Supplementary information for

Nonhomogeneous volume conduction effects affecting needle electromyography: an analytical and simulation study

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This PDF file includes Supplementary Parts A, B and C.

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Part A. Lemma 1

Surface Ψ is an infinity plane in $\mathbb{R}^3$, which divide full space into two domains Ω_1 and Ω_2. Point Q at position $\mathbf{r}_\mathrm{Q}$ is on Ψ (i.e., $\mathbf{r}_\mathrm{Q} \in \Psi$), while two different points S at $\mathbf{r}_\mathrm{S}$ and E at $\mathbf{r}_\mathrm{E}$ are arbitrarily located in $\mathbb{R}^3$ (see Figure A1 A and B). When S and E are inside the same domain, point S' at $\mathbf{r}_{\mathrm{S}'}$ is the mirrored point of S with respect to plane Ψ. Vector $\mathbf{n}_{21}(\mathbf{r}_\mathrm{Q})$ is a normal vector of surface Ψ at $\mathbf{r}_\mathrm{Q}$ pointing from Ω_2 to Ω_1. Then the 1st-order geometrical parameter $K_\Psi^{(1)}(\mathbf{r}_\mathrm{E}, \mathbf{r}_\mathrm{S})$ defined in (9) can be calculated for planar boundary as

$$
\begin{aligned}
&-\frac{1}{2\pi}\iint_\Psi \frac{\partial\left(|\mathbf{r}_\mathrm{Q}-\mathbf{r}_\mathrm{S}|\right)}{\partial \mathbf{n}_{21}(\mathbf{r}_\mathrm{Q})}\cdot\frac{\mathbf{n}_{21}(\mathbf{r}_\mathrm{Q})\,\mathrm{d}\Psi(\mathbf{r}_\mathrm{Q})}{|\mathbf{r}_\mathrm{Q}-\mathbf{r}_\mathrm{S}|^2\,|\mathbf{r}_\mathrm{E}-\mathbf{r}_\mathrm{Q}|}\\
&=\begin{cases} 0 & \text{if} \quad \mathbf{r}_\mathrm{S} \text{ on } \Psi \\ \frac{(-1)^{i-1}}{|\mathbf{r}_\mathrm{E}-\mathbf{r}_{\mathrm{S}'}|} & \text{if} \quad \mathbf{r}_\mathrm{S} \text{ and } \mathbf{r}_\mathrm{E} \text{ in } \Omega_i \\ \frac{(-1)^{i-1}}{|\mathbf{r}_\mathrm{E}-\mathbf{r}_\mathrm{S}|} & \text{else } \mathbf{r}_\mathrm{S} \text{ in } \Omega_i \text{ and } \mathbf{r}_\mathrm{E} \text{ in } \Omega_{3-i} \text{ or on } \Psi, \end{cases}
\end{aligned} \tag{A1}
$$

where $i \in \{1, 2\}$.

Proof. Here we introduce an auxiliary model of a charged plane Ψ in vacuum to prove (A1) indirectly. Considering plane Ψ as $z = 0$ in Cartesian coordinates (x, y, z) originated at point C with coordinates $\mathbf{r}_\mathrm{C} = (0, 0, 0)$. First, we place point S on the z-axis at $\mathbf{r}_\mathrm{S} = (0, 0, z_\mathrm{S})$. Point E at $\mathbf{r}_\mathrm{E} := (0, y_{,\mathrm{E}}, z_\mathrm{E})$ is placed on the plane $x = 0$. An arbitrary point Q is placed on plane Ψ at $\mathbf{r}_\mathrm{Q} := (x, y, 0)$. Next, we assume that a charge density function $q(\mathbf{r}_\mathrm{Q}) \in \mathbb{R}$ (C m^{-2}) is distributed across the surface Ψ, defined as

$$
q(\mathbf{r}_\mathrm{Q}) := \frac{\partial\left(|\mathbf{r}_\mathrm{Q}-\mathbf{r}_\mathrm{S}|\right)}{\partial \mathbf{n}_{21}(\mathbf{r}_\mathrm{Q})}\cdot\frac{\mathbf{n}_{21}(\mathbf{r}_\mathrm{Q})}{|\mathbf{r}_\mathrm{Q}-\mathbf{r}_\mathrm{S}|^2}.
$$

The existence of $q(\mathbf{r}_\mathrm{Q})$ on the entire charged plane Ψ leads to a static electrical potential distribution $U(\mathbf{r}) \in \mathbb{R}$ (V) in $\mathbb{R}^3$. Then the static electrical potential evaluated at $\mathbf{r}_\mathrm{E}$ is deduced as

$$
U(\mathbf{r}_\mathrm{E}) = \frac{1}{4\pi\varepsilon_0}\iint_\Psi q(\mathbf{r}_\mathrm{Q})\frac{\mathrm{d}\Psi(\mathbf{r}_\mathrm{Q})}{|\mathbf{r}_\mathrm{E}-\mathbf{r}_\mathrm{Q}|}, \tag{A2}
$$

which is similar to the left hand side of (A1). Thereby, one can prove (A1) with the solution to static electrical potential $U(\mathbf{r}_\mathrm{E})$. Next, we derive the expression of $U(\mathbf{r}_\mathrm{E})$ based on (1) $\mathbf{r}_\mathrm{S}$ on Ψ (2) $\mathbf{r}_\mathrm{S}$ and $\mathbf{r}_\mathrm{E}$ in the same domain Ω_i (3) $\mathbf{r}_\mathrm{S}$ in Ω_i and $\mathbf{r}_\mathrm{E}$ in Ω_{3-i} or on Ψ .

