/4,13/4,15*3/4,16/2,18/4,21,22/2); ME: identical to (HSΔ7/4,12/4,13/8,16/2,18/4,22/4);

(3) $(H_1,H_2),(H_3)$, probability $\frac{v}{4(v-1)}$, MS: identical to (HSΔ4/2,5/2,9/2,10,13/2,15,17,19/2,21,22/2);

ME: identical to (HSΔ5/2,13/4,17,19/2,22/4).

The transition matrix $\mathbf{\Omega}$ for the MS or the ME mating system is not shown, but the matrices $\mathbf{T}$ and $\mathbf{S}$ in the principal part of $\mathbf{\Omega}$ are listed in Appendix I.

Appendix G. DR mating system

In the dioecious mating systems, no matter whether it is DR or DH, each individual is formed by a sperm and an egg that are independently sampled from the sperm and the egg pools, respectively. We will discuss the double non-identity coefficients one by one for the DR mating system in this appendix.

For simplicity, we will use the symbol $\{*\}$ to be the identifier of an item, e.g., the item {HSΘ2} means that its contents are the same as those in HSΘ2 except for the weight, and we also use the symbol MEΘ_1' to represent Θ_1' in ME, and so on.

Θ_1': identical to MEΘ_1'.

Θ_2': identical to MEΘ_2'.

Γ_1': identical to MEΓ_1'.

Γ_2': identical to MEΓ_2'.

Γ_3':
{HSΓ1} $H_1 \equiv H_2 \equiv H_3$, weight $2(1+f^2)$;
{HSΓ2} $H_1 \equiv H_2 \asymp H_3$ or $H_1 \equiv H_3 \asymp H_2$, weight $4(1+f^2)(v-1)$;
{HSΓ3} $H_1 \asymp H_2 \equiv H_3$, weight $2(1+f^2)(v-1)$;
{HSΓ4} $H_1 \asymp H_2 \asymp H_3$, weight $2(1+f^2)(v-1)(v-2)$;
{HSΓ5} $H_1 \equiv H_2 \sim H_3$ or $H_1 \equiv H_3 \sim H_2$, weight $2(1+f)^2 vN_e - 4(1+f^2)v$;
{HSΓ6} $H_2 \equiv H_3 \sim H_1$, weight $(1+f)^2 vN_e - 2(1+f^2)v$;
{HSΓ7} $H_1 \asymp H_2 \sim H_3$ or $H_1 \asymp H_3 \sim H_2$,
weight $2(1+f)^2(v-1)vN_e - 4(1+f^2)(v-1)v$;
{HSΓ8} $H_1 \sim H_2 \asymp H_3$, weight $(1+f)^2(v-1)vN_e - 2(1+f^2)(v-1)v$;
{HSΓ9} $H_1 \sim H_2 \sim H_3$, weight $(1+f)^2 v^2 N_e^2 - 3(1+f)^2 v^2 N_e + 4(1+f^2)v^2$;

then $\Gamma_3' = \mathbf{W}_\gamma^* \mathbf{\Gamma}^T / \mathbf{W}_\gamma^* \mathbf{1}$, where $\mathbf{W}_\gamma^*$ is the row vector consisting of the above nine weights.

Γ_4': identical to MEΓ_4'.

Δ_1':
(1) $(H_1, H_3), (H_2, H_4)$, probability $\frac{(v/2-1)^2}{(v-1)^2}$, identical to (HSΔ2,9,10/2,15/2,16/2,21);
(2) $(H_1, H_3), (H_2)(H_4)$ or $(H_1), (H_3), (H_2, H_4)$, probability $\frac{2(v/2-1)(v/2)}{(v-1)^2}$,
identical to (HSΔ7/2,13/4,18/4,22/2);
(3) $(H_1), (H_3), (H_2), (H_4)$, probability $\frac{(v/2)^2}{(v-1)^2}$,
{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $8f$;
{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $16f(v-1)$;
{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $2(1+f)^2 vN_e - 16fv$;
{HSΔ11/2} $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $8f$;

{HSΔ12/2} $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$, weight $16f(v-1)$;
{HSΔ14/2} $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$,
weight $2(1+f)^2 vN_e - 16fv$;
{HSΔ16/2} $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $8f(v-1)^2$;
{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $8f(v-1)^2$;
{HSΔ18/2} $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$,
weight $2(1+f)^2(v-1)vN_e - 16f(v-1)v$;
{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$,
weight $2(1+f)^2(v-1)vN_e - 16f(v-1)v$;
{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $(1+f)^2v^2N_e^2 - 4(1+f)^2v^2N_e + 16fv^2$.

Δ_2':
(1) $(H_1, H_2), (H_3, H_4)$, probability $\frac{(v/2-1)^2}{(v-1)^2}$, identical to (HSΔ10,15,17,21);
(2) $(H_1, H_2), (H_3), (H_4)$ or $(H_1), (H_2), (H_3, H_4)$, probability $\frac{2(v/2-1)(v/2)}{(v-1)^2}$,
identical to (HSΔ13/2,19/2,22/2);
(3) $(H_1), (H_2), (H_3), (H_4)$, probability $\frac{(v/2)^2}{(v-1)^2}$,
{HSΔ11} $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $16f$;
{HSΔ12} $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$
or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $32f(v-1)$;
{HSΔ14} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$
or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $4(1+f)^2vN_e - 32fv$;
{HSΔ16} $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $16f(v-1)^2$;
{HSΔ18} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$
or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $4(1+f)^2(v-1)vN_e - 32f(v-1)v$;
{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $(1+f)^2v^2N_e^2 - 4(1+f)^2v^2N_e + 16fv^2$;

Δ_3':
(1) (H_1, H_3), probability $\frac{v/2-1}{v-1}$,
{HSΔ2} $H_1 \equiv H_2 \asymp H_3 \equiv H_4$, weight $2(1+f^2)$;
{HSΔ4/2} $H_1 \equiv H_2 \equiv H_4 \asymp H_3$ or $H_1 \asymp H_2 \equiv H_3 \equiv H_4$, weight $4(1+f^2)$;
{HSΔ7/2} $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$,
weight $2(1+f)^2vN_e - 4(1+f^2)v$;
{HSΔ9} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \asymp H_4$, weight $4(1+f^2)(v-2)$;
{HSΔ10/2} $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $2(1+f^2)$;
{HSΔ12/4} $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $(1+f)^2vN_e - 2(1+f^2)v$;
{HSΔ13/4} $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$,
weight $2(1+f)^2vN_e - 4(1+f^2)v$;
{HSΔ15*3/4} $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$,
weight $6(1+f^2)(v-2)$;
{HSΔ16/2} $H_1 \asymp H_3 \sim H_2 \asymp H_4$, weight $(1+f)^2(v-1)vN_e - 2(1+f^2)(v-1)v$;
{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$,
weight $(1+f)^2v^2N_e^2 - 3(1+f)^2v^2N_e + 4(1+f^2)v^2$;
{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $2(1+f^2)(v-2)(v-3)$;
{HSΔ22/2} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$,
weight $2(1+f)^2(v-2)vN_e - 4(1+f^2)(v-2)v$;
(2) $(H_1), (H_3)$, probability $\frac{v/2}{v-1}$,
{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $4f$;
{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $8f(v-1)$;

{HSΔ6/2} $H_2 \equiv H_3 \equiv H_4 \sim H_1$ or $H_1 \equiv H_2 \equiv H_4 \sim H_3$, weight $4(1+f^2)$;

{HSΔ7/2} $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_4 \sim H_3$, weight $4(1+f^2)(v-1)$;

{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $2(1+f)^2 vN_e - 4(1+f)^2 v$;

{HSΔ11/2} $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $4f$;

{HSΔ12/2} $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$, weight $8f(v-1)$;

{HSΔ13/2} $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $8(1+f^2)(v-1)$;

{HSΔ14*3/4} $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $3(1+f)^2 vN_e - 8(1+f+f^2)v$;

{HSΔ16/2} $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $4f(v-1)^2$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $4f(v-1)^2$;

{HSΔ18*3/4} $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $3(1+f)^2(v-1)vN_e - 8(1+f+f^2)(v-1)v$;

{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $2(1+f)^2(v-1)vN_e - 4(1+f)^2(v-1)v$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $(1+f)^2 v^2 N_e^2 - 5(1+f)^2 v^2 N_e + 8(1+f+f^2)v^2$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_4 \sim H_3$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$, weight $4(1+f^2)(v-1)(v-2)$;

Δ_4':

(1) (H_1, H_2), probability $\frac{v/2-1}{v-1}$,

{HSΔ4/2} $H_1 \asymp H_2 \equiv H_3 \equiv H_4$ or $H_2 \asymp H_1 \equiv H_3 \equiv H_4$, weight $4(1+f^2)$;

{HSΔ5/2} $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $(1+f)^2 vN_e - 2(1+f^2)v$;

{HSΔ9/2} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$, weight $2(1+f^2)(v-2)$;

{HSΔ10} $H_1 \equiv H_3 \asymp H_2 \equiv H_4$ or $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $4(1+f^2)$;

{HSΔ13/2} $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$, weight $4(1+f)^2 vN_e - 8(1+f^2)v$;

{HSΔ15} $H_1 \equiv H_3 \asymp H_2 \asymp H_4$ or $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $8(1+f^2)(v-2)$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $(1+f)^2(v-1)vN_e - 2(1+f^2)(v-1)v$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$, weight $(1+f)^2 v^2 N_e^2 - 3(1+f)^2 v^2 N_e + 4(1+f^2)v^2$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $2(1+f^2)(v-2)(v-3)$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$, weight $2(1+f)^2(v-2)vN_e - 4(1+f^2)(v-2)v$;

(2) $(H_1), (H_2)$, probability $\frac{v/2}{v-1}$,

{HSΔ6/2} $H_1 \equiv H_3 \equiv H_4 \sim H_2$ or $H_2 \equiv H_3 \equiv H_4 \sim H_1$, weight $4(1+f^2)$;

{HSΔ7/2} $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_2 \sim H_1 \asymp H_3 \equiv H_4$, weight $4(1+f^2)(v-1)$;

{HSΔ8/2} $H_1 \sim H_2 \sim H_3 \equiv H_4$, weight $(1+f)^2 vN_e - 4(1+f^2)v$;

{HSΔ11} $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $8f$;

{HSΔ12} $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$ or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $16f(v-1)$;

{HSΔ13/2} $H_1 \equiv H_3 \asymp H_4 \sim H_2$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $8(1+f^2)(v-1)$;

{HSΔ14} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $4(1+f)^2 vN_e - 8(1+f)^2 v$;

{HSΔ16} $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $8f(v-1)^2$;

{HSΔ18} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $4(1+f)^2(v-1)vN_e - 8(1+f)^2(v-1)v$;

{HSΔ19/2} $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $(1+f)^2(v-1)vN_e - 4(1+f^2)(v-1)v$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $(1+f)^2v^2N_e^2 - 5(1+f)^2v^2N_e + 8(1+f+f^2)v^2$;

{HSΔ22/2} $H_1 \asymp H_3 \asymp H_4 \sim H_2$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$, weight $4(1+f^2)(v-1)(v-2)$;

Δ_5':

{HSΔ1} $H_1 \equiv H_2 \equiv H_3 \equiv H_4$, weight $4(1-f+f^2)$;

{HSΔ2} $H_1 \equiv H_2 \asymp H_3 \equiv H_4$, weight $4(1-f+f^2)(v-1)$;

{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $(1+f)^2vN_e - 4(1-f+f^2)v$;

{HSΔ4} $H_1 \equiv H_2 \equiv H_3 \asymp H_4$ or $H_1 \equiv H_2 \equiv H_4 \asymp H_3$ or $H_1 \asymp H_2 \equiv H_3 \equiv H_4$ or $H_2 \asymp H_1 \equiv H_3 \equiv H_4$, weight $16(1-f+f^2)(v-1)$;

{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $2(1+f)^2N_e(v-1)v - 8(1-f+f^2)(v-1)v$;

{HSΔ6} $H_1 \equiv H_3 \equiv H_4 \sim H_2$ or $H_2 \equiv H_3 \equiv H_4 \sim H_1$ or $H_1 \equiv H_2 \equiv H_3 \sim H_4$ or $H_1 \equiv H_2 \equiv H_4 \sim H_3$, weight $8(1+f^2)N_ev - 16(1-f+f^2)v$;

{HSΔ7} $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$ or $H_1 \equiv H_2 \asymp H_4 \sim H_3$, weight $8(1+f^2)N_ev(v-1) - 16(1-f+f^2)v(v-1)$;

{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $2(1+f)^2N_e^2v^2 - 2(5+2f+5f^2)N_ev^2 + 16(1-f+f^2)v^2$;

{HSΔ9} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \asymp H_4$, weight $8(1-f+f^2)(v-1)(v-2)$;

{HSΔ10} $H_1 \equiv H_3 \asymp H_2 \equiv H_4$ or $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $8(1-f+f^2)(v-1)$;

{HSΔ11} $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $2(1+f)^2N_ev - 8(1-f+f^2)v$;

{HSΔ12} $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$ or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $4(1+f)^2N_e(v-1)v - 16(1-f+f^2)(v-1)v$;

{HSΔ13} $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$ or $H_1 \equiv H_3 \asymp H_4 \sim H_2$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $16(1+f^2)N_e(v-1)v - 32(1-f+f^2)(v-1)v$;

{HSΔ14} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $4(1+f)^2N_e^2v^2 - 4(5+2f+5f^2)N_ev^2 + 32(1-f+f^2)v^2$;

{HSΔ15} $H_1 \equiv H_3 \asymp H_2 \asymp H_4$ or $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $16(1-f+f^2)(v-1)(v-2)$;

{HSΔ16} $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $2(1+f)^2N_e(v-1)^2v - 8(1-f+f^2)(v-1)^2v$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $(1+f)^2N_e(v-1)^2v - 4(1-f+f^2)(v-1)^2v$;

{HSΔ18} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $4(1+f)^2N_e^2v^2(v-1) - 4(5+2f+5f^2)N_ev^2(v-1) + 32(1-f+f^2)v^2(v-1)$;

{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $2(1+f)^2N_e^2(v-1)v^2 - 2(5+2f+5f^2)N_e(v-1)v^2 + 16(1-f+f^2)(v-1)v^2$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $(1+f)^2N_e^3v^3 - 6(1+f)^2N_e^2v^3 +$

$(19 + 6f + 19f^2)N_e v^3 - 24(1 - f + f^2)v^3$

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $4(1 - f + f^2)(v - 1)(v - 2)(v - 3)$;

{HSΔ22} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$, weight $8(1 + f^2)N_e(v - 1)(v - 2)v - 16(1 - f + f^2)(v - 1)(v - 2)v$,

then $\Delta_5' = \mathbf{W}_\delta^* \boldsymbol{\Delta}^T / \mathbf{W}_\delta^* \mathbf{1}$ where $\mathbf{W}_\delta^*$ is the row vector consisting of the above 22 weights.

Δ_6': identical to MEΔ_6'.

Δ_7': identical to MEΔ_7'.

The transition matrix $\boldsymbol{\Omega}$ for the DR mating system is not shown, but the matrices $\mathbf{T}$ and $\mathbf{S}$ in the principal part of $\boldsymbol{\Omega}$ are listed in Appendix I.

Appendix H. DH mating system

For the DH mating system, because each individual remains in a reproductive unit for its entire lifetime, the offspring produced within each reproductive unit are either full- or half-sibs. We will denote $[H, H', \cdots]$ for which those haplotypes within square brackets are from the same reproductive unit.