1.1 Point S and E in the same domain

To obtain the expression of electrical potential at arbitrary position $\mathbf{r}_\mathrm{E}$, an auxiliary point E' at $\mathbf{r}_{\mathrm{E}'} := (0, 0, \mathrm{sgn}(z_\mathrm{E})R_\mathrm{E})$ on z-axis is introduced (see Figure A1 A), where $R_\mathrm{E} := |\mathbf{r}_\mathrm{E} - \mathbf{r}_\mathrm{C}|$ is the distance of line segment $|EC|$ and $\mathrm{sgn}(x)$ is the sign function. Since this model has axial symmetry with respect to z-axis, the general solution to $U(\mathbf{r}_\mathrm{E})$ then satisfies

$$
U(\mathbf{r}_\mathrm{E}) = \sum_{n=0}^{\infty}\left(A_n R_\mathrm{E}^n + \frac{B_n}{R_\mathrm{E}^{n+1}}\right)P_n(\cos\theta_\mathrm{C}), \tag{A3}
$$

where A_n, B_n are a series of constants; $\theta_\mathrm{C} \in [0, \pi]$ (rad s^{-1}) is the angle between line segments $|SC|$ and $|EC|$; $P_n(x)$ are Legendre polynomials. When E and E' are overlap (i.e., $\theta_\mathrm{C} = 0$),

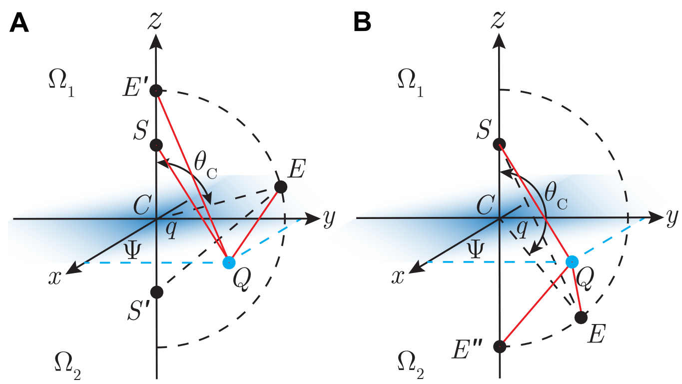

Figure A1. Auxiliary model of charged plane $\Psi : z = 0$ in vacuum. Boundary Ψ divide $\mathbb{R}^3$ into two domains $\Omega_1 : z \geq 0$ and $\Omega_2 : z \leq 0$. Position S with coordinates $\mathbf{r}_{\rm S} := (0, 0, z_{\rm S})$ is on the positive z-axis of the Cartesian coordinates originated at point C. The position Q has coordinates $\mathbf{r}_{\rm Q} := (x, y, 0)$ is an arbitrary point on plane Ψ. The charge density $q(\mathbf{r}_{\rm Q})$ is distributed on Ψ. Positions E, E' and E'' are three points on the plane $x = 0$, which satisfy the line segments $|EC| = |E'C| = |E''C|$. Point E' is on the positive z-axis (A) while E'' is on the negative z-axis (B). Angle $\theta_{\rm C}$ is the angle between line segments $|SC|$ and $|EC|$. Point S' at $(0, 0, -z_{\rm S})$ is a mirrored point of S with respect to Ψ in (A).

then (A3) is rewritten as

$$U(\mathbf{r}_{\rm E'}) = \sum_{n=0}^{\infty} \left(A_n R_{\rm E}^n + \frac{B_n}{R_{\rm E}^{n+1}} \right). \tag{A4}$$

According to (A2) in Cartesian coordinates, electrical potential at E' can also be expressed as

$$U\left(\mathbf{r}_{\rm E'}\right) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{\partial R_{\rm SQ}}{\partial z_{\rm S}} \frac{{\rm d}x{\rm d}y}{R_{\rm SQ}^2 R_{\rm QE'}}, \tag{A5}$$

in which $R_{\rm SQ} := |\mathbf{r}_{\rm Q} - \mathbf{r}_{\rm S}| = \sqrt{x^2 + y^2 + z_{\rm S}^2}$ is the distance between S and Q; $R_{\rm QE'} := |\mathbf{r}_{\rm E'} - \mathbf{r}_{\rm Q}| = \sqrt{x^2 + y^2 + R_{\rm E}^2}$ is the distance between Q and E'. Equation (A5) can be further simplified as

$$U\left(\mathbf{r}_{\rm E'}\right) = -\frac{{\rm sgn}(z_{\rm S})}{2\varepsilon_0} \frac{1}{|z_{\rm S}| + R_{\rm E}}. \tag{A6}$$

When $|z_{\rm S}| \neq R_{\rm E}$, (A6) can be rewritten using Taylor series as

$$U\left(\mathbf{r}_{\rm E'}\right) = \begin{cases} -\frac{{\rm sgn}(z_{\rm S})}{2\varepsilon_0 R_{\rm E}} \sum_{n=0}^{\infty} \left(-\frac{|z_{\rm S}|}{R_{\rm E}}\right)^n & \text{if } R_{\rm E} > |z_{\rm S}| \\ -\frac{{\rm sgn}(z_{\rm S})}{2\varepsilon_0 |z_{\rm S}|} \sum_{n=0}^{\infty} \left(-\frac{R_{\rm E}}{|z_{\rm S}|}\right)^n & \text{if } R_{\rm E} < |z_{\rm S}|. \end{cases} \tag{A7}$$

Comparing (A7) with (A4) we have

$$\begin{cases} A_n = 0,\ B_n = -\frac{{\rm sgn}(z_{\rm S})}{2\varepsilon_0} \left(-|z_{\rm S}|\right)^n & \text{if } R_{\rm E} > |z_{\rm S}| \\ A_n = \frac{{\rm sgn}(z_{\rm S})}{2\varepsilon_0} \left(-|z_{\rm S}|\right)^{-n-1},\ B_n = 0 & \text{if } R_{\rm E} < |z_{\rm S}|. \end{cases} \tag{A8}$$

Then, the Legendre polynomials can also be written as

$$\frac{1}{\sqrt{x^2 - 2xt + 1}} = \sum_{n=0}^{\infty} x^n P_n\left(t\right), \tag{A9}$$

which we can plug into (A3) using (A8) as well to obtain

$$U(\mathbf{r}_\mathrm{E}) = -\frac{\mathrm{sgn}(z_\mathrm{S})}{2\varepsilon_0\sqrt{R_\mathrm{E}^2 - 2R_\mathrm{E}|z_\mathrm{S}|\cos(\pi-\theta_\mathrm{C}) + z_\mathrm{S}^2}}. \tag{A10}$$

When $|z_\mathrm{S}| = R_\mathrm{E}$, then

$$U(\mathbf{r}_\mathrm{E})\big|_{|z_\mathrm{S}|=R_\mathrm{E}} \in \left[\lim_{\Delta\to 0} U(\mathbf{r}_\mathrm{E})\big|_{|z_\mathrm{S}|=R_\mathrm{E}-\Delta}, \lim_{\Delta\to 0} U(\mathbf{r}_\mathrm{E})\big|_{|z_\mathrm{S}|=R_\mathrm{E}+\Delta}\right].$$

According to (A10) and the Squeeze theorem, (A10) also satisfies that $|z_\mathrm{S}| = R_\mathrm{E}$.

As illustrated in Figure A1 A, we introduce a auxiliary point S' at $\mathbf{r}_\mathrm{S} = (0, 0, -z_\mathrm{S})$ to simplify (A10). Point S' is the mirrored image of S with respect to plane Ψ. Then the distance between S' and E is described as

$$R_\mathrm{S'E} := |\mathbf{r}_\mathrm{E} - \mathbf{r}_\mathrm{S'}| = \sqrt{R_\mathrm{E}^2 - 2R_\mathrm{E}|z_\mathrm{S}|\cos(\pi-\theta_\mathrm{C}) + z_\mathrm{S}^2}.$$

Finally, we can write the electrical potential at $\mathbf{r}_\mathrm{E}$ as

$$U(\mathbf{r}_\mathrm{E}) = -\frac{\mathrm{sgn}(z_\mathrm{S})}{2\varepsilon_0 R_\mathrm{S'E}}. \tag{A11}$$

1.2 Point E on plane Ψ or in different domain from S

To obtain the expression of electrical potential at arbitrary position $\mathbf{r}_\mathrm{E}$, an auxiliary point E'' at $\mathbf{r}_{\mathrm{E}''} := (0, 0, \mathrm{sgn}(z_\mathrm{E})R_\mathrm{E})$ on z-axis is introduced (see Figure A1 B). When $\theta_\mathrm{C} = \pi$, then (A3) is

$$U(\mathbf{r}_{\mathrm{E}''}) = \sum_{n=0}^{\infty} (-1)^n \left(A_n R_\mathrm{E}^n + \frac{B_n}{R_\mathrm{E}^{n+1}}\right). \tag{A12}$$

Replacing E' with E'' in (A6), the electrical potential at E'' can be re-written as

$$U(\mathbf{r}_{\mathrm{E}''}) = \frac{\mathrm{sgn}(z_\mathrm{S})}{2\varepsilon_0}\frac{1}{|z_\mathrm{S}| + R_\mathrm{E}}. \tag{A13}$$

Similar mathematical operations as (A7) (A8) (A9) (A10) can be performed to (A12) and (A13). Then we can find general expression for electrical potential $U(\mathbf{r}_\mathrm{E})$, namely,

$$U(\mathbf{r}_\mathrm{E}) = -\frac{\mathrm{sgn}(z_\mathrm{S})}{2\varepsilon_0\sqrt{R_\mathrm{E}^2 - 2R_\mathrm{E}|z_\mathrm{S}|\cos\theta_\mathrm{C} + z_\mathrm{S}^2}} = -\frac{\mathrm{sgn}(z_\mathrm{S})}{2\varepsilon_0 R_\mathrm{SE}} \tag{A14}$$

where $R_\mathrm{SE} := |\mathbf{r}_\mathrm{E} - \mathbf{r}_\mathrm{S}| = \sqrt{R_\mathrm{E}^2 - 2R_\mathrm{E}|z_\mathrm{S}|\cos\mathrm{C} + z_\mathrm{S}^2}$ is the distance between S and E.

1.3 Point S on plane Ψ

From the assumption we have $\mathbf{r}_\mathrm{S} \in \Psi$. Meanwhile according to the definition of $\mathbf{r}_\mathrm{Q}$, we can find that $\mathbf{r}_\mathrm{Q} \in \Psi$. It follows that

$$\frac{\partial(|\mathbf{r}_\mathrm{Q} - \mathbf{r}_\mathrm{S}|)}{\partial \mathbf{n}_{21}(\mathbf{r}_\mathrm{Q})} \equiv 0.$$

According to (A2) we have

$$U(\mathbf{r}_\mathrm{E}) \equiv 0. \tag{A15}$$

One can find (A1) equating the right hand sides of (A11), (A14), (A15) and (A2).