Θ_1': identical to MEΘ_1'.

Θ_2': identical to MEΘ_2'.

Γ_1': identical to MEΓ_1'.

Γ_2': identical to MEΓ_2'.

Γ_3':

(1) $[H_1, H_2, H_3]$, probability $\frac{1}{M^2}$,

{HSΓ1} $H_1 \equiv H_2 \equiv H_3$, weight $\frac{1}{8v^2} + \frac{1}{8f^2v^2}$;

{HSΓ2} $H_1 \equiv H_2 \asymp H_3$ or $H_1 \equiv H_3 \asymp H_2$ weight $\frac{v-1}{4v^2} + \frac{v-1}{4f^2v^2}$;

{HSΓ3} $H_1 \asymp H_2 \equiv H_3$, weight $\frac{v-1}{8v^2} + \frac{v-1}{8f^2v^2}$;

{HSΓ4} $H_1 \asymp H_2 \asymp H_3$, weight $\frac{(v-1)(v-2)}{8v^2} + \frac{(v-1)(v-2)}{8f^2v^2}$;

{HSΓ5} $H_1 \equiv H_2 \sim H_3$ or $H_1 \equiv H_3 \sim H_2$, weight $\frac{1}{4v} + \frac{1}{4fv} + \frac{f-1}{4f^2v}$;

{HSΓ6} $H_2 \equiv H_3 \sim H_1$, weight $\frac{1}{8v} + \frac{1}{8fv} + \frac{f-1}{8f^2v}$;

{HSΓ7} $H_1 \asymp H_2 \sim H_3$ or $H_1 \asymp H_3 \sim H_2$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv} + \frac{(v-1)(f-1)}{4f^2v}$;

{HSΓ8} $H_1 \sim H_2 \asymp H_3$, weight $\frac{v-1}{8v} + \frac{v-1}{8fv} + \frac{(v-1)(f-1)}{8f^2v}$;

{HSΓ9} $H_1 \sim H_2 \sim H_3$, weight $\frac{3(f-1)}{8f} + \frac{(f-1)(f-2)}{8f^2}$;

(2) $[H_1, H_2], [H_3]$ or $[H_1, H_3], [H_2]$, probability $\frac{2(M-1)}{M^2}$,

{HSΓ5/2} $H_1 \equiv H_2 \sim H_3$ or $H_1 \equiv H_3 \sim H_2$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΓ7/2} $H_1 \asymp H_2 \sim H_3$ or $H_1 \asymp H_3 \sim H_2$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΓ9} $H_1 \sim H_2 \sim H_3$, weight $\frac{f-1}{4f} + \frac{1}{2}$;

(3) $[H_2, H_3], [H_1]$, probability $\frac{M-1}{M^2}$,

{HSΓ6} $H_2 \equiv H_3 \sim H_1$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΓ8} $H_1 \sim H_2 \asymp H_3$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΓ9} $H_1 \sim H_2 \sim H_3$, weight $\frac{f-1}{4f} + \frac{1}{2}$;

(4) $[H_1], [H_2], [H_3]$, probability $\frac{(M-1)(M-2)}{M^2}$, identical to (HSΓ9).

Γ_4':

(1) (H_1, H_2, H_3), probability $\frac{(v/2-1)(v/2-2)}{(v-1)(v-2)}$,

(a) not recombined, probability $1 - c$, double non-identity Γ_4;

(b) recombined, probability c, double non-identity Δ_6;

(2) $(H_1, H_2), (H_3)$ or $(H_1, H_3), (H_2)$, probability $\frac{2(v/2-1)(v/2)}{(v-1)(v-2)}$, identical to (MOΓ7/2);

(3) $(H_1), (H_2, H_3)$, probability $\frac{(v/2-1)(v/2)}{(v-1)(v-2)}$, identical to (MOΓ8).

Δ_1':

(1) $[(H_1, H_3), (H_2, H_4)]$, probability $\frac{1}{M}\frac{(v/2-1)^2}{(v-1)^2}$,

{HSΔ2} $H_1 \equiv H_2 \asymp H_3 \equiv H_4$, weight $\frac{1}{4v(v-1)} + \frac{1}{4fv(v-1)}$;

{HSΔ9} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \asymp H_4$, weight $\frac{v-2}{2v(v-1)} + \frac{v-2}{2fv(v-1)}$;

{HSΔ10/2} $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $\frac{1}{4v(v-1)} + \frac{1}{4fv(v-1)}$;

{HSΔ15/2} $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$, weight $\frac{v-2}{2v(v-1)} + \frac{v-2}{2fv(v-1)}$;

{HSΔ16/2} $H_1 \asymp H_3 \sim H_2 \asymp H_4$, weight $\frac{1}{2} + \frac{f-1}{4f}$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $\frac{(v-2)(v-3)}{4v(v-1)} + \frac{(v-2)(v-3)}{4fv(v-1)}$;

(2) $[(H_1, H_3), (H_2), (H_4)]$ or $[(H_1), (H_3), (H_2, H_4)]$, probability $\frac{2}{M}\frac{(v/2-1)(v/2)}{(v-1)^2}$,

{HSΔ7/2} $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$, weight $\frac{1}{2v} + \frac{1}{2fv}$;

{HSΔ13/4} $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$, weight $\frac{1}{2v} + \frac{1}{2fv}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$, weight $\frac{f-1}{2f}$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$, weight $\frac{v-2}{2v} + \frac{v-2}{2fv}$;

(3) $[(H_1), (H_3), (H_2), (H_4)]$, probability $\frac{1}{M}\frac{(v/2)^2}{(v-1)^2}$,

{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{2fv^2}$;

{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $\frac{v-1}{fv^2}$;

{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{2fv}$;

{HSΔ11/2} $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $\frac{1}{2fv^2}$;

{HSΔ12/2} $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$, weight $\frac{v-1}{fv^2}$;

{HSΔ14/2} $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$, weight $\frac{f-1}{2fv}$;

{HSΔ16/2} $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $\frac{(v-1)^2}{2fv^2}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{(v-1)^2}{2fv^2}$;

{HSΔ18/2} $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$, weight $\frac{(v-1)(f-1)}{2vf}$;

{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{(v-1)(f-1)}{2vf}$;

(4) $[(H_1, H_3)], [(H_2, H_4)]$, probability $\frac{M-1}{M}\frac{(v/2-1)^2}{(v-1)^2}$, identical to (HSΔ16/2);

(5) $[(H_1,H_3)],[(H_2),(H_4)]$ or $[(H_1),(H_3)],[(H_2,H_4)]$, probability $\frac{2(M-1)}{M}\frac{(v/2-1)(v/2)}{(v-1)^2}$, identical to (HSΔ18/4);

(6) $[(H_1),(H_3)],[(H_2),(H_4)]$, probability $\frac{M-1}{M}\frac{(v/2)^2}{(v-1)^2}$, identical to (HSΔ20).

Δ_2':

(1) $[(H_1,H_2),(H_3,H_4)]$, probability $\frac{1}{M}\frac{(v/2-1)^2}{(v-1)^2}$,

{HSΔ10} $H_1 \equiv H_3 \asymp H_2 \equiv H_4$ or $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $\frac{1}{2v(v-1)}+\frac{1}{2fv(v-1)}$;

{HSΔ15} $H_1 \equiv H_3 \asymp H_2 \asymp H_4$ or $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $\frac{v-2}{v(v-1)}+\frac{v-2}{fv(v-1)}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{1}{2}+\frac{f-1}{4f}$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $\frac{(v-2)(v-3)}{4v(v-1)}+\frac{(v-2)(v-3)}{4fv(v-1)}$;

(2) $[(H_1,H_2),(H_3),(H_4)]$ or $[(H_1),(H_2),(H_3,H_4)]$, probability $\frac{2}{M}\frac{(v/2-1)(v/2)}{(v-1)^2}$,

{HSΔ13/2} $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$, weight $\frac{1}{v}+\frac{1}{vf}$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{2f}$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$, weight $\frac{v-2}{2v}+\frac{v-2}{2vf}$;

(3) $[(H_1),(H_2),(H_3),(H_4)]$, probability $\frac{1}{M}\frac{(v/2)^2}{(v-1)^2}$,

{HSΔ11} $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $\frac{1}{fv^2}$;

{HSΔ12} $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$ or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $\frac{2(v-1)}{fv^2}$;

{HSΔ14} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{f-1}{fv}$;

{HSΔ16} $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $\frac{(v-1)^2}{fv^2}$;

{HSΔ18} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{(f-1)(v-1)}{fv}$;

(4) $[(H_1,H_2)],[(H_3,H_4)]$, probability $\frac{M-1}{M}\frac{(v/2-1)^2}{(v-1)^2}$, identical to (HSΔ17);

(5) $[(H_1,H_2)],[(H_3),(H_4)]$ or $[(H_1),(H_2)],[(H_3,H_4)]$, probability $\frac{2(M-1)}{M}\frac{(v/2-1)(v/2)}{(v-1)^2}$, identical to (HSΔ19/2);

(6) $[(H_1),(H_2)],[(H_3),(H_4)]$, probability $\frac{M-1}{M}\frac{(v/2)^2}{(v-1)^2}$, identical to (HSΔ20).

Δ_3':

(1) $[(H_1,H_3),H_2,H_4]$, probability $\frac{1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ2} $H_1 \equiv H_2 \asymp H_3 \equiv H_4$, weight $\frac{1}{8v^2}+\frac{1}{8v^2f^2}$;

{HSΔ4/2} $H_1 \equiv H_2 \equiv H_4 \asymp H_3$ or $H_1 \asymp H_2 \equiv H_3 \equiv H_4$, weight $\frac{1}{4v^2}+\frac{1}{4v^2f^2}$;

{HSΔ7/2} $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$, weight $\frac{1}{4v}+\frac{1}{4vf}+\frac{f-1}{4vf^2}$;

{HSΔ9} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \asymp H_4$, weight $\frac{v-2}{4v^2}+\frac{v-2}{4v^2f^2}$;

{HSΔ10/2} $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $\frac{1}{8v^2}+\frac{1}{8v^2f^2}$;

{HSΔ12/4} $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $\frac{1}{8v}+\frac{1}{8fv}+\frac{f-1}{8f^2v}$;

{HSΔ13/4} $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$, weight $\frac{1}{4v} + \frac{1}{4fv} + \frac{f-1}{4f^2v}$;

{HSΔ15*3/4} $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $\frac{3(v-2)}{8v^2} + \frac{3(v-2)}{8f^2v^2}$;

{HSΔ16/2} $H_1 \asymp H_3 \sim H_2 \asymp H_4$, weight $\frac{v-1}{8v} + \frac{v-1}{8fv} + \frac{(f-1)(v-1)}{8f^2v}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$, weight $\frac{3(f-1)}{8f} + \frac{(f-1)(f-2)}{8f^2}$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $\frac{(v-2)(v-3)}{8v^2} + \frac{(v-2)(v-3)}{8f^2v^2}$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$, weight $\frac{v-2}{4v} + \frac{v-2}{4fv} + \frac{(v-2)(f-1)}{4f^2v}$;

(2) $[(H_1),(H_3),H_2,H_4]$, probability $\frac{1}{M^2}\frac{v/2}{v-1}$,

{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{4fv^2}$;

{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $\frac{v-1}{2fv^2}$;

{HSΔ6/2} $H_2 \equiv H_3 \equiv H_4 \sim H_1$ or $H_1 \equiv H_2 \equiv H_4 \sim H_3$, weight $\frac{1}{4v^2} + \frac{1}{4f^2v^2}$;

{HSΔ7/2} $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_4 \sim H_3$, weight $\frac{v-1}{4v^2} + \frac{v-1}{4f^2v^2}$;

{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{4fv} + \frac{f-1}{4f^2v}$;

{HSΔ11/2} $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $\frac{1}{4fv^2}$;

{HSΔ12/2} $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$, weight $\frac{v-1}{2fv^2}$;

{HSΔ13/2} $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $\frac{v-1}{2v^2} + \frac{v-1}{2f^2v^2}$;

{HSΔ14*3/4} $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{f-1}{4fv} + \frac{f-1}{2f^2v}$;

{HSΔ16/2} $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $\frac{(v-1)^2}{4fv^2}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{(v-1)^2}{4fv^2}$;

{HSΔ18*3/4} $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{(v-1)(f-1)}{4fv} + \frac{(v-1)(f-1)}{2f^2v}$;

{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{(v-1)(f-1)}{4fv} + \frac{(v-1)(f-1)}{4f^2v}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{(f-1)(f-2)}{4f^2}$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_4 \sim H_3$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$, weight $\frac{(v-1)(v-2)}{4v^2} + \frac{(v-1)(v-2)}{4f^2v^2}$;

(3) $[(H_1,H_3)],[H_2,H_4]$, probability $\frac{M-1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ12/4} $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ16/2} $H_1 \asymp H_3 \sim H_2 \asymp H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$, weight $\frac{f-1}{4f} + \frac{1}{2}$;

(4) $[(H_1),(H_3)],[H_2,H_4]$, probability $\frac{M-1}{M^2}\frac{v/2}{v-1}$,

{HSΔ14/4} $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ18/4} $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{4f} + \frac{1}{2}$;

(5) $[(H_1,H_3),H_2],[H_4]$, probability $\frac{M-1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ7/4} $H_1 \equiv H_2 \asymp H_3 \sim H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ13/8} $H_2 \equiv H_3 \asymp H_1 \sim H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$, weight $\frac{f-1}{4f} + \frac{1}{2}$;

{HSΔ22/4} $H_1 \asymp H_2 \asymp H_3 \sim H_4$, weight $\frac{v-2}{4v} + \frac{v-2}{4fv}$;

(6) $[(H_1), (H_3), H_2], [H_4]$, probability $\frac{M-1}{M^2}\frac{v/2}{v-1}$,

{HSΔ8/2} $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ14/4} $H_2 \equiv H_3 \sim H_1 \sim H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ18/4} $H_2 \asymp H_3 \sim H_1 \sim H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{2f}$;

(7) $[(H_1, H_3), H_4], [H_2]$, probability $\frac{M-1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ7/4} $H_2 \sim H_1 \asymp H_3 \equiv H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ13/8} $H_1 \equiv H_4 \asymp H_3 \sim H_2$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$, weight $\frac{f-1}{4f} + \frac{1}{2}$;

{HSΔ22/4} $H_1 \asymp H_3 \asymp H_4 \sim H_2$, weight $\frac{v-2}{4v} + \frac{v-2}{4fv}$;

(8) $[(H_1), (H_3), H_4], [H_2]$, probability $\frac{M-1}{M^2}\frac{v/2}{v-1}$,

{HSΔ8/2} $H_1 \sim H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ14/4} $H_1 \equiv H_4 \sim H_2 \sim H_3$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ18/4} $H_1 \asymp H_4 \sim H_2 \sim H_3$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ19/2} $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{2f}$;

(9) $[(H_1, H_3)], [H_2], [H_4]$, probability $\frac{(M-1)(M-2)}{M^2}\frac{v/2-1}{v-1}$, identical to (HSΔ18/4);

(10) $[(H_1), (H_3)], [H_2], [H_4]$, probability $\frac{(M-1)(M-2)}{M^2}\frac{v/2}{v-1}$, identical to (HSΔ20).

Δ_4':

(1) $[(H_1, H_2), H_3, H_4]$, probability $\frac{1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ4/2} $H_1 \asymp H_2 \equiv H_3 \equiv H_4$ or $H_2 \asymp H_1 \equiv H_3 \equiv H_4$, weight $\frac{1}{4v^2} + \frac{1}{4f^2v^2}$;

{HSΔ5/2} $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{8v} + \frac{1}{8fv} + \frac{f-1}{8f^2v}$;

{HSΔ9/2} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$, weight $\frac{v-2}{8v^2} + \frac{v-2}{8f^2v^2}$;

{HSΔ10} $H_1 \equiv H_3 \asymp H_2 \equiv H_4$ or $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $\frac{1}{4v^2} + \frac{1}{4f^2v^2}$;

{HSΔ13/2} $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$, weight $\frac{1}{2v} + \frac{1}{2fv} + \frac{f-1}{2f^2v}$;

{HSΔ15} $H_1 \equiv H_3 \asymp H_2 \asymp H_4$ or $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $\frac{v-2}{2v^2} + \frac{v-2}{2f^2v^2}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{v-1}{8v} + \frac{v-1}{8fv} + \frac{(f-1)(v-1)}{8f^2v}$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$, weight $\frac{3(f-1)}{8f} + \frac{(f-1)(f-2)}{8f^2}$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $\frac{(v-2)(v-3)}{8v^2} + \frac{(v-2)(v-3)}{8f^2v^2}$;

{HSΔ22/2} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$, weight $\frac{v-2}{4v}+\frac{v-2}{4fv}+\frac{(v-2)(f-1)}{4f^2v}$;

(2) $[(H_1),(H_2),H_3,H_4]$, probability $\frac{1}{M^2}\frac{v/2}{v-1}$,

{HSΔ6/2} $H_1\equiv H_3\equiv H_4\sim H_2$ or $H_2\equiv H_3\equiv H_4\sim H_1$, weight $\frac{1}{4v^2}+\frac{1}{4f^2v^2}$;

{HSΔ7/2} $H_1\sim H_2\asymp H_3\equiv H_4$ or $H_2\sim H_1\asymp H_3\equiv H_4$, weight $\frac{v-1}{4v^2}+\frac{v-1}{4f^2v^2}$;

{HSΔ8/2} $H_1\sim H_2\sim H_3\equiv H_4$, weight $\frac{f-1}{4f^2v}$;

{HSΔ11} $H_1\equiv H_3\sim H_2\equiv H_4$ or $H_1\equiv H_4\sim H_2\equiv H_3$, weight $\frac{1}{2fv^2}$;

{HSΔ12} $H_1\equiv H_3\sim H_2\asymp H_4$ or $H_1\equiv H_4\sim H_2\asymp H_3$ or $H_2\equiv H_3\sim H_1\asymp H_4$ or $H_2\equiv H_4\sim H_1\asymp H_3$, weight $\frac{v-1}{fv^2}$;

{HSΔ13/2} $H_1\equiv H_3\asymp H_4\sim H_2$ or $H_1\equiv H_4\asymp H_3\sim H_2$ or $H_2\equiv H_3\asymp H_4\sim H_1$ or $H_2\equiv H_4\asymp H_3\sim H_1$, weight $\frac{v-1}{2v^2}+\frac{v-1}{2f^2v^2}$;

{HSΔ14} $H_1\equiv H_3\sim H_2\sim H_4$ or $H_1\equiv H_4\sim H_2\sim H_3$ or $H_2\equiv H_3\sim H_1\sim H_4$ or $H_2\equiv H_4\sim H_1\sim H_3$, weight $\frac{f-1}{2vf}+\frac{f-1}{2f^2v}$;

{HSΔ16} $H_1\asymp H_3\sim H_2\asymp H_4$ or $H_1\asymp H_4\sim H_2\asymp H_3$, weight $\frac{(v-1)^2}{2fv^2}$;

{HSΔ18} $H_1\asymp H_3\sim H_2\sim H_4$ or $H_1\asymp H_4\sim H_2\sim H_3$ or $H_2\asymp H_3\sim H_1\sim H_4$ or $H_2\asymp H_4\sim H_1\sim H_3$, weight $\frac{(v-1)(f-1)}{2fv}+\frac{(v-1)(f-1)}{2f^2v}$;

{HSΔ19/2} $H_1\sim H_2\sim H_3\asymp H_4$, weight $\frac{(f-1)(v-1)}{4f^2v}$;

{HSΔ20} $H_1\sim H_2\sim H_3\sim H_4$, weight $\frac{(f-1)(f-2)}{4f^2}$;

{HSΔ22/2} $H_1\asymp H_3\asymp H_4\sim H_2$ or $H_2\asymp H_3\asymp H_4\sim H_1$, weight $\frac{(v-1)(v-2)}{4v^2}+\frac{(v-1)(v-2)}{4f^2v^2}$;

(3) $[(H_1,H_2)],[H_3,H_4]$, probability $\frac{M-1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ5/2} $H_1\asymp H_2\sim H_3\equiv H_4$, weight $\frac{1}{4v}+\frac{1}{4fv}$;

{HSΔ17} $H_1\asymp H_2\sim H_3\asymp H_4$, weight $\frac{v-1}{4v}+\frac{v-1}{4fv}$;

{HSΔ19/2} $H_1\asymp H_2\sim H_3\sim H_4$, weight $\frac{f-1}{4f}+\frac{1}{2}$;

(4) $[(H_1),(H_2)],[H_3,H_4]$, probability $\frac{M-1}{M^2}\frac{v/2}{v-1}$,

{HSΔ8/2} $H_1\sim H_2\sim H_3\equiv H_4$, weight $\frac{1}{4v}+\frac{1}{4fv}$;

{HSΔ19/2} $H_1\sim H_2\sim H_3\asymp H_4$, weight $\frac{v-1}{4v}+\frac{v-1}{4fv}$;

{HSΔ20} $H_1\sim H_2\sim H_3\sim H_4$, weight $\frac{f-1}{4f}+\frac{1}{2}$;

(5) $[(H_1,H_2),H_3],[H_4]$, probability $\frac{M-1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ13/4} $H_1\equiv H_3\asymp H_2\sim H_4$ or $H_2\equiv H_3\asymp H_1\sim H_4$, weight $\frac{1}{2v}+\frac{1}{2fv}$;

{HSΔ19/2} $H_1\asymp H_2\sim H_3\sim H_4$, weight $\frac{f-1}{4f}+\frac{1}{2}$;

{HSΔ22/4} $H_1\asymp H_2\asymp H_3\sim H_4$, weight $\frac{v-2}{4v}+\frac{v-2}{4fv}$;

(6) $[(H_1),(H_2),H_3],[H_4]$, probability $\frac{M-1}{M^2}\frac{v/2}{v-1}$,

{HSΔ14/2} $H_1\equiv H_3\sim H_2\sim H_4$ or $H_2\equiv H_3\sim H_1\sim H_4$, weight $\frac{1}{4v}+\frac{1}{4fv}$;

{HSΔ18/2} $H_1\asymp H_3\sim H_2\sim H_4$ or $H_2\asymp H_3\sim H_1\sim H_4$, weight $\frac{v-1}{4v}+\frac{v-1}{4fv}$;

{HSΔ20} $H_1\sim H_2\sim H_3\sim H_4$, weight $\frac{f-1}{4f}+\frac{1}{2}$;

(7) $[(H_1,H_2),H_4],[H_3]$, probability $\frac{M-1}{M^2}\frac{v/2-1}{v-1}$,

{HSΔ13/4} $H_1\equiv H_4\asymp H_2\sim H_3$ or $H_2\equiv H_4\asymp H_1\sim H_3$, weight $\frac{1}{2v}+\frac{1}{2fv}$;

{HSΔ19/2} $H_1\asymp H_2\sim H_3\sim H_4$, weight $\frac{f-1}{4f}+\frac{1}{2}$;

{HSΔ22/4} $H_1 \asymp H_2 \asymp H_4 \sim H_3$, weight $\frac{v-2}{4v} + \frac{v-2}{4fv}$;

(8) $[(H_1),(H_2),H_4],[H_3]$, probability $\frac{M-1}{M^2}\frac{v/2}{v-1}$,

{HSΔ14/2} $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{1}{2v} + \frac{1}{2fv}$;

{HSΔ18/2} $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{v-1}{2v} + \frac{v-1}{2fv}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{2f}$;

(9) $[(H_1,H_2)],[H_3],[H_4]$, probability $\frac{(M-1)(M-2)}{M^2}\frac{v/2-1}{v-1}$, identical to (HSΔ19/2);

(10) $[(H_1),(H_2)],[H_3],[H_4]$, probability $\frac{(M-1)(M-2)}{M^2}\frac{v/2}{v-1}$, identical to (HSΔ20).

Δ_5':

(1) $[H_1,H_2,H_3,H_4]$, probability $\frac{1}{M^3}$,

{HSΔ1} $H_1 \equiv H_2 \equiv H_3 \equiv H_4$, weight $\frac{1}{16v^3} + \frac{1}{16f^3v^3}$;

{HSΔ2} $H_1 \equiv H_2 \asymp H_3 \equiv H_4$, weight $\frac{v-1}{16v^3} + \frac{v-1}{16f^3v^3}$;

{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{8fv^2} + \frac{f-1}{16f^3v^2}$;

{HSΔ4} $H_1 \equiv H_2 \equiv H_3 \asymp H_4$ or $H_1 \equiv H_2 \equiv H_4 \asymp H_3$ or $H_1 \asymp H_2 \equiv H_3 \equiv H_4$ or $H_2 \asymp H_1 \equiv H_3 \equiv H_4$, weight $\frac{v-1}{4v^3} + \frac{v-1}{4f^3v^3}$;

{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $\frac{v-1}{4fv^2} + \frac{(f-1)(v-1)}{8f^3v^2}$;

{HSΔ6} $H_1 \equiv H_3 \equiv H_4 \sim H_2$ or $H_2 \equiv H_3 \equiv H_4 \sim H_1$ or $H_1 \equiv H_2 \equiv H_3 \sim H_4$ or $H_1 \equiv H_2 \equiv H_4 \sim H_3$, weight $\frac{1}{4v^2} + \frac{1}{4f^2v^2} + \frac{f-1}{4f^3v^2}$;

{HSΔ7} $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$ or $H_1 \equiv H_2 \asymp H_4 \sim H_3$, weight $\frac{v-1}{4v^2} + \frac{v-1}{4f^2v^2} + \frac{(f-1)(v-1)}{4f^3v^2}$;

{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{8fv} + \frac{f-1}{4f^2v} + \frac{(f-1)(f-2)}{8f^3v}$;

{HSΔ9} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \asymp H_4$, weight $\frac{(v-1)(v-2)}{8v^3} + \frac{(v-1)(v-2)}{8f^3v^3}$;

{HSΔ10} $H_1 \equiv H_3 \asymp H_2 \equiv H_4$ or $H_1 \equiv H_4 \asymp H_2 \equiv H_3$, weight $\frac{v-1}{8v^3} + \frac{v-1}{8f^3v^3}$;

{HSΔ11} $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $\frac{1}{4fv^2} + \frac{f-1}{8f^3v^2}$;

{HSΔ12} $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$ or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $\frac{v-1}{2fv^2} + \frac{(f-1)(v-1)}{4f^3v^2}$;

{HSΔ13} $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$ or $H_1 \equiv H_3 \asymp H_4 \sim H_2$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $\frac{v-1}{2v^2} + \frac{v-1}{2f^2v^2} + \frac{(f-1)(v-1)}{2f^3v^2}$;

{HSΔ14} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{f-1}{4fv} + \frac{f-1}{2f^2v} + \frac{(f-1)(f-2)}{4f^3v}$;

{HSΔ15} $H_1 \equiv H_3 \asymp H_2 \asymp H_4$ or $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_3 \asymp H_1 \asymp H_4$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $\frac{(v-1)(v-2)}{4v^3} + \frac{(v-1)(v-2)}{4f^3v^3}$;

{HSΔ16} $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $\frac{(v-1)^2}{4fv^2} + \frac{(f-1)(v-1)^2}{8f^3v^2}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{(v-1)^2}{8fv^2} + \frac{(f-1)(v-1)^2}{16f^3v^2}$;

{HSΔ18} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{(f-1)(v-1)}{4fv} + \frac{(f-1)(v-1)}{2f^2v} + \frac{(f-1)(f-2)(v-1)}{4f^3v}$;

{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{(f-1)(v-1)}{8fv}+\frac{(f-1)(v-1)}{4f^2v}+\frac{(f-1)(f-2)(v-1)}{8f^3v}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{(f-1)(f-2)}{4f^2}+\frac{(f-1)(f-2)(f-3)}{16f^3}$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $\frac{(v-1)(v-2)(v-3)}{16v^3}+\frac{(v-1)(v-2)(v-3)}{16f^3v^3}$;

{HSΔ22} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$, weight $\frac{(v-1)(v-2)}{4v^2}+\frac{(v-1)(v-2)}{4f^2v^2}+\frac{(f-1)(v-1)(v-2)}{4f^3v^2}$;

(2) $[H_1,H_2,H_3],[H_4]$ or $[H_1,H_2,H_4],[H_3]$ or $[H_1,H_3,H_4],[H_2]$ or $[H_2,H_3,H_4],[H_1]$, probability $\frac{4(M-1)}{M^3}$,

{HSΔ6/4} $H_1 \equiv H_3 \equiv H_4 \sim H_2$ or $H_2 \equiv H_3 \equiv H_4 \sim H_1$ or $H_1 \equiv H_2 \equiv H_3 \sim H_4$ or $H_1 \equiv H_2 \equiv H_4 \sim H_3$, weight $\frac{1}{8v^2}+\frac{1}{8f^2v^2}$;

{HSΔ7/4} $H_1 \sim H_2 \asymp H_3 \equiv H_4$ or $H_2 \sim H_1 \asymp H_3 \equiv H_4$ or $H_1 \equiv H_2 \asymp H_3 \sim H_4$ or $H_1 \equiv H_2 \asymp H_4 \sim H_3$, weight $\frac{v-1}{8v^2}+\frac{v-1}{8f^2v^2}$;

{HSΔ8/2} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{1}{8v}+\frac{1}{8fv}+\frac{f-1}{8f^2v}$;

{HSΔ13/4} $H_1 \equiv H_3 \asymp H_2 \sim H_4$ or $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_3 \asymp H_1 \sim H_4$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$ or $H_1 \equiv H_3 \asymp H_4 \sim H_2$ or $H_1 \equiv H_4 \asymp H_3 \sim H_2$ or $H_2 \equiv H_3 \asymp H_4 \sim H_1$ or $H_2 \equiv H_4 \asymp H_3 \sim H_1$, weight $\frac{v-1}{4v^2}+\frac{v-1}{4f^2v^2}$;

{HSΔ14/2} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{1}{4v}+\frac{1}{4fv}+\frac{f-1}{4f^2v}$;

{HSΔ18/2} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{v-1}{4v}+\frac{v-1}{4fv}+\frac{(f-1)(v-1)}{4f^2v}$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{v-1}{8v}+\frac{v-1}{8fv}+\frac{(f-1)(v-1)}{8f^2v}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{3(f-1)}{8f}+\frac{(f-1)(f-2)}{8f^2}$;

{HSΔ22/4} $H_1 \asymp H_2 \asymp H_3 \sim H_4$ or $H_1 \asymp H_2 \asymp H_4 \sim H_3$ or $H_1 \asymp H_3 \asymp H_4 \sim H_2$ or $H_2 \asymp H_3 \asymp H_4 \sim H_1$, weight $\frac{(v-1)(v-2)}{8v^2}+\frac{(v-1)(v-2)}{8f^2v^2}$;