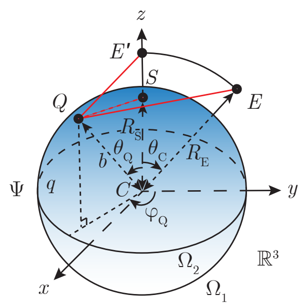

Figure B1. Auxiliary model of charged spherical shell in vacuum. A nonuniform charged spherical shell Ψ is centered at the origin C of spherical coordinates (r, θ, φ) in infinity vacuum $\mathbb{R}^3$ with b the radius of Ψ. Boundary Ψ divide the full space into two domains $\Omega_1 : r \geq b$ and $\Omega_2 : r \leq b$. The position Q has coordinates $\mathbf{r}_{\mathrm{Q}} := (b, \theta_{\mathrm{Q}}, \varphi_{\mathrm{Q}})$ on Ψ. The charge density $q(\mathbf{r}_{\mathrm{Q}})$ is distributed on surface Ψ. The position S, E and E' have coordinates $\mathbf{r}_{\mathrm{S}} := (R_{\mathrm{S}}, 0, 0)$, $\mathbf{r}_{\mathrm{E}} := (R_{\mathrm{E}}, \theta_{\mathrm{C}}, 0)$ and $\mathbf{r}_{\mathrm{E}'} := (R_{\mathrm{E}}, 0, 0)$, which satisfy $|EC| = |E'C|$ and S, E' are on positive z-axis. The length R_{S} is the distance between S and C; the length R_{E} is the distance between E and C. The angle θ_{C} is defined between line segment $|SC|$ and $|EC|$; θ_{Q} is the angle between line segment $|SC|$ and $|EC|$.

Part B. Lemma 2

Surface Ψ is a spherical surface centered at C with radius b in $\mathbb{R}^3$, which divide the full space into two domains Ω_1 and Ω_2. Point Q at position $\mathbf{r}_{\mathrm{Q}}$ is on Ψ (i.e., $\mathbf{r}_{\mathrm{Q}} \in \Psi$), while two different points S at $\mathbf{r}_{\mathrm{S}}$ and E at $\mathbf{r}_{\mathrm{E}}$ are arbitrarily located in $\mathbb{R}^3$ (see Figure B1). Vector $\mathbf{n}_{21}(\mathbf{r}_{\mathrm{Q}})$ is a normal vector of surface Ψ at $\mathbf{r}_{\mathrm{Q}}$ pointing from Ω_2 to Ω_1. R_{S} and R_{E} are are the distance between S and C, E and C, respectively. Then the 1st-order geometrical parameter $K_{\Psi}^{(1)}(\mathbf{r}_{\mathrm{E}}, \mathbf{r}_{\mathrm{S}})$ defined in (9) can be calculated as

$$-\frac{1}{2\pi}\iint_{\Psi} \frac{\partial\left(|\mathbf{r}_{\mathrm{Q}}-\mathbf{r}_{\mathrm{S}}|\right)}{\partial \mathbf{n}_{21}(\mathbf{r}_{\mathrm{Q}})} \cdot \frac{\mathbf{n}_{21}(\mathbf{r}_{\mathrm{Q}})\,\mathrm{d}\Psi(\mathbf{r}_{\mathrm{Q}})}{|\mathbf{r}_{\mathrm{Q}}-\mathbf{r}_{\mathrm{S}}|^2\,|\mathbf{r}_{\mathrm{E}}-\mathbf{r}_{\mathrm{Q}}|} \tag{B1}$$

$$= \begin{cases} \frac{b}{R_{\mathrm{S}}R_{\mathrm{E}}}\sum_{n=0}^{\infty}\frac{2n}{2n+1}\left(\frac{b^2}{R_{\mathrm{S}}R_{\mathrm{E}}}\right)^n P_n(\cos\theta_{\mathrm{C}}) & \text{if} \quad R_{\mathrm{S}} > b, R_{\mathrm{E}} \geq b \\ \frac{1}{R_{\mathrm{S}}}\sum_{n=0}^{\infty}\frac{2n}{2n+1}\left(\frac{R_{\mathrm{E}}}{R_{\mathrm{S}}}\right)^n P_n(\cos\theta_{\mathrm{C}}) & \text{if} \quad R_{\mathrm{S}} > b, R_{\mathrm{E}} \leq b \\ -\frac{1}{R_{\mathrm{E}}}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{b}{R_{\mathrm{E}}}\right)^n P_n(\cos\theta_{\mathrm{C}}) & \text{if} \quad R_{\mathrm{S}} = b, R_{\mathrm{E}} \geq b \\ -\frac{1}{b}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{R_{\mathrm{E}}}{b}\right)^n P_n(\cos\theta_{\mathrm{C}}) & \text{if} \quad R_{\mathrm{S}} = b, R_{\mathrm{E}} \leq b \\ -\frac{1}{R_{\mathrm{E}}}\sum_{n=0}^{\infty}\frac{2n+2}{2n+1}\left(\frac{R_{\mathrm{S}}}{R_{\mathrm{E}}}\right)^n P_n(\cos\theta_{\mathrm{C}}) & \text{if} \quad R_{\mathrm{S}} < b, R_{\mathrm{E}} \geq b \\ -\frac{1}{b}\sum_{n=0}^{\infty}\frac{2n+2}{2n+1}\left(\frac{R_{\mathrm{S}}R_{\mathrm{E}}}{b^2}\right)^n P_n(\cos\theta_{\mathrm{C}}) & \text{else } R_{\mathrm{S}} < b, R_{\mathrm{E}} \leq b. \end{cases} \tag{B2}$$

Proof. Firstly, we establish a spherical coordinates in $\mathbb{R}^3$ originated at the center C of spherical surface Ψ. The position $\mathbf{r} := (r, \theta, \varphi)$ is defined in spherical coordinates. For convenience, we consider S with coordinates $\mathbf{r}_{\mathrm{S}} := (R_{\mathrm{S}}, 0, 0)$ on the positive z-axis, while E with coordinates $\mathbf{r}_{\mathrm{E}} := (R_{\mathrm{E}}, \theta_{\mathrm{C}}, 0)$ is placed on the plane $\varphi = 0$. Point Q with coordinates $\mathbf{r}_{\mathrm{Q}} := (b, \theta_{\mathrm{Q}}, \varphi_{\mathrm{Q}})$ is an arbitrary point on Ψ. Angle θ_{C} is the angle between line segment $|SC|$ and $|EC|$, θ_{Q} is the angle between line segment $|SC|$ and $|QC|$. Next, we follow a similar procedure as we did in Lemma 1, proving (B2) indirectly with an auxiliary physical model of charged spherical surface Ψ in vacuum.