(3) $[H_1,H_2],[H_3,H_4]$, probability $\frac{(M-1)}{M^3}$,

{HSΔ3} $H_1 \equiv H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{16v^2}+\frac{1}{8fv^2}+\frac{1}{16f^2v^2}$;

{HSΔ5} $H_1 \equiv H_2 \sim H_3 \asymp H_4$ or $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $\frac{v-1}{8v^2}+\frac{v-1}{4fv^2}+\frac{v-1}{8f^2v^2}$;

{HSΔ8} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{1}{4v}+\frac{1}{4fv}+\frac{f-1}{8fv}+\frac{f-1}{8f^2v}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{(v-1)^2}{16v^2}+\frac{(v-1)^2}{8fv^2}+\frac{(v-1)^2}{16f^2v^2}$;

{HSΔ19} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{v-1}{4v}+\frac{v-1}{4fv}+\frac{(f-1)(v-1)}{8fv}+\frac{(f-1)(v-1)}{8f^2v}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{1}{4}+\frac{f-1}{4f}+\frac{(f-1)^2}{16f^2}$;

(4) $[H_1,H_3],[H_2,H_4]$ or $[H_1,H_4],[H_2,H_3]$, probability $\frac{2(M-1)}{M^3}$,

{HSΔ11/2} $H_1 \equiv H_3 \sim H_2 \equiv H_4$ or $H_1 \equiv H_4 \sim H_2 \equiv H_3$, weight $\frac{1}{16v^2}+\frac{1}{8fv^2}+\frac{1}{16f^2v^2}$;

{HSΔ12/2} $H_1 \equiv H_3 \sim H_2 \asymp H_4$ or $H_1 \equiv H_4 \sim H_2 \asymp H_3$ or $H_2 \equiv H_3 \sim H_1 \asymp H_4$ or $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $\frac{v-1}{8v^2}+\frac{v-1}{4fv^2}+\frac{v-1}{8f^2v^2}$;

{HSΔ14/2} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{1}{4v}+\frac{1}{4fv}+\frac{f-1}{8fv}+\frac{f-1}{8f^2v}$;

{HSΔ16/2} $H_1 \asymp H_3 \sim H_2 \asymp H_4$ or $H_1 \asymp H_4 \sim H_2 \asymp H_3$, weight $\frac{(v-1)^2}{16v^2}+\frac{(v-1)^2}{8fv^2}+\frac{(v-1)^2}{16f^2v^2}$;

{HSΔ18/2} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$

or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv} + \frac{(f-1)(v-1)}{8fv} + \frac{(f-1)(v-1)}{8f^2 v}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{1}{4} + \frac{f-1}{4f} + \frac{(f-1)^2}{16f^2}$;

(5) $[H_1, H_2], [H_3], [H_4]$ or $[H_3, H_4], [H_1], [H_2]$, probability $\frac{2(M-1)(M-2)}{M^3}$,

{HSΔ8/2} $H_1 \sim H_2 \sim H_3 \equiv H_4$ or $H_1 \equiv H_2 \sim H_3 \sim H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$ or $H_1 \sim H_2 \sim H_3 \asymp H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{1}{2} + \frac{f-1}{4f}$;

(6) $[H_1, H_3], [H_2], [H_4]$ or $[H_1, H_4], [H_2], [H_3]$ or $[H_2, H_3], [H_1], [H_4]$ or $[H_2, H_4], [H_1], [H_3]$, probability $\frac{4(M-1)(M-2)}{M^3}$,

{HSΔ14/4} $H_1 \equiv H_3 \sim H_2 \sim H_4$ or $H_1 \equiv H_4 \sim H_2 \sim H_3$ or $H_2 \equiv H_3 \sim H_1 \sim H_4$ or $H_2 \equiv H_4 \sim H_1 \sim H_3$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$ or $H_1 \asymp H_4 \sim H_2 \sim H_3$ or $H_2 \asymp H_3 \sim H_1 \sim H_4$ or $H_2 \asymp H_4 \sim H_1 \sim H_3$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ20} $H_1 \sim H_2 \sim H_3 \sim H_4$, weight $\frac{1}{2} + \frac{f-1}{4f}$;

(7) $[H_1], [H_2], [H_3], [H_4]$, probability $\frac{(M-1)(M-2)(M-3)}{M^3}$, identical to (HSΔ20).

Δ'_6:

(1) (H_1, H_2, H_3, H_4), probability $\frac{(v/2-1)(v/2-2)(v/2-3)}{(v-1)(v-2)(v-3)}$, identical to (HSΔ21);

(2) $(H_1, H_2, H_3), (H_4)$ or $(H_1, H_2, H_4), (H_3)$ or $(H_1, H_3, H_4), (H_2)$ or $(H_2, H_3, H_4), (H_1)$, probability $\frac{4(v/2-1)(v/2-2)(v/2)}{(v-1)(v-2)(v-3)}$, identical to (HSΔ22/4);

(3) $(H_1, H_2), (H_3, H_4)$, probability $\frac{(v/2-1)^2\,(v/2)}{(v-1)(v-2)(v-3)}$, identical to (HSΔ17);

(4) $(H_1, H_3), (H_2, H_4)$ or $(H_1, H_4), (H_2, H_3)$, probability $\frac{2(v/2-1)^2\,(v/2)}{(v-1)(v-2)(v-3)}$, identical to (HSΔ16/2).

Δ'_7:

(1) $[(H_1, H_2, H_3), H_4]$, probability $\frac{1}{M}\frac{(v/2-1)(v/2-2)}{(v-1)(v-2)}$,

{HSΔ9/2} $H_1 \asymp H_2 \asymp H_3 \equiv H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ15/2} $H_1 \equiv H_4 \asymp H_2 \asymp H_3$ or $H_2 \equiv H_4 \asymp H_1 \asymp H_3$, weight $\frac{1}{2v} + \frac{1}{2fv}$;

{HSΔ21} $H_1 \asymp H_2 \asymp H_3 \asymp H_4$, weight $\frac{v-3}{4v} + \frac{v-3}{4fv}$;

{HSΔ22/4} $H_1 \asymp H_2 \asymp H_3 \sim H_4$, weight $\frac{1}{2} + \frac{f-1}{4f}$;

(2) $[(H_1, H_2, H_3)], [H_4]$, probability $\frac{M-1}{M}\frac{(v/2-1)(v/2-2)}{(v-1)(v-2)}$, identical to (HSΔ22/4);

(3) $[(H_1, H_2), (H_3), H_4]$, probability $\frac{1}{M}\frac{(v/2-1)(v/2)}{(v-1)(v-2)}$,

{HSΔ5/2} $H_1 \asymp H_2 \sim H_3 \equiv H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ13/4} $H_1 \equiv H_4 \asymp H_2 \sim H_3$ or $H_2 \equiv H_4 \asymp H_1 \sim H_3$, weight $\frac{1}{2v} + \frac{1}{2fv}$;

{HSΔ17} $H_1 \asymp H_2 \sim H_3 \asymp H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ19/2} $H_1 \asymp H_2 \sim H_3 \sim H_4$, weight $\frac{f-1}{2f}$;

{HSΔ22/4} $H_1 \asymp H_2 \asymp H_4 \sim H_3$, weight $\frac{v-2}{4v} + \frac{v-2}{4fv}$;

(4) $[(H_1, H_2), (H_3)], [H_4]$, probability $\frac{M-1}{M}\frac{(v/2-1)(v/2)}{(v-1)(v-2)}$, identical to (HSΔ19/2);

(5) $[(H_1, H_3), (H_2), H_4]$ or $[(H_2, H_3), (H_1), H_4]$, probability $\frac{2}{M}\frac{(v/2-1)(v/2)}{(v-1)(v-2)}$,

{HSΔ7/4} $H_2 \sim H_1 \asymp H_3 \equiv H_4$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ12/4} $H_2 \equiv H_4 \sim H_1 \asymp H_3$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ13/8} $H_1 \equiv H_4 \asymp H_3 \sim H_2$, weight $\frac{1}{4v} + \frac{1}{4fv}$;

{HSΔ16/2} $H_1 \asymp H_3 \sim H_2 \asymp H_4$, weight $\frac{v-1}{4v} + \frac{v-1}{4fv}$;

{HSΔ18/4} $H_1 \asymp H_3 \sim H_2 \sim H_4$, weight $\frac{f-1}{2f}$;

{HSΔ22/4} $H_1 \asymp H_3 \asymp H_4 \sim H_2$, weight $\frac{v-2}{4v} + \frac{v-2}{4fv}$;

(6) $[(H_1, H_3), (H_2)], [H_4]$ or $[(H_2, H_3), (H_1)], [H_4]$, probability $\frac{2(M-1)}{M} \frac{(v/2-1)(v/2)}{(v-1)(v-2)}$, identical to (HSΔ18/4).

The transition matrix **Ω** for the DH mating system is not shown, but the matrices **T** and **S** in the principal part of **Ω** are listed in Appendix I.

Appendix I. T and S for various mating systems

HS mating system

$$\mathbf{T}_{\mathrm{HS}}=\begin{bmatrix}
\frac{c_1^2 v_2}{2v_1} & \frac{v c_1^2}{2v_1} & -\frac{c v c_1}{v_1} & 0 & 0 & -\frac{c c_1 v_2}{v_1} & \frac{c^2 v}{2v_1} & 0 & 0 & 0 & 0 & \frac{c^2 v_2}{2v_1} & 0\\
0 & c_1^2 & -2cc_1 & 0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{c_1 v_2}{2v_1} & 0 & -\frac{v c_1}{2v_1} & 0 & \frac{c v_2}{2v_1} & 0 & \frac{cv}{2v_1} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{c_1 v_2}{2v_1} & -\frac{v c_1}{2v_1} & 0 & 0 & 0 & \frac{cv}{2v_1} & 0 & 0 & 0 & \frac{c v_2}{2v_1}\\
0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{v c_1}{4v_1} & -\frac{v c_1}{2v_1} & 0 & -\frac{c_1 v_4}{4v_1} & \frac{cv}{4v_1} & 0 & 0 & 0 & 0 & \frac{c v_4}{4v_1} & \frac{cv}{2v_1}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & 0 & \frac{v^2}{4v_1^2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & \frac{v^2}{4v_1^2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & 0 & \frac{v}{2v_1} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & \frac{v}{2v_1} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v v_2}{4v_1 v_3} & \frac{v v_2}{8v_1 v_3} & 0 & 0 & 0 & \frac{v_4 v_6}{8v_1 v_3} & \frac{v v_4}{2v_1 v_3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v}{2v_1} & \frac{v}{4v_1} & 0 & 0 & \frac{v_4}{4v_1}
\end{bmatrix},$$

$$
\mathbf{S}_{\mathrm{HS}} = \begin{bmatrix}
\frac{1}{2}+\frac{c}{2}\left(c_2+\frac{c}{v_1^2}\right) & -\frac{vc_1^2}{2v_1} & \frac{cvc_1}{v_1} & 0 & 0 & \frac{cv_2(v_1-cv_2)}{v_1^2} & -\frac{c^2v}{2v_1} & 0 & 0 & 0 & 0 & \frac{c^2v_2v_3}{2v_1^2} & 0 \\
c_1^2-\frac{1+2cc_1}{v}+\frac{c^2}{v_1} & -c_1^2 & 2cc_1 & 0 & 0 & \frac{2cv_2(v_1-cv_2)}{vv_1} & -c^2 & 0 & 0 & 0 & 0 & \frac{c^2v_2v_3}{vv_1} & 0 \\
\frac{cv_2}{2vv_1^2} & -\frac{c_1}{2v_1} & \frac{1}{2v_1} & \frac{c-c_1v_1}{v_1} & \frac{3vc_1}{2v_1} & \frac{v_2^2(v_1-cv_3)}{2vv_1^2} & \frac{c}{2v_1} & 0 & \frac{-3cv}{2v_1} & 0 & 0 & \frac{cv_2^2v_3}{2vv_1^2} & \frac{cv_2}{v_1} \\
-\frac{c_1v_2}{2vv_1} & -\frac{c_1}{2v_1} & \frac{v_1-cv_2}{2v_1} & \frac{v-cv_2}{2v_1} & \frac{3vc_1}{2v_1} & \frac{(v_2-cv_4)v_2}{2v_1v} & \frac{c}{2} & 0 & \frac{-3cv}{2v_1} & 0 & 0 & \frac{cv_2v_3}{2v_1v} & \frac{cv_2}{2v_1} \\
0 & -\frac{c_1}{v} & \frac{v_1-cv_2}{v} & \frac{2v_1-2cv_2}{v} & 3c_1 & 0 & \frac{cv_1}{v} & 0 & -3c & 0 & 0 & 0 & \frac{2cv_2}{v} \\
\frac{3c+2v_1-2cv}{4v_1^2} & 0 & \frac{vc_1}{4v_1} & \frac{vc_1}{2v_1} & 0 & \frac{3v_1v_2-c(3v_5v+14)}{4v_1^2} & -\frac{cv}{4v_1} & 0 & 0 & 0 & 0 & \frac{cv_3(3v-4)}{4v_1^2} & -\frac{cv}{2v_1} \\
\frac{v_2^2}{4vv_1^3} & 0 & \frac{v_2}{2v_1^2} & \frac{v_2}{v_1^2} & \frac{v}{v_1^2} & \frac{v_2^3}{2vv_1^3} & \frac{vv_2}{4v_1^2} & 0 & -\frac{vv_4}{2v_1^2} & \frac{v}{2v_1} & -\frac{3v^2}{2v_1^2} & \frac{v_2^3v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
\frac{v_2^2}{2vv_1^3} & 0 & 0 & \frac{2v_2}{v_1^2} & \frac{v}{v_1^2} & \frac{v_2^3}{vv_1^3} & 0 & \frac{vv_2}{4v_1^2} & \frac{v}{v_1} & \frac{(5-2v)v}{2v_1^2} & -\frac{3v^2}{2v_1^2} & \frac{v_2^3v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
0 & 0 & \frac{1}{v}-\frac{1}{2v_1} & \frac{v_2}{vv_1} & \frac{2}{v_1} & 0 & \frac{1}{2}-\frac{1}{v} & 0 & \frac{2+v}{2v_1} & 1 & -\frac{3v}{v_1} & 0 & 1-\frac{4}{v}+\frac{1}{v_1} \\
0 & 0 & 0 & \frac{2v_2}{vv_1} & \frac{2}{v_1} & 0 & 0 & \frac{1}{2}-\frac{1}{v} & 2 & -\frac{v_4}{2v_1} & -\frac{3v}{v_1} & 0 & 1-\frac{4}{v}+\frac{1}{v_1} \\
0 & 0 & 0 & 0 & \frac{4}{v} & 0 & 0 & 0 & 4-\frac{4}{v} & 2-\frac{2}{v} & -6 & 0 & 0 \\
\frac{v_2}{2v_1^2v_3} & 0 & 0 & 0 & 0 & \frac{8+v(2v-9)}{v_1^2v_3} & -\frac{vv_2}{4v_1v_3} & -\frac{vv_2}{8v_1v_3} & 0 & 0 & 0 & \frac{28+v(7v-32)}{8v_1^2} & -\frac{vv_4}{2v_1v_3} \\
0 & 0 & \frac{1}{2v_1} & \frac{2}{v_1} & 0 & \frac{v_4}{2vv_1} & \frac{1}{2} & \frac{1}{4} & -\frac{3v}{2v_1} & -\frac{3v}{4v_1} & 0 & \frac{5+v}{4v_1}-\frac{3}{v} & \frac{8-5v}{4-4v}
\end{bmatrix},
$$

MS mating system

$$\mathbf{T}_{\mathrm{MS}} = \begin{bmatrix}
\frac{c_1^2 v_2}{2v_1} & \frac{v c_1^2}{2v_1} & -\frac{c v c_1}{v_1} & 0 & 0 & -\frac{c c_1 v_2}{v_1} & \frac{c^2 v}{2v_1} & 0 & 0 & 0 & 0 & \frac{c^2 v_2}{2v_1} & 0 \\
0 & c_1^2 & -2cc_1 & 0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{c_1 v_2}{2v_1} & 0 & -\frac{v c_1}{2v_1} & 0 & \frac{c v_2}{2v_1} & 0 & \frac{cv}{2v_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{c_1 v_2}{2v_1} & -\frac{v c_1}{2v_1} & 0 & 0 & 0 & \frac{cv}{2v_1} & 0 & 0 & 0 & \frac{c v_2}{2v_1} \\
0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{v c_1}{4v_1} & -\frac{v c_1}{2v_1} & 0 & -\frac{c_1 v_4}{4v_1} & \frac{cv}{4v_1} & 0 & 0 & 0 & 0 & \frac{c v_4}{4v_1} & \frac{cv}{2v_1} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & 0 & \frac{v^2}{4v_1^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & \frac{v^2}{4v_1^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & 0 & \frac{v}{2v_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & \frac{v}{2v_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v v_2}{4v_1 v_3} & \frac{v v_2}{8v_1 v_3} & 0 & 0 & 0 & \frac{v_4 v_6}{8v_1 v_3} & \frac{v v_4}{2v_1 v_3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v}{2v_1} & \frac{v}{4v_1} & 0 & 0 & \frac{v_4}{4v_1}
\end{bmatrix},$$

$$
\mathbf{S}_{\mathrm{MS}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_1^2 - \frac{1+2cc_1}{v} + \frac{c^2}{v_1} & -c_1^2 & 2cc_1 & 0 & 0 & \frac{2cv_2(v_1 - cv_2)}{vv_1} & -c^2 & 0 & 0 & 0 & 0 & \frac{c^2 v_2 v_3}{vv_1} & 0 \\
\frac{cv_2}{2vv_1^2} & 0 & \frac{c_1 v_2}{2v_1} & \frac{v_1 + c - cv_1}{v_1} & \frac{vc_1}{v_1} & \frac{v_2^2(v_1 - cv_3)}{2vv_1^2} & -\frac{cv_2}{2v_1} & 0 & -\frac{cv}{v_1} & 0 & 0 & \frac{cv_2^2 v_3}{2vv_1^2} & \frac{cv_2}{v_1} \\
-\frac{c_1 v_2}{2vv_1} & -\frac{c_1}{2v_1} & \frac{v_1 - cv_2}{2v_1} & \frac{1}{2v_1} & \frac{vc_1}{v_1} & \frac{v_2(v_2 - cv_4)}{2v_1 v} & \frac{c}{2} & 0 & -\frac{cv}{v_1} & 0 & 0 & \frac{cv_2 v_3}{2v_1 v} & 0 \\
0 & -\frac{c_1}{v} & \frac{v_1 - cv_2}{v} & \frac{2v_1 - 2cv_2}{v} & 3c_1 & 0 & \frac{cv_1}{v} & 0 & -3c & 0 & 0 & 0 & \frac{2cv_2}{v} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{v_2^2}{4vv_1^3} & 0 & 0 & \frac{v_2}{v_1^2} & \frac{v}{2v_1^2} & \frac{v_2^3}{2vv_1^3} & -\frac{v_2^2}{4v_1^2} & 0 & -\frac{vv_3}{2v_1^2} & \frac{v}{2v_1} & -\frac{v^2}{v_1^2} & \frac{v_2^3 v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
\frac{v_2^2}{2vv_1^3} & 0 & 0 & \frac{2v_2}{v_1^2} & \frac{v}{v_1^2} & \frac{v_2^3}{vv_1^3} & 0 & -\frac{v_2^2}{4v_1^2} & \frac{v}{v_1} & \frac{1 - v_1^2}{v_1^2} & -\frac{v^2}{v_1^2} & \frac{v_2^3 v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
0 & 0 & \frac{v_2}{2vv_1} & \frac{v_2}{vv_1} & \frac{3}{2v_1} & 0 & \frac{v_2}{2v} & 0 & \frac{3}{2v_1} & 1 & -\frac{5v}{2v_1} & 0 & 1 - \frac{4}{v} + \frac{1}{v_1} \\
0 & 0 & 0 & \frac{2v_2}{vv_1} & \frac{2}{v_1} & 0 & 0 & \frac{v_2}{2v} & 2 & \frac{5 - 2v}{2v_1} & -\frac{5v}{2v_1} & 0 & 1 - \frac{4}{v} + \frac{1}{v_1} \\
0 & 0 & 0 & 0 & \frac{4}{v} & 0 & 0 & 0 & \frac{4v_1}{v} & \frac{2v_1}{v} & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2v_1} & \frac{1}{v_1} & 0 & \frac{v_4}{2vv_1} & \frac{1}{2} & \frac{1}{4} & -\frac{v}{v_1} & -\frac{v}{2v_1} & 0 & \frac{v+5}{4v_1} - \frac{3}{v} & \frac{1}{2}
\end{bmatrix},
$$

ME and DR mating systems

$$
\mathbf{T}_{\mathrm{ME/DR}} = \begin{bmatrix}
\frac{c_1^2 v_2}{2v_1} & \frac{v c_1^2}{2v_1} & -\frac{c v c_1}{v_1} & 0 & 0 & -\frac{c c_1 v_2}{v_1} & \frac{c^2 v}{2v_1} & 0 & 0 & 0 & 0 & \frac{c^2 v_2}{2v_1} & 0 \\
0 & c_1^2 & -2cc_1 & 0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{c_1 v_2}{2v_1} & 0 & -\frac{v c_1}{2v_1} & 0 & \frac{c v_2}{2v_1} & 0 & \frac{c v}{2v_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{c_1 v_2}{2v_1} & -\frac{v c_1}{2v_1} & 0 & 0 & 0 & \frac{c v}{2v_1} & 0 & 0 & 0 & \frac{c v_2}{2v_1} \\
0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{v c_1}{4v_1} & -\frac{v c_1}{2v_1} & 0 & -\frac{c_1 v_4}{4v_1} & \frac{c v}{4v_1} & 0 & 0 & 0 & 0 & \frac{c v_4}{4v_1} & \frac{c v}{2v_1} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & 0 & \frac{v^2}{4v_1^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & \frac{v^2}{4v_1^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & 0 & \frac{v}{2v_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & \frac{v}{2v_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v v_2}{4v_1 v_3} & \frac{v v_2}{8v_1 v_3} & 0 & 0 & 0 & \frac{v_4 v_6}{8v_1 v_3} & \frac{v v_4}{2v_1 v_3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v}{2v_1} & \frac{v}{4v_1} & 0 & 0 & \frac{v_4}{4v_1}
\end{bmatrix},
$$

$$
\mathbf{S}_{\mathrm{ME/DR}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_1^2 - \frac{1+2cc_1}{v} + \frac{c^2}{v_1} & -c_1^2 & 2cc_1 & 0 & 0 & \frac{2cv_2(v_1 - cv_2)}{vv_1} & -c^2 & 0 & 0 & 0 & 0 & \frac{c^2 v_2 v_3}{vv_1} & 0 \\
\frac{cv_2}{2vv_1^2} & 0 & \frac{c_1 v_2}{2v_1} & \frac{v_1 + c - cv_1}{v_1} & \frac{vc_1}{v_1} & \frac{v_2^2(v_1 - cv_3)}{2vv_1^2} & -\frac{cv_2}{2v_1} & 0 & -\frac{cv}{v_1} & 0 & 0 & \frac{cv_2^2 v_3}{2vv_1^2} & \frac{cv_2}{v_1} \\
-\frac{c_1 v_2}{2vv_1} & -\frac{c_1}{2v_1} & \frac{v_1 - cv_2}{2v_1} & \frac{1}{2v_1} & \frac{vc_1}{v_1} & \frac{v_2(v_2 - cv_4)}{2v_1 v} & \frac{c}{2} & 0 & -\frac{cv}{v_1} & 0 & 0 & \frac{cv_2 v_3}{2v_1 v} & 0 \\
0 & -\frac{c_1}{v} & \frac{v_1 - cv_2}{v} & \frac{2v_1 - 2cv_2}{v} & 3c_1 & 0 & \frac{cv_1}{v} & 0 & -3c & 0 & 0 & 0 & \frac{2cv_2}{v} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{v_2^2}{4vv_1^3} & 0 & 0 & \frac{v_2}{v_1^2} & \frac{v}{2v_1^2} & \frac{v_2^3}{2vv_1^3} & -\frac{v_2^2}{4v_1^2} & 0 & -\frac{vv_3}{2v_1^2} & \frac{v}{2v_1} & -\frac{v^2}{v_1^2} & \frac{v_2^3 v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
\frac{v_2^2}{2vv_1^3} & 0 & 0 & \frac{2v_2}{v_1^2} & \frac{v}{v_1^2} & \frac{v_2^3}{vv_1^3} & 0 & -\frac{v_2^2}{4v_1^2} & \frac{v}{v_1} & \frac{1 - v_1^2}{v_1^2} & -\frac{v^2}{v_1^2} & \frac{v_2^3 v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
0 & 0 & \frac{v_2}{2vv_1} & \frac{v_2}{vv_1} & \frac{3}{2v_1} & 0 & \frac{v_2}{2v} & 0 & \frac{3}{2v_1} & 1 & -\frac{5v}{2v_1} & 0 & 1 - \frac{4}{v} + \frac{1}{v_1} \\
0 & 0 & 0 & \frac{2v_2}{vv_1} & \frac{2}{v_1} & 0 & 0 & \frac{v_2}{2v} & 2 & \frac{5 - 2v}{2v_1} & -\frac{5v}{2v_1} & 0 & 1 - \frac{4}{v} + \frac{1}{v_1} \\
0 & 0 & 0 & 0 & \frac{4}{v} & 0 & 0 & 0 & \frac{4v_1}{v} & \frac{2v_1}{v} & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2v_1} & \frac{1}{v_1} & 0 & \frac{v_4}{2vv_1} & \frac{1}{2} & \frac{1}{4} & -\frac{v}{v_1} & -\frac{v}{2v_1} & 0 & \frac{v+5}{4v_1} - \frac{3}{v} & \frac{1}{2}
\end{bmatrix},
$$

DH mating system

$$
\mathbf{T}_{\mathrm{DH}}=\begin{bmatrix}
\frac{c_1^2 v_2}{2v_1} & \frac{v c_1^2}{2v_1} & -\frac{c v c_1}{v_1} & 0 & 0 & -\frac{c c_1 v_2}{v_1} & \frac{c^2 v}{2v_1} & 0 & 0 & 0 & 0 & \frac{c^2 v_2}{2v_1} & 0 \\
0 & c_1^2 & -2cc_1 & 0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{c_1 v_2}{2v_1} & 0 & -\frac{v c_1}{2v_1} & 0 & \frac{c v_2}{2v_1} & 0 & \frac{cv}{2v_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{c_1 v_2}{2v_1} & -\frac{v c_1}{2v_1} & 0 & 0 & 0 & \frac{cv}{2v_1} & 0 & 0 & 0 & \frac{c v_2}{2v_1} \\
0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{v c_1}{4v_1} & -\frac{v c_1}{2v_1} & 0 & -\frac{c_1 v_4}{4v_1} & \frac{cv}{4v_1} & 0 & 0 & 0 & 0 & \frac{c v_4}{4v_1} & \frac{cv}{2v_1} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & 0 & \frac{v^2}{4v_1^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2^2}{4v_1^2} & 0 & \frac{v v_2}{2v_1^2} & \frac{v^2}{4v_1^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & 0 & \frac{v}{2v_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v_2}{2v_1} & \frac{v}{2v_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{v v_2}{4v_1 v_3} & \frac{v v_2}{8v_1 v_3} & 0 & 0 & 0 & \frac{v_4 v_6}{8v_1 v_3} & \frac{v v_4}{2v_1 v_3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{v}{2v_1} & \frac{v}{4v_1} & 0 & 0 & \frac{v_4}{4v_1}
\end{bmatrix},
$$

$$
\mathbf{S}_{\mathrm{DH}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_1^2 - \frac{1+2cc_1}{v} + \frac{c^2}{v_1} & -c_1^2 & 2cc_1 & 0 & 0 & \frac{2cv_2(v_1 - cv_2)}{vv_1} & -c^2 & 0 & 0 & 0 & 0 & \frac{cv_2^2 v_3}{2vv_1^2} & \frac{cv_2}{v_1} \\
\frac{cv_2}{2vv_1^2} & 0 & \frac{c_1 v_2}{2v_1} & \frac{v_1 + c(1-v_1)}{v_1} & \frac{vc_1}{v_1} & \frac{v_2^2(v_1 - cv_3)}{2vv_1^2} & -\frac{cv_2}{2v_1} & 0 & -\frac{cv}{v_1} & 0 & 0 & \frac{cv_2^2 v_3}{2vv_1^2} & \frac{cv_2}{v_1} \\
-\frac{c_1 v_2}{2vv_1} & -\frac{c_1}{2v_1} & \frac{v_1 - cv_2}{2v_1} & \frac{1}{2v_1} & \frac{vc_1}{v_1} & \frac{v_2(v_2 - cv_4)}{2vv_1} & \frac{c}{2} & 0 & -\frac{cv}{v_1} & 0 & 0 & \frac{cv_2 v_3}{2v_1 v} & 0 \\
0 & -\frac{c_1}{v} & \frac{v_1 - cv_2}{v} & \frac{2v_1 - 2cv_2}{v} & 3c_1 & 0 & \frac{cv_1}{v} & 0 & -3c & 0 & 0 & 0 & \frac{2cv_2}{v} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{v_2^2}{4vv_1^3} & \frac{1}{2(1+f)v_1^2} & \frac{1}{(1+f)v_1} & \frac{v_2}{v_1^2} & \frac{vf_1}{2(1+f)v_1^2} & \frac{v_2^3}{2vv_1^3} & \frac{v^2 - fv_2^2 - 2}{4(1+f)v_1^2} & \frac{1}{2+2f} & \frac{v[5+3f-(3+f)v]}{2(1+f)v_1^2} & \frac{vf_1}{2(1+f)v_1} & -\frac{fv^2}{(1+f)v_1^2} & \frac{v_2^3 v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
\frac{v_2^2}{2vv_1^3} & \frac{1}{(1+f)v_1^2} & \frac{2}{(1+f)v_1} & \frac{2v_2}{v_1^2} & \frac{vf_1}{(1+f)v_1^2} & \frac{v_2^3}{vv_1^3} & \frac{1}{1+f} & -\frac{v_2^2}{4v_1^2} & \frac{vf_1}{(1+f)v_1} & \frac{1-v_1^2}{v_1^2} & -\frac{fv^2}{(1+f)v_1^2} & \frac{v_2^3 v_3}{4vv_1^3} & \frac{v_2^2}{v_1^2} \\
0 & 0 & \frac{1}{v} - \frac{1}{2v_1} & \frac{v_2}{vv_1} & \frac{3}{2v_1} & 0 & \frac{1}{2} - \frac{1}{v} & 0 & \frac{3}{2v_1} & 1 & -\frac{5v}{2v_1} & 0 & 1 - \frac{4}{v} + \frac{1}{v_1} \\
0 & 0 & 0 & \frac{2v_2}{vv_1} & \frac{3}{2v_1} & 0 & 0 & \frac{1}{2} - \frac{1}{v} & \frac{3}{2} & \frac{3-2v_1}{2v_1} & -\frac{2v}{v_1} & 0 & 1 - \frac{4}{v} + \frac{1}{v_1} \\
0 & 0 & 0 & 0 & \frac{4}{v} & 0 & 0 & 0 & 4 - \frac{4}{v} & 2 - \frac{2}{v} & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2v_1} & \frac{1}{v_1} & 0 & \frac{v_4}{2vv_1} & \frac{1}{2} & \frac{1}{4} & -\frac{v}{v_1} & -\frac{v}{2v_1} & 0 & -\frac{3}{v} + \frac{5+v}{4v_1} & \frac{1}{2}
\end{bmatrix},
$$

Appendix J. Approximations of d^2 and δ^2

The equality $\mathbf{T1} = \mathbf{1}$ still holds for all five mating systems. Therefore, the approximations of d^2 and δ^2 can be derived with the same method. The following matrix equation holds if N_e is large enough:

$$(\mathbf{S} - r\boldsymbol{I})\mathbf{1} = (\boldsymbol{I} - \mathbf{T})\mathbf{x},$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_{13}]^T$. From Appendix I, we show that the elements in the 11^{th} row of $\boldsymbol{I} - \mathbf{T}$ are all zero, and the 11^{th} element in the vector $(\mathbf{S} - r\boldsymbol{I})\mathbf{1}$ is $-\frac{v}{2} - r$ for all five mating systems. Therefore, the 11^{th} equation in this matrix equation is $-\frac{v}{2} - r = 0$, and thus $r = -2/v$. For the other 13 unknowns $x_1, x_2, \cdots, x_{13}$, the 13 differences

$$x_1 - x_{11},\ x_2 - x_{11}, \cdots,\ x_{13} - x_{11}$$

can be solved for each mating system. The appropriate expressions are listed in Table S4.