2.1 Neither point S nor E is on spherical surface Ψ

Considering a charge density function $q(\mathbf{r}_{\mathrm{Q}})$ on the spherical surface Ψ, as illustrated in Figure B1, we already know the electrical potential $U(\mathbf{r}_{\mathrm{E}})$ is the key to give the proof from Lemma 1. Introduce an auxiliary point E' with coordinates $\mathbf{r}_{\mathrm{E}'} := (R_{\mathrm{E}}, 0, 0)$ on positive z-axis, which satisfy $|E'C| = |EC|$. From (A2), $U(\mathbf{r}_{\mathrm{E}'})$ can be expressed as

$$U\left(\mathbf{r}_{\mathrm{E}'}\right) = \frac{1}{4\pi\varepsilon_0} \int_{-\pi}^{\pi} \int_{0}^{\pi} \frac{\partial\left(|\mathbf{r}_{\mathrm{Q}} - \mathbf{r}_{\mathrm{S}}|\right)}{\partial b} \frac{b^2 \sin\theta_{\mathrm{Q}} \mathrm{d}\theta_{\mathrm{Q}} \mathrm{d}\varphi_{\mathrm{Q}}}{|\mathbf{r}_{\mathrm{Q}} - \mathbf{r}_{\mathrm{S}}|^2 \, |\mathbf{r}_{\mathrm{E}'} - \mathbf{r}_{\mathrm{Q}}|} \tag{B3}$$

where $|\mathbf{r}_{\mathrm{Q}} - \mathbf{r}_{\mathrm{S}}| = \sqrt{R_{\mathrm{S}}^2 + b^2 - 2bR_{\mathrm{S}}\cos\theta_{\mathrm{Q}}}$ and $|\mathbf{r}_{\mathrm{E}'} - \mathbf{r}_{\mathrm{Q}}| = \sqrt{R_{\mathrm{E}}^2 + b^2 - 2bR_{\mathrm{E}}\cos\theta_{\mathrm{Q}}}$. Equation (B3) can be further simplified as

$$U\left(\mathbf{r}_{\mathrm{E}'}\right) = \begin{cases} \frac{1}{4\varepsilon_0}\left(\frac{2b}{b^2 - R_{\mathrm{S}}R_{\mathrm{E}}} + \frac{1}{\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}} \ln\left(\frac{\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}+b}{\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}-b}\right)\right) & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} > b \\ \frac{1}{4\varepsilon_0}\left(\frac{2}{R_{\mathrm{E}} - R_{\mathrm{S}}} + \frac{1}{\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}} \ln\left(\frac{\sqrt{R_{\mathrm{S}}}+\sqrt{R_{\mathrm{E}}}}{\sqrt{R_{\mathrm{S}}}-\sqrt{R_{\mathrm{E}}}}\right)\right) & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} < b \\ \frac{1}{4\varepsilon_0}\left(\frac{2}{R_{\mathrm{E}} - R_{\mathrm{S}}} + \frac{1}{\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}} \ln\left(\frac{\sqrt{R_{\mathrm{E}}}+\sqrt{R_{\mathrm{S}}}}{\sqrt{R_{\mathrm{E}}}-\sqrt{R_{\mathrm{S}}}}\right)\right) & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} > b \\ \frac{1}{4\varepsilon_0}\left(\frac{2b}{b^2 - R_{\mathrm{S}}R_{\mathrm{E}}} + \frac{1}{\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}} \ln\left(\frac{b+\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}}{b-\sqrt{R_{\mathrm{S}}R_{\mathrm{E}}}}\right)\right) & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} < b. \end{cases} \tag{B4}$$

Expression (B4) can be rewritten using Taylor series as

$$U\left(\mathbf{r}_{\mathrm{E}'}\right) = \begin{cases} -\frac{b}{2\varepsilon_0 R_{\mathrm{S}} R_{\mathrm{E}}} \sum_{n=0}^{\infty} \frac{2n}{2n+1} \left(\frac{b^2}{R_{\mathrm{S}}R_{\mathrm{E}}}\right)^n & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} > b \\ -\frac{1}{2\varepsilon_0 R_{\mathrm{S}}} \sum_{n=0}^{\infty} \frac{2n}{2n+1} \left(\frac{R_{\mathrm{E}}}{R_{\mathrm{S}}}\right)^n & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} < b \\ \frac{1}{2\varepsilon_0 R_{\mathrm{E}}} \sum_{n=0}^{\infty} \frac{2n+2}{2n+1} \left(\frac{R_{\mathrm{S}}}{R_{\mathrm{E}}}\right)^n & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} > b \\ \frac{1}{2\varepsilon_0 b} \sum_{n=0}^{\infty} \frac{2n+2}{2n+1} \left(\frac{R_{\mathrm{S}}R_{\mathrm{E}}}{b^2}\right)^n & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} < b. \end{cases} \tag{B5}$$