Let $a_i = x_i - x_{11}$, then $x_i = a_i + x_{11}$, $i = 1, 2, \cdots, 13$. The solutions of the matrix equation for each mating system can be expressed as

$$r = -2/v,\ \ x_1 = a_1 + \zeta,\ \ x_2 = a_2 + \zeta, \cdots,\ x_{13} = a_{13} + \zeta,$$

where ζ is any number. Now, if we let ζ be 0, we obtain a special solution as follows:

$$r = -2/v \ \text{ and } \ \mathbf{x} = [a_1, a_2, \cdots, a_{13}]^T.$$

In addition, $v = 1 + N_e^{-1}r + O(N^{-2})$ and $\boldsymbol{\omega} = \mathbf{1} + N_e^{-1}\mathbf{x} + \boldsymbol{O}(N^{-2})$, it follows

$$v \approx \frac{N_e v - 2}{N_e v} \ \text{ and } \ \boldsymbol{\omega} \approx \left[\frac{N_e + a_1}{N_e}, \frac{N_e + a_2}{N_e}, \cdots, \frac{N_e + a_{13}}{N_e}\right]^T.$$

Similarly, by substituting the approximation of $\boldsymbol{\omega}$ into Equation (4), we can calculate the approximations of various moments.

Appendix K. LD moments under pair sampling of clones

Denoting p_{ij} as the probability of there are j pairs of clones in i individuals. Such that

$$p_{ij} = \frac{2^{i-2j}\binom{n/2}{j}\binom{n/2-j}{i-2j}}{\binom{n}{i}} \quad (i \geq 2j),$$

In the following example, we will use stars to denote the symbols under non-random sampling. The observed expectation of each allele configuration under pair sampling of clones are:

Digenic:

$E_1^* = E_1$

$E_2^* = E_2$

$$E_3^* = p_{21}\frac{1}{nv^2}(C_1E_1 + C_2E_2) + p_{20}E_3$$

Trigenic:

$$E_4^* = E_4$$

$$E_5^* = p_{21}\frac{1}{nv^2(v-1)}(2C_4E_4 + C_9E_9) + p_{20}E_5$$

$$E_6^* = p_{21}\frac{1}{nv^2}(C_1E_1 + C_4E_4) + p_{20}E_6$$

$$E_7^* = p_{21}\frac{1}{nv^2(v-1)}(C_2E_2 + C_4E_4 + C_9E_9) + p_{20}E_7$$

$$E_8^* = p_{31}\frac{1}{3n(n-1)v^3}(C_3E_3 + C_5E_5 + 2C_6E_6 + 2C_7E_7) + p_{30}E_8$$

$$E_9^* = E_9$$

Quadgenic:

Dihaplotypic:

$$E_{10}^* = E_{10}$$

$$E_{11}^* = p_{21}\frac{1}{nv^2}(C_1E_1 + C_{10}E_{10}) + p_{20}E_{11}$$

Trihaplotypic:

$$E_{12}^* = p_{21}\frac{1}{nv^2(v-1)}(2C_4E_4 + C_{15}E_{15}) + p_{20}E_{12}$$

$$E_{13}^* = p_{21}\frac{1}{nv^2(v-1)}(C_4E_4 + C_{10}E_{10} + C_{15}E_{15}) + p_{20}E_{13}$$

$$E_{14}^* = p_{31}\frac{1}{3n(n-1)v^3}(2C_6E_6 + 2C_{13}E_{13} + C_{11}E_{11} + C_{12}E_{12}) + p_{30}E_{14}$$

$$E_{15}^* = E_{15}$$

Quadhaplotypic:

$$E_{16}^* = p_{21}\frac{1}{nv^2(v-1)^2}(C_2E_2 + 2C_9E_9 + C_{10}E_{10} + 2C_{15}E_{15} + C_{21}E_{21}) + p_{20}E_{16}$$

$$E_{17}^* = p_{21}\frac{1}{nv^2(v-1)^2}(2C_{10}E_{10} + 4C_{15}E_{15} + C_{21}E_{21}) + p_{20}E_{17}$$

$$E_{18}^* = p_{31}\frac{1}{3n(n-1)v^3(v-1)}(2C_7E_7 + 2C_{13}E_{13} + 2C_{22}E_{22} + C_{12}E_{12} + C_{16}E_{16}) + p_{30}E_{18}$$

$$E_{19}^* = p_{31}\frac{1}{3n(n-1)v^3(v-1)}(4C_{13}E_{13} + 2C_{22}E_{22} + C_5E_5 + C_{17}E_{17}) + p_{30}E_{19}$$

$$E_{20}^* = P_{42}\frac{1}{3n(n-1)v^4}(C_3E_3 + 2C_5E_5 + C_{17}E_{17} + 2C_{11}E_{11} + 4C_{12}E_{12} + 2C_{16}E_{16})$$

$$+ P_{41}\frac{1}{3n(n-1)(n-2)v^4}(C_8E_8 + C_{19}E_{19} + 2C_{14}E_{14} + 2C_{18}E_{18}) + P_{40}E_{20}$$

$$E_{21}^* = E_{21}$$

$$E_{22}^* = p_{21}\frac{1}{nv^2(v-1)(v-2)}(C_9E_9 + 2C_{15}E_{15} + C_{21}E_{21}) + p_{20}E_{22}$$

Using these expectations, the LD moment can be derived by the same method in Appendix C. The principal parts of $\mathbf{A}$, are shown in Tables S8 and S9. It can be found $\mathbf{A}_1^*$ is identical to $\mathbf{A}_1$, while $\mathbf{A}_2^*$ is distinct from $\mathbf{A}_2$ in two aspects: the coefficients of double non-identities are changes; extra terms of single non-identities, heterozygosities and allele probabilities are presented. Because they are not linear functions of double non-identities d^2 and δ^2 cannot be derived with our previous methods.

Supplementary Tables

Table S1. Elements in combination matrix $\mathbf{A_1}/\boldsymbol{Q}$

	$\mathrm{E}(\widehat{D}_w^2)$	$\mathrm{E}(\widehat{D}_b^2)$	$\mathrm{E}(\widehat{D}_w\widehat{D}_b)$	$\mathrm{E}(\widehat{D}^2)$	$\mathrm{E}(\widehat{\Delta}^2)$	$\mathrm{E}(\hat{Q})$	$\mathrm{E}(\hat{R})$
Θ_1	0	0	0	0	0	0	0
Θ_2	1	0	0	1	1	0	0
Γ_1	-2	0	1	0	$2v_1$	0	0
Γ_2	0	0	0	0	0	0	0
Γ_3	0	0	-1	-2	$-2v$	0	0
Γ_4	0	0	0	0	0	0	0
Δ_1	1	1	-1	0	v_1^2	0	0
Δ_2	0	0	0	0	0	0	v_1^2
Δ_3	0	-2	1	0	$-2v_1v$	0	0
Δ_4	0	0	0	0	0	0	$-2v_1v$
Δ_5	0	1	0	1	v^2	1	v^2
Δ_6	0	0	0	0	0	0	0
Δ_7	0	0	0	0	0	0	0

Table S2. Elements in combination matrix $\mathbf{A_2}/\boldsymbol{Q}$

	$\mathrm{E}(\widehat{D}_w^2)$	$\mathrm{E}(\widehat{D}_b^2)$	$\mathrm{E}(\widehat{D}_w\widehat{D}_b)$	$\mathrm{E}(\widehat{D}^2)$	$\mathrm{E}(\widehat{\Delta}^2)$	$\mathrm{E}(\hat{Q})$	$\mathrm{E}(\hat{R})$
Θ_1	$\frac{v_1^2+1}{vv_1}$	$\frac{1}{vv_1}$	$-\frac{1}{vv_1}$	$\frac{v_1}{v}$	$\frac{2v_1}{v}$	0	$\frac{2v_1}{v}$
Θ_2	-1	0	$-\frac{1}{v}$	$-\frac{2+v}{v}$	-3	0	0
Γ_1	2	$-\frac{2}{v}$	$-\frac{2v_1}{v}$	$-\frac{2v_1}{v}$	$-6v_1$	0	0
Γ_2	0	$-\frac{4}{v}$	$-\frac{2v_2}{v}$	$-\frac{4v_1}{v}$	$-8v_1$	0	$-8v_1$
Γ_3	0	$\frac{4}{v}$	3	$\frac{4+6v}{v}$	$10v$	$\frac{4}{v}$	$4v$
Γ_4	$-\frac{2v_2^2}{vv_1}$	$\frac{2v_2}{vv_1}$	$\frac{v_2v_3}{vv_1}$	0	$\frac{4v_1v_2}{v}$	0	$\frac{4v_1v_2}{v}$
Δ_1	-1	$\frac{2-3v}{v}$	$\frac{2v-1}{v}$	0	$-3v_1^2$	0	0
Δ_2	0	0	0	0	0	0	$-3v_1^2$
Δ_3	0	$\frac{10v-4}{v}$	-3	$\frac{4v_1}{v}$	$10vv_1$	$\frac{4v_1}{v}$	$4vv_1$
Δ_4	0	$\frac{2v_1}{v}$	0	$\frac{2v_1}{v}$	$2vv_1$	$\frac{2v_1}{v}$	$8vv_1$
Δ_5	0	-6	0	-6	$-6v^2$	-6	$-6v^2$
Δ_6	$\frac{v_2v_3}{vv_1}$	$\frac{v_2v_3}{vv_1}$	$-\frac{v_2v_3}{vv_1}$	0	$\frac{v_1v_2v_3}{v}$	0	$\frac{v_1v_2v_3}{v}$
Δ_7	0	$-\frac{4v_2}{v}$	$\frac{2v_2}{v}$	0	$-4v_1v_2$	0	$-4v_1v_2$

Table S3. Essential factors to form $\mathbf{\Omega}^{T}$ for HS mating system

	Θ_1' to Θ_2'	Γ_1' to Γ_4'	Δ_1' to Δ_7'
Θ_1	$c^2+c_1^2v_1^2$	v_1-cv_2	$2v_1$
Θ_2	$vc_1^2N_1v_1$	$-c_1N_1v$	$2vN_1$
Γ_1	$-2cvc_1N_1v_1$	$vN_1(v_1-cv_2)$	$4vN_1v_1$
Γ_2	0	$2vN_1(v_1-cv_2)$	$8vN_1v_1$
Γ_3	0	$-v^2c_1N_1N_2$	$4v^2N_1N_2$
Γ_4	$-2cv_2(cv_2-v_1)$	$(v_1-cv_4)v_2$	$4v_1v_2$
Δ_1	$c^2vN_1v_1$	cvN_1v_1	$2vN_1v_1^2$
Δ_2	0	0	$vN_1v_1^2$
Δ_3	0	$cv^2N_1N_2$	$4v^2N_1N_2v_1$
Δ_4	0	0	$2v^2N_1N_2v_1$
Δ_5	0	0	$v^3N_1N_2N_3$
Δ_6	$c^2v_2v_3$	cv_2v_3	$v_1v_2v_3$
Δ_7	0	$2cvN_1v_2$	$4vN_1v_1v_2$
Divisor	Nvv_1	N^2v^2	N^3v^3

There are 13 columns for $\mathbf{\Omega}^{T}$, the first two columns are the same, each of which is Nvv_1 times of the combination coefficients of Θ_1' or Θ_2'. Similarly, the next four columns are the same, so are the last seven columns. Moreover, $c-1$ is denoted by c_1, $N-i$ by N_i and $v-i$ by v_i, $i=1,2,3$.

Table S4. Expressions of $x_1-x_{11}, x_2-x_{11}, \cdots, x_{13}-x_{11}$

	HS	MS	ME/ DR	DH
x_1-x_{11}	$\frac{2c+c_1^2v-1}{c_2cv_1v}$	*	*	*
x_2-x_{11}	$\frac{2c+c_1^2v-1}{c_2cv_1v}$	*	*	*
x_3-x_{11}	0	$\frac{4cv_1v_2}{v^2(cv_2+v)(3v-4)}$	$\frac{4cv_1v_2}{v^2(cv_2+v)(3v-4)}$	$\frac{2cv_2(2fv_1+3v-2)}{(1+f)v^2(cv_2+v)(3v-4)}$
x_4-x_{11}	0	$\frac{v_2}{v^2}$	$\frac{2v_1}{v^2}$	$\frac{2v_1}{v^2}$
x_5-x_{11}	0	0	0	0
x_6-x_{11}	0	*	*	*
x_7-x_{11}	0	$\frac{v_2^2}{v_1v^2(3v-4)}$	$\frac{v_2^2}{v_1v^2(3v-4)}$	$\frac{fv_2^2+3v^2-6v+4}{(1+f)v_1v^2(3v-4)}$
x_8-x_{11}	0	$\frac{2v_2}{v^2}$	$\frac{4v_1}{v^2}$	$\frac{4v_1}{v^2}$
x_9-x_{11}	0	0	0	0
$x_{10}-x_{11}$	0	$\frac{v_2}{v^2}$	$\frac{2v_1}{v^2}$	$\frac{2v_1}{v^2}$
$x_{11}-x_{11}$	0	0	0	0
$x_{12}-x_{11}$	0	*	*	*
$x_{13}-x_{11}$	0	$\frac{v_2}{v^2}$	$\frac{2v_1}{v^2}$	$\frac{2v_1}{v^2}$

The expressions represented by $*$ are too long and are placed in the supplementary files.