Comparing (B5) and (A4), we have

$$\begin{cases} A_n = 0, B_n = -\frac{1}{2\varepsilon_0} \frac{2n}{2n+1} \frac{b^{2n+1}}{R_{\mathrm{S}}^{n+1}} & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} > b \\ A_n = -\frac{1}{2\varepsilon_0} \frac{2n}{2n+1} \frac{1}{R_{\mathrm{S}}^{n+1}}, B_n = 0 & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} < b \\ A_n = 0, B_n = \frac{1}{2\varepsilon_0} \frac{2n+2}{2n+1} R_{\mathrm{S}}^n & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} > b \\ A_n = \frac{1}{2\varepsilon_0} \frac{2n+2}{2n+1} \frac{R_{\mathrm{S}}^n}{b^{2n+1}}, B_n = 0 & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} < b. \end{cases} \tag{B6}$$

Substituting (B6) to (A3) gives

$$U\left(\mathbf{r}_{\mathrm{E}}\right) = U\left(\mathbf{r}_{\mathrm{E}'}\right) P_n(\cos\theta_{\mathrm{C}}). \tag{B7}$$

2.2 Either point S or E is on spherical surface Ψ Next we consider the cases when $R_{\mathrm{S}} = b$ or $R_{\mathrm{E}} = b$, which can be yield from the limit when R_{S} or R_{E} approaches b in (B7). The electrical potential on these singularity is defined as

$$U(\mathbf{r}_{\mathrm{E}}) := \begin{cases} \lim_{\Delta\to 0} \frac{U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}=b+\Delta, R_{\mathrm{E}}>b} + U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}=b-\Delta, R_{\mathrm{E}}>b}}{2} & \text{if } R_{\mathrm{S}} = b, R_{\mathrm{E}} > b \\ \lim_{\Delta\to 0} \frac{U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}=b+\Delta, R_{\mathrm{E}}<b} + U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}=b-\Delta, R_{\mathrm{E}}<b}}{2} & \text{if } R_{\mathrm{S}} = b, R_{\mathrm{E}} < b \\ \lim_{\Delta\to 0} \frac{U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}>b, R_{\mathrm{E}}=b+\Delta} + U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}>b, R_{\mathrm{E}}=b-\Delta}}{2} & \text{if } R_{\mathrm{S}} > b, R_{\mathrm{E}} = b \\ \lim_{\Delta\to 0} \frac{U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}<b, R_{\mathrm{E}}=b+\Delta} + U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}<b, R_{\mathrm{E}}=b-\Delta}}{2} & \text{if } R_{\mathrm{S}} < b, R_{\mathrm{E}} = b \\ \lim_{\Delta\to 0} \frac{U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}=R_{\mathrm{E}}=b+\Delta} + U(\mathbf{r}_{\mathrm{E}})|_{R_{\mathrm{S}}=R_{\mathrm{E}}=b-\Delta}}{2} & \text{if } R_{\mathrm{S}} = b, R_{\mathrm{E}} = b \end{cases} \tag{B8}$$

Substituting (B5), (B7) into (B8) we have

$$U(\mathbf{r}_\mathrm{E}) = \begin{cases} \frac{1}{2\varepsilon_0 R_\mathrm{E}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{b}{R_\mathrm{E}}\right)^n P_n\left(\cos\theta_\mathrm{C}\right) & \text{if } R_\mathrm{S} = b, R_\mathrm{E} > b \\ \frac{1}{2\varepsilon_0 b} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{R_\mathrm{E}}{b}\right)^n P_n\left(\cos\theta_\mathrm{C}\right) & \text{if } R_\mathrm{S} = b, R_\mathrm{E} < b \\ -\frac{1}{2\varepsilon_0 R_\mathrm{S}} \sum_{n=0}^{\infty} \frac{2n}{2n+1} \left(\frac{b}{R_\mathrm{S}}\right)^n P_n\left(\cos\theta_\mathrm{C}\right) & \text{if } R_\mathrm{S} > b, R_\mathrm{E} = b \\ \frac{1}{2\varepsilon_0 b} \sum_{n=0}^{\infty} \frac{2n+2}{2n+1} \left(\frac{R_\mathrm{S}}{b}\right)^n P_n\left(\cos\theta_\mathrm{C}\right) & \text{if } R_\mathrm{S} < b, R_\mathrm{E} = b \\ \frac{1}{2\varepsilon_0 b} \sum_{n=0}^{\infty} \frac{1}{2n+1} P_n\left(\cos\theta_\mathrm{C}\right) & \text{if } R_\mathrm{S} = b, R_\mathrm{E} = b. \end{cases} \tag{B9}$$

One can find (B2) equating the right hand sides of (B7), (B9) and (A2).