Table S5. Approximations of $\boldsymbol{d^2}$

v	HS	MS/ME/DR	DH
2	$\frac{1-2c+2c^2}{4cN_e-2c^2N_e}$	$\frac{1-2c+2c^2}{4cN_e-2c^2N_e}$	$-\frac{1+f-2c(1+f)+c^2(3+2f)}{2(-2+c)c(1+f)N_e}$
4	$\frac{3-6c+4c^2}{24cN_e-12c^2N_e}$	$\frac{16-24c+12c^2+5c^3}{128cN_e-32c^3N_e}$	$-\frac{16(1+f)-24c(1+f)+4c^2(5+3f)+c^3(3+5f)}{32c(-4+c^2)(1+f)N_e}$
6	$\frac{5-10c+6c^2}{60cN_e-30c^2N_e}$	$\frac{63-84c+14c^2+32c^3}{126c(6+c-2c^2)N_e}$	$-\frac{63(1+f)-84c(1+f)+7c^2(5+2f)+c^3(26+32f)}{126c(-6-c+2c^2)(1+f)N_e}$
8	$\frac{7-14c+8c^2}{112cN_e-56c^2N_e}$	$\frac{160-200c-10c^2+99c^3}{2560cN_e+640c^2N_e-960c^3N_e}$	$-\frac{-10c^2(-3+f)+160(1+f)-200c(1+f)+c^3(87+99f)}{320c(-8-2c+3c^2)(1+f)N_e}$
10	$\frac{9-18c+10c^2}{180cN_e-90c^2N_e}$	$\frac{-325+390c+78c^2-224c^3}{650c(-10-3c+4c^2)N_e}$	$\frac{-325(1+f)+390c(1+f)+13c^2(1+6f)-4c^3(51+56f)}{650c(-10-3c+4c^2)(1+f)N_e}$

$1/(vn-1)$ should be added to each expression in order to correct for finite sample size.

Table S6. Approximations of $\boldsymbol{\delta^2}$

v	HS	MS/ME/DR	DH
2	$\frac{1-2c+2c^2}{4cN_e-2c^2N_e}$	$\frac{-1+2c-2c^2}{2(-2+c)c(N_e-v^2\eta)}$	$-\frac{1+f-2cf+2c^2(1+f)}{2(-2+c)c(1+f)(N_e-v^2\eta)}$
4	$\frac{3-6c+4c^2}{24cN_e-12c^2N_e}$	$\frac{-4+3c-12c^2+4c^3}{2c(-4+c^2)(4N_e-v^2\eta)}$	$\frac{3c(-5+f)-4(1+f)+4c^3(3+f)-4c^2(5+3f)}{2c(-4+c^2)(1+f)(4N_e-v^2\eta)}$
6	$\frac{5-10c+6c^2}{60cN_e-30c^2N_e}$	$\frac{9(-7-4c-26c^2+12c^3)}{14c(-6-c+2c^2)(9N_e-v^2\eta)}$	$\frac{9[-7(1+f)+12c^3(3+f)-2c(27+2f)-2c^2(25+13f)]}{14c(-6-c+2c^2)(1+f)(9N_e-v^2\eta)}$
8	$\frac{7-14c+8c^2}{112cN_e-56c^2N_e}$	$\frac{4(-10-19c-44c^2+24c^3)}{5c(-8-2c+3c^2)(16N_e-v^2\eta)}$	$\frac{4[-10(1+f)+24c^3(3+f)-4c^2(23+11f)-c(117+19f)]}{5c(-8-2c+3c^2)(1+f)(16N_e-v^2\eta)}$
10	$\frac{9-18c+10c^2}{180cN_e-90c^2N_e}$	$\frac{25(-13-42c-66c^2+40c^3)}{26c(-10-3c+4c^2)(25N_e-v^2\eta)}$	$\frac{25[-13(1+f)+40c^3(3+f)-6c(34+7f)-2c^2(73+33f)]}{26c(-10-3c+4c^2)(1+f)(25N_e-v^2\eta)}$

Where $\eta=\frac{2(v-2)(v-1)}{v^2}$ for the MS mating system, or $\eta=\frac{4(v-1)^2}{v^2}$ for the ME/DR/DH mating systems. $1/(n-1)$ should be added to each expression to correct for finite sample size.

Table S7. Exact d^2 and δ^2

Mating System	v	$d^2_{c=0.5}$	$d^2_{c=1-1/v}$	Error rate	$\delta^2_{c=0.5}$	$\delta^2_{c=1-1/v}$	Error rate
	2	0.0083	0.0083	0.00%	0.0134	0.0134	0.00%
	4	0.0036	0.0032	13.95%	0.0112	0.0108	4.13%
HS	6	0.0023	0.0020	19.45%	0.0108	0.0104	3.69%
	8	0.0017	0.0014	22.49%	0.0106	0.0103	3.11%
	10	0.0014	0.0011	24.43%	0.0105	0.0102	2.66%
	2	0.0083	0.0083	0.00%	0.0134	0.0134	0.00%
	4	0.0038	0.0033	13.06%	0.0134	0.0134	0.13%
MS	6	0.0024	0.0020	18.63%	0.0134	0.0135	-0.41%
	8	0.0018	0.0015	21.81%	0.0134	0.0135	-0.51%
	10	0.0014	0.0011	23.85%	0.0134	0.0135	-0.51%
	2	0.0083	0.0083	0.00%	0.0133	0.0133	0.00%
	4	0.0037	0.0033	12.99%	0.0133	0.0133	0.06%
ME	6	0.0024	0.0020	18.55%	0.0133	0.0133	-0.47%
	8	0.0018	0.0014	21.71%	0.0133	0.0133	-0.57%
	10	0.0014	0.0011	23.74%	0.0133	0.0133	-0.56%
	2	0.0083	0.0083	0.00%	0.0133	0.0133	0.00%
	4	0.0037	0.0033	12.96%	0.0133	0.0133	0.04%
DR ($f = 1$)	6	0.0024	0.0020	18.52%	0.0133	0.0134	-0.48%
	8	0.0018	0.0014	21.69%	0.0133	0.0134	-0.58%
	10	0.0014	0.0011	23.72%	0.0133	0.0134	-0.56%
	2	0.0083	0.0083	0.00%	0.0133	0.0133	0.00%
	4	0.0037	0.0033	12.95%	0.0133	0.0133	0.04%
DR ($f = 2$)	6	0.0024	0.0020	18.51%	0.0133	0.0134	-0.49%
	8	0.0018	0.0014	21.68%	0.0133	0.0134	-0.58%
	10	0.0014	0.0011	23.72%	0.0133	0.0134	-0.57%
	2	0.0082	0.0082	0.00%	0.0133	0.0133	0.00%
	4	0.0037	0.0033	12.93%	0.0133	0.0133	0.02%
DR ($f = 5$)	6	0.0024	0.0020	18.50%	0.0133	0.0133	-0.50%
	8	0.0018	0.0014	21.67%	0.0133	0.0134	-0.59%
	10	0.0014	0.0011	23.71%	0.0133	0.0134	-0.58%
	2	0.0091	0.0091	0.00%	0.0166	0.0166	0.00%
	4	0.0039	0.0035	10.13%	0.0166	0.0168	-1.20%
DH ($f = 1$)	6	0.0024	0.0021	16.04%	0.0165	0.0168	-1.35%
	8	0.0018	0.0015	19.59%	0.0165	0.0167	-1.22%
	10	0.0014	0.0012	21.92%	0.0165	0.0167	-1.07%
	2	0.0088	0.0088	0.00%	0.0155	0.0155	0.00%
	4	0.0038	0.0034	11.04%	0.0154	0.0156	-0.85%
DH ($f = 2$)	6	0.0024	0.0021	16.85%	0.0154	0.0156	-1.10%
	8	0.0018	0.0015	20.28%	0.0154	0.0156	-1.03%
	10	0.0014	0.0011	22.51%	0.0154	0.0156	-0.93%
	2	0.0085	0.0085	0.00%	0.0144	0.0144	0.00%
	4	0.0038	0.0034	11.96%	0.0143	0.0144	-0.45%
DH ($f = 5$)	6	0.0024	0.0020	17.66%	0.0143	0.0145	-0.83%
	8	0.0018	0.0015	20.96%	0.0143	0.0145	-0.83%
	10	0.0014	0.0011	23.10%	0.0143	0.0144	-0.77%

Where the effective population size N_e and the sample size n are 100, and the error rate is calculated by $(d^2_{c=0.5} - d^2_{c=1-1/v})/d^2_{c=1-1/v}$ or $(\delta^2_{c=0.5} - \delta^2_{c=1-1/v})/\delta^2_{c=1-1/v}$.

Table S8. Elements in combination matrix $\mathbf{A}_1^*/Q$

	$E(\hat{D}_w^{*2})$	$E(\hat{D}_b^{*2})$	$E(\hat{D}_w^*\hat{D}_b^*)$	$E(\hat{D}^{*2})$	$E(\hat{\Delta}^{*2})$	$E(\hat{Q}^*)$	$E(\hat{R}^*)$
Θ_1	0	0	0	0	0	0	0
Θ_2	1	0	0	1	1	0	0
Γ_1	-2	0	1	0	$2v_1$	0	0
Γ_2	0	0	0	0	0	0	0
Γ_3	0	0	-1	-2	$-2v$	0	0
Γ_4	0	0	0	0	0	0	0
Δ_1	1	1	-1	0	v_1^2	0	0
Δ_2	0	0	0	0	0	0	v_1^2
Δ_3	0	-2	1	0	$-2v_1v$	0	0
Δ_4	0	0	0	0	0	0	$-2v_1v$
Δ_5	0	1	0	1	v^2	1	v^2
Δ_6	0	0	0	0	0	0	0
Δ_7	0	0	0	0	0	0	0

Table S9. Elements in combination matrix $\mathbf{A}_2^*$

	$E(\hat{D}_w^{*2})$	$E(\hat{D}_b^{*2})$	$E(\hat{D}_w^*\hat{D}_b^*)$	$E(\hat{D}^{*2})$	$E(\hat{\Delta}^{*2})$	$E(\hat{Q}^*)$	$E(\hat{R}^*)$
λP	$\frac{1}{v_1}$	$\frac{v_1v_2+1}{vv_1}$	$-\frac{1}{vv_1}$	$\frac{v_1}{v}$	v_1^2	$-\frac{v_1}{v}$	$-v_1(2v-1)$
$\lambda\Pi$	0	1	0	1	v^2	-1	$-v^2$
pq	$-\frac{2}{v_1}$	$-\frac{2v_1^2+2}{vv_1}$	$\frac{2}{vv_1}$	-2	$-2v^2+2v-2$	$\frac{2v+1}{v}$	$2-2v+3v^2$
Θ_1Q	$\frac{2v_1^2+2}{vv_1}$	$\frac{2}{vv_1}$	$-\frac{2}{vv_1}$	$\frac{2v_1}{v}$	$\frac{4v_1}{v}$	0	$\frac{4v_1}{v}$
Θ_2Q	-2	0	$-\frac{2}{v}$	$-\frac{2(v+2)}{v}$	-6	0	0
Γ_1Q	4	$-\frac{4}{v}$	$-\frac{4v_1}{v}$	$-\frac{4v_1}{v}$	$-12v_1$	0	0
Γ_2Q	0	$-\frac{12}{v}$	$-\frac{6v_2}{v}$	$-\frac{12v_1}{v}$	$-24v_1$	0	$-16v_1$
Γ_3Q	0	$\frac{8}{v}$	6	$\frac{12v+8}{v}$	$20v$	$\frac{8}{v}$	$8v$
Γ_4Q	$-\frac{4v_2^2}{vv_1}$	$\frac{4v_2}{vv_1}$	$\frac{2v_2v_3}{vv_1}$	0	$\frac{8v_1v_2}{v}$	0	$\frac{8v_1v_2}{v}$
Δ_1Q	-2	$\frac{4-6v}{v}$	$\frac{4v-2}{v}$	0	$-6v_1^2$	0	0
Δ_2Q	0	0	0	0	0	0	$-5v_1^2$
Δ_3Q	0	$\frac{20v-8}{v}$	-6	$\frac{8v_1}{v}$	$20vv_1$	$\frac{8v_1}{v}$	$8vv_1$
Δ_4Q	0	$\frac{6v_1}{v}$	0	$\frac{6v_1}{v}$	$6vv_1$	$\frac{6v_1}{v}$	$18vv_1$
Δ_5Q	0	-12	0	-12	$-12v^2$	-12	$-12v^2$
Δ_6Q	$\frac{2v_2v_3}{vv_1}$	$\frac{2v_2v_3}{vv_1}$	$-\frac{2v_2v_3}{vv_1}$	0	$\frac{2v_1v_2v_3}{v}$	0	$\frac{2v_1v_2v_3}{v}$
Δ_7Q	0	$-\frac{12v_2}{v}$	$\frac{6v_2}{v}$	0	$12v_1v_2$	0	$-8v_1v_2$

Where $\lambda = 2pq - p^2q - pq^2$.

Supplementary Figures

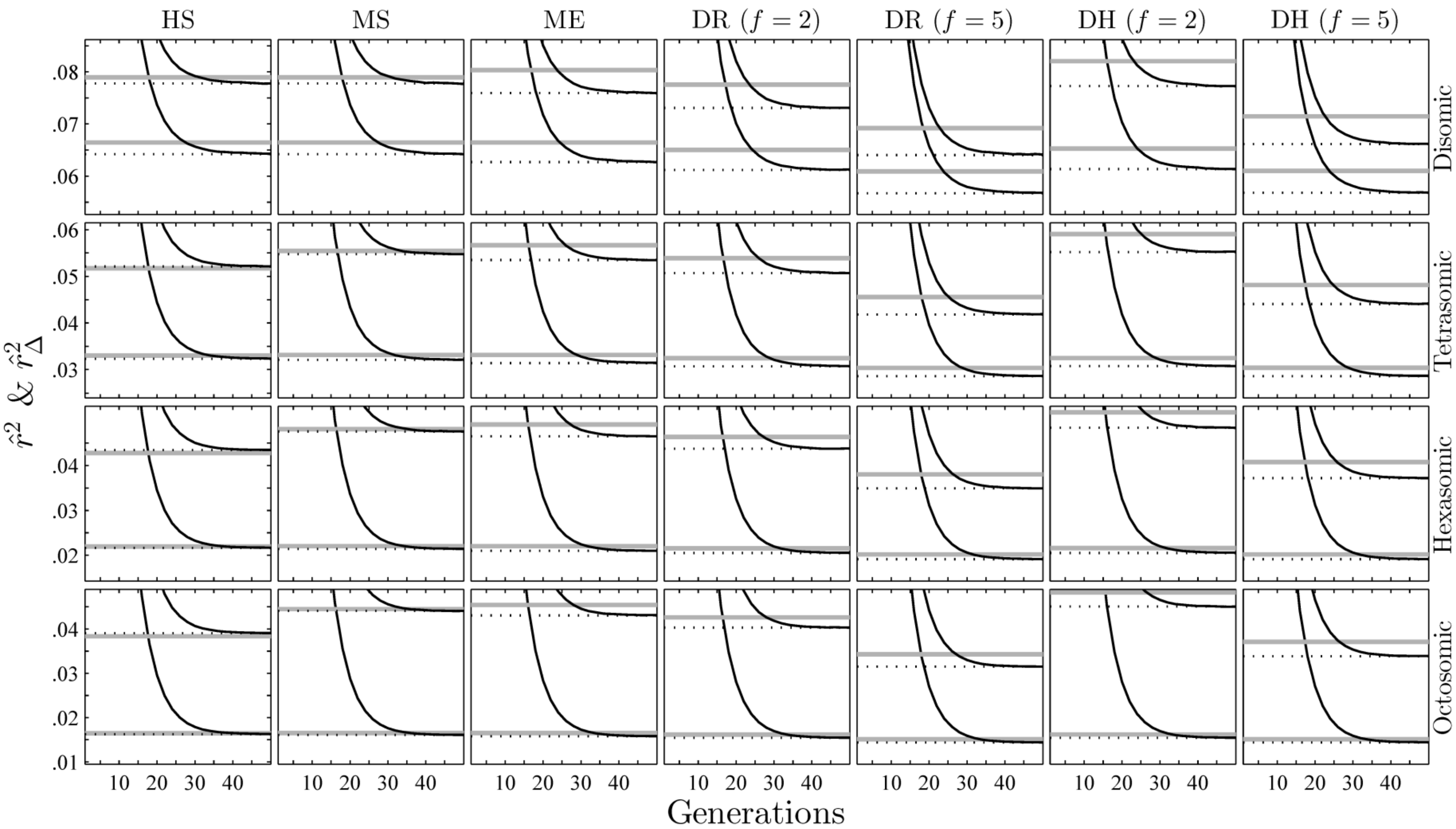

Figure S1. The behaviors of $\hat{r}^2$ and $\hat{r}_\Delta^2$ for various mating systems (set $N_e = 40$, $v = 2, 4, 6$ or 8, $L = 200$ and $c = 0.1$; for DR and DH, also set $f = 2$ or 5). Each column shows the results under a different mating system. Each row shows the results under a different ploidy level. Solid gray lines denote the approximate d^2 or δ^2, dotted gray lines denote of the exact d^2 or δ^2, and solid lines denote $\hat{r}^2$ and $\hat{r}_\Delta^2$, where the lines denoting δ^2 (or $\hat{r}_\Delta^2$) are above those denoting d^2 (or $\hat{r}^2$) for each situation.

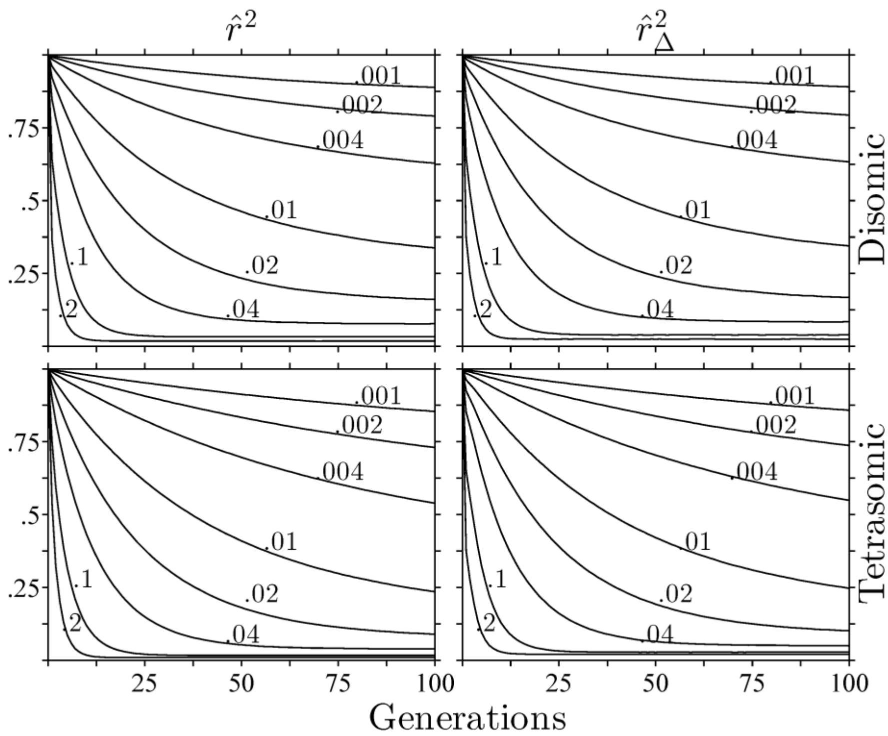

Figure S2. The behaviors of $\hat{r}^2$ and $\hat{r}_\Delta^2$ for the MS mating system under different recombination frequencies (set $N_e = 80$ and $L = 200$). The number above a line represents a recombination frequency.

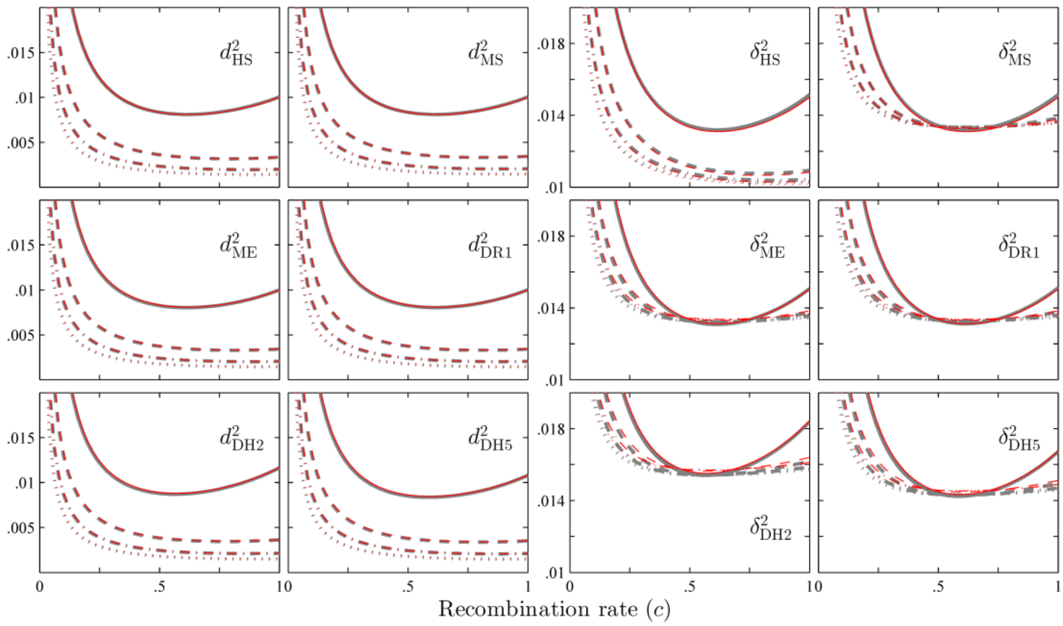

Figure S3. The relationship between d^2 (or δ^2) and the recombination frequency c for various mating systems (set $N_e = 100$ and $n = 100$). The line styles are the same as those in Figure 2. Each subscript of d^2 or δ^2 denotes a mating system, e.g., the subscript DR1 is the DR mating system with $f = 1$.

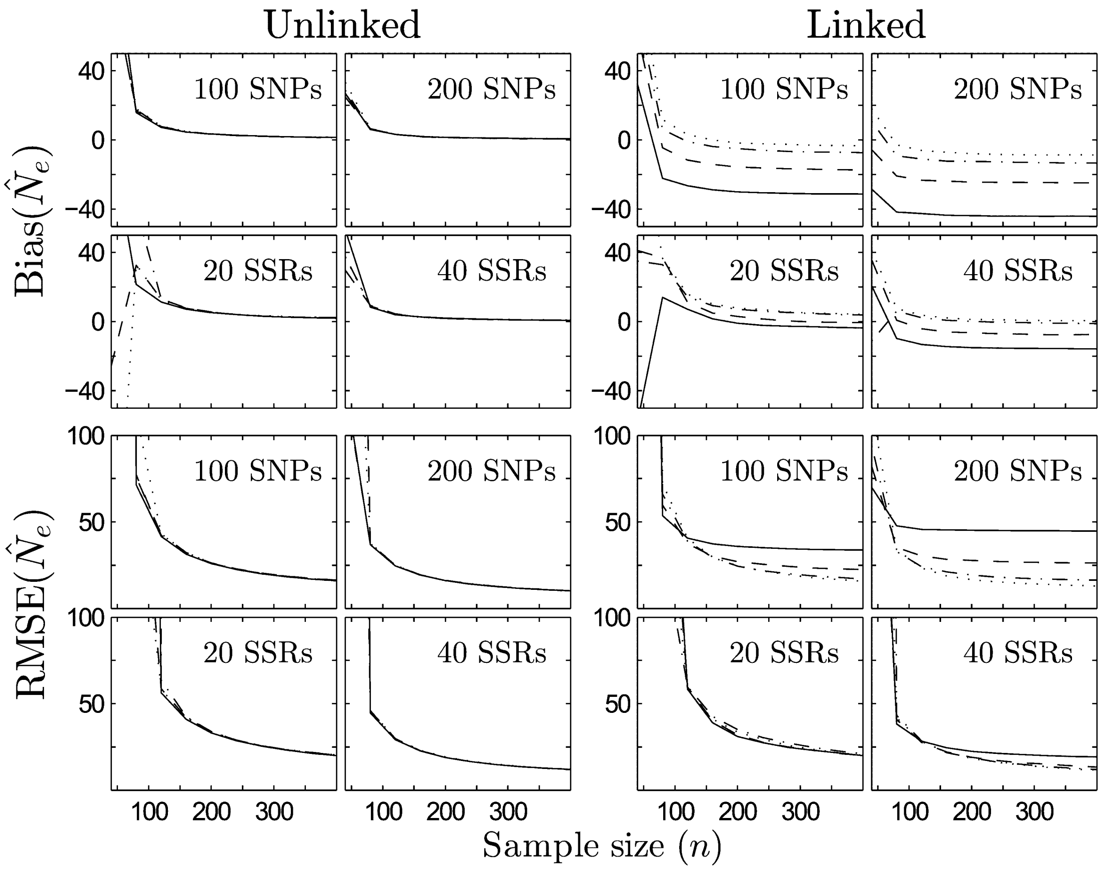

Figure S4. The bias and RMSE of $\widehat{N}_e$. The first and second columns show the results for unlinked and linked loci, respectively. The line styles and the remaining simulation configurations are as for Figure 3.

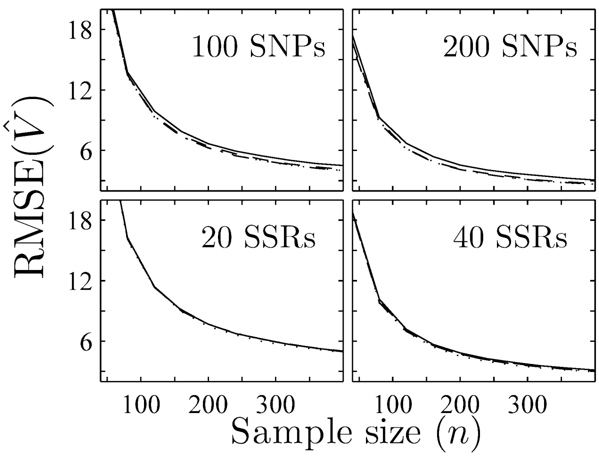

Figure S5. The relationship between the RMSE of $\hat{V}$ and the sample size n. The figure layout and the line style are the same as those in Figure 3.

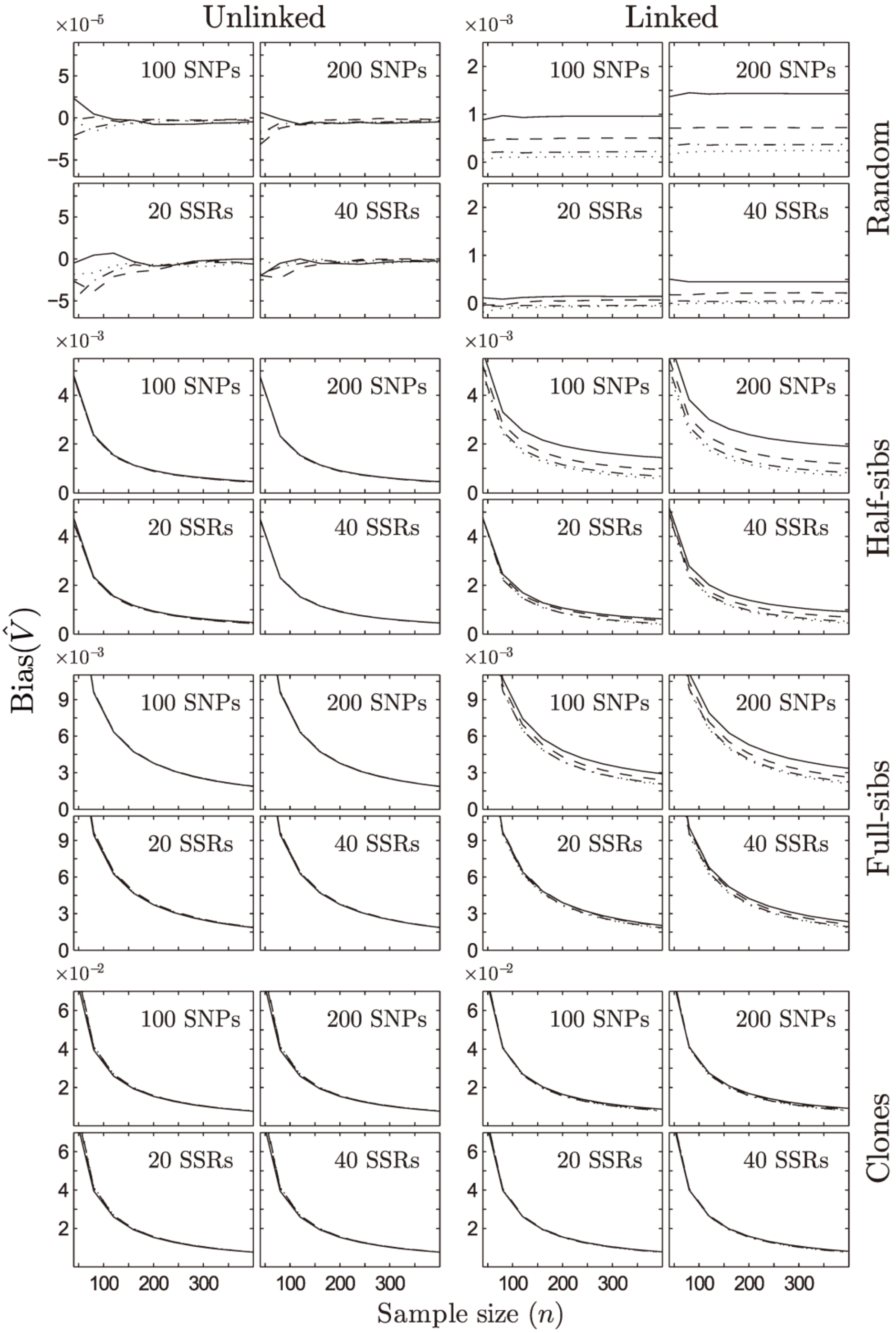

Figure S6. The bias of $\hat{V}$ for different types of loci under different sampling strategies. The first and second columns show the results for unlinked and linked loci, respectively. Four sampling strategies are compared: random sampling, pair sampling of half-sibs, full-sibs and clones, with the results for each shown on different rows. The line styles and the remaining simulation configurations are as for Figure 3.