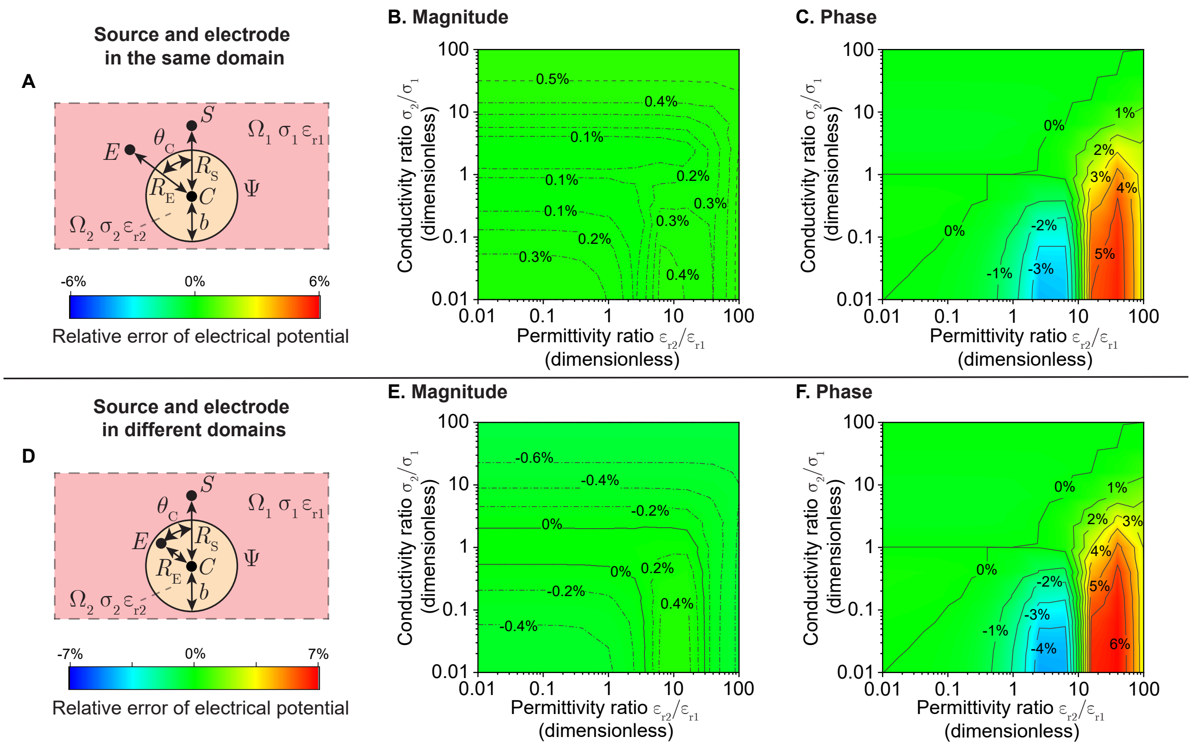

Figure B2. Relative error distribution of electrical potential between theory and FEM with varying nonhomogeneous electrical property in case study 2. A spherical surface Ψ with radius b centered at point C divides full space into two domain Ω_1 and Ω_2 with conductivity $\sigma_{\{1,2\}}$ and relative permittivity $\varepsilon_{\{\mathrm{r1,r2}\}}$, respectively. Sinusoidal current ($I = 20$ nA of 1 kHz) source S and potential recording electrode E are placed in the same domain (A) and in different domains (D). Angle θ_C is the angle between of line segment $|EC|$ and $|SC|$, and R_S, R_E are distance from S, E to point C, respectively. Relative error distributions of potential magnitude and phase when E in Ω_1 with $R_\mathrm{E} = 8$ mm (B, C) and E in Ω_2 with $R_\mathrm{E} = 3$ mm (E, F) changing the conductivity $\sigma_2 = [0.01 \cdot \sigma_1, 100 \cdot \sigma_1]$ and relative permittivity $\varepsilon_{\mathrm{r2}} = [0.01 \cdot \varepsilon_{\mathrm{r1}}, 100 \cdot \varepsilon_{\mathrm{r1}}]$. Additional simulation parameters: $b = 5$ mm, $R_\mathrm{S} = 10$ mm, $\theta_\mathrm{C} = 90°$, $\sigma_1 = 0.431$ S/m, $\varepsilon_1 = 8.67 \times 10^5$ (dimensionless).

Part C. Accuracy analysis

We proposed (9) to calculate the 1st-order approximated electrical potential in a nonhomogeneous conducting volume. Therefore, the accuracy of the approximation is determined by how electrically similar or different the tissues within the domain are. Here, we compare the 1st-order approximation with finite element model simulation considering the same case study 1 and 2 to assess the accuracy when varying the tissues' electrical properties.

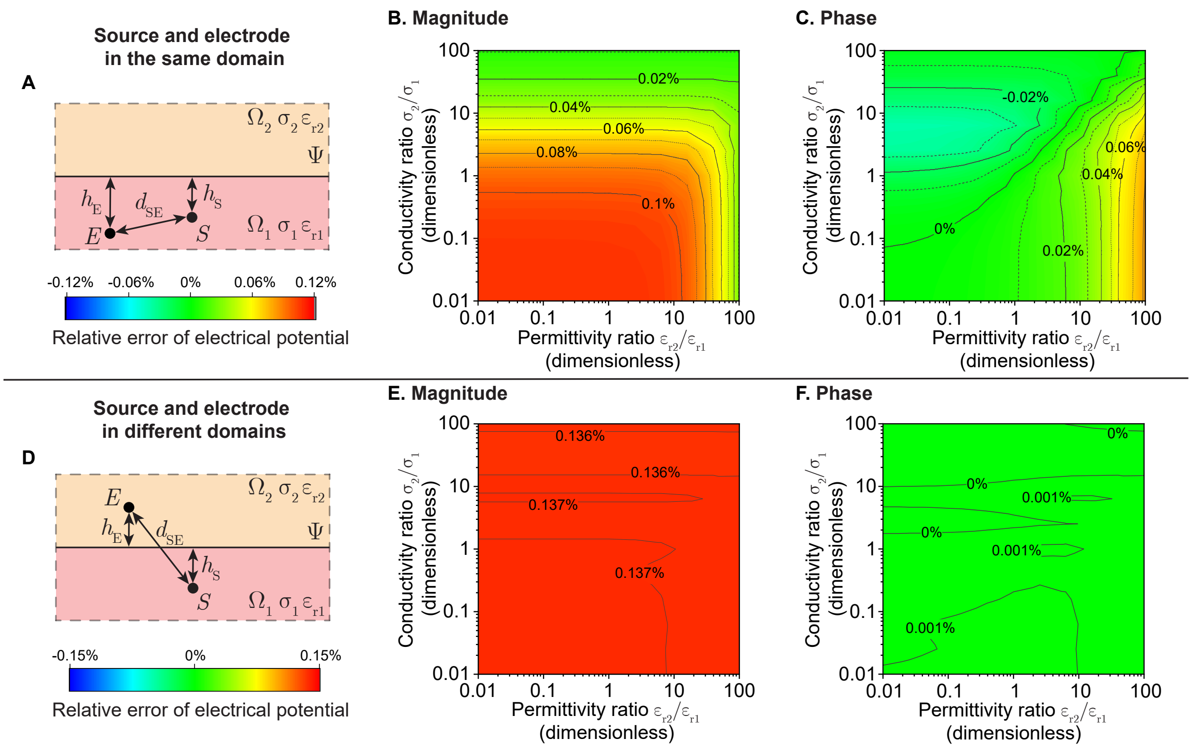

Figure C1. Relative error distribution of electrical potential between theory and FEM with varying nonhomogeneous electrical property in case study 1. A plane Ψ divides full space into two domain Ω_1 and Ω_2 with conductivity $\sigma_{\{1,2\}}$ and relative permittivity $\varepsilon_{\{\mathrm{r}1,\mathrm{r}2\}}$, respectively. Sinusoidal current ($I = 20$ nA at 1 kHz) source S and potential recording electrode E are placed in the same domain (A) and in different domains (D). Distance d_{SE} is the length of line segment $|SE|$, and h_{S}, h_{E} are distance from S, E to boundary Ψ, respectively. Relative error distributions of potential magnitude and phase when E in Ω_1 with $d_{\mathrm{SE}} = 10$ mm (B, C) and E in Ω_2 with $d_{\mathrm{SE}} = 15$ mm (E, F) changing the conductivity $\sigma_2 = [0.01 \cdot \sigma_1, 100 \cdot \sigma_1]$ and relative permittivity $\varepsilon_{\mathrm{r}2} = [0.01 \cdot \varepsilon_{\mathrm{r}1}, 100 \cdot \varepsilon_{\mathrm{r}1}]$. Additional simulation parameters: $h_{\mathrm{S}} = h_{\mathrm{E}} = 5$ mm, $\sigma_1 = 0.431$ S/m, $\varepsilon_1 = 8.67 \times 10^5$ (dimensionless).

3.1 Simulation configuration

A surface Ψ divide full space into two domain $\Omega_{\{1,2\}}$ with conductivity $\sigma_{\{1,2\}}$ and relative permittivity $\varepsilon_{\{\mathrm{r}1,\mathrm{r}2\}}$, respectively. The electrical property in Ω_1 is that of isotropic muscle, i.e., $\sigma_1 = 4.31 \times 10^{-1}$ S/m, $\varepsilon_{\mathrm{r}1} = 8.67 \times 10^5$ (dimensionless), while the electrical property in Ω_2 changes as $\sigma_2 = [0.01 \cdot \sigma_1, 100 \cdot \sigma_1]$, $\varepsilon_2 = [0.01 \cdot \varepsilon_1, 100 \cdot \varepsilon_1]$. A point-like source S located in Ω_1 generating sinusoidal current (I=20 nA at 1 kHz). An electrode E records the electrical potential U. To evaluate the accuracy, we define the magnitude error $e_{\mathrm{mag}} := (|U_{\mathrm{Theory}}| - |U_{\mathrm{FEM}}|)/|U_{\mathrm{FEM}}|$ and phase error $e_{\mathrm{phase}} := (\mathrm{Arg}\{U_{\mathrm{Theory}}\} - \mathrm{Arg}\{U_{\mathrm{FEM}}\})/\mathrm{Arg}\{U_{\mathrm{FEM}}\}$ of electrical potential, where $\mathrm{Arg}\{\cdot\}$ is the argument of a complex value.

In case study 1, the potential recording electrode E can be placed in domain Ω_1 (see Figure C1 A) and in Ω_2 (see Figure C1 D). Geometrical parameters are set as constants: distance $d_{\mathrm{SE}} = 10$ mm is the length of line segment $|SE|$, and $h_{\mathrm{S}} = 5$ mm, $h_{\mathrm{E}} = 5$ mm are distance from S, E to planar boundary Ψ, respectively.

In case study 2, boundary Ψ is a spherical surface with radius $b = 5$ mm, and length $R_{\mathrm{S}} = 10$ mm, R_{E} are distance from S, E to spherical center point C. Electrode E can also be placed in domain Ω_1 with $R_{\mathrm{E}} = 8$ mm (see Figure B2 A) and in Ω_2 with $R_{\mathrm{E}} = 3$ mm (see Figure B2 D). Angle $\theta_{\mathrm{C}} = 90°$ is the angle between of line segment $|EC|$ and $|SC|$.

3.2. Accuracy

Figure C1 and B2 plot the accuracy of the 1st-order electrical potential predictions compared to FEM simulation in case study 1 and 2, respectively. The conductivity and relative permittivity properties in domain Ω_2 change from 0.01 to 100 times to that in Ω_1 while keeping the geometrical parameters constants. The relative error of magnitude and phase are shown when electrode E located in domain Ω_1 (Figure C1 B and C in case study 1, Figure B2 B and C in case study 2) and in Ω_2 (Figure C1 E and F in case study 1, Figure B2 E and F in case study 2). The relative errors for magnitude and phase are $\leq 0.14\%$, $\leq 0.09\%$ in case study 1, and $\leq 0.8\%$, $\leq 7\%$ in case study 2, respectively. Of note, when electrical property in domain Ω_2 is set as fat (i.e., $\sigma_2/\sigma_1 = 0.052$ and $\varepsilon_2/\varepsilon_1 = 0.028$), the relative errors of magnitude and phase of electrical potential are both $\leq 0.5\%$.