PNAS
www.pnas.org

Supplementary Information for

Skin effect as a probe of transport regimes in Weyl semimetals

Paweł Matus, Renato M. A. Dantas, Roderich Moessner and Piotr Surówka

Corresponding Author Piotr Surówka.
E-mail: surowka@pks.mpg.de

This PDF file includes:

Supplementary text
Fig. S1 (not allowed for Brief Reports)
Table S1 (not allowed for Brief Reports)
SI References

Supporting Information Text

Higher order corrections to energy and Berry curvature

Second-order corrections to the semiclassical equations of motion in the presence of the magnetic field were derived within the wave-packet approach in (1, 2). These corrections modify the Berry curvature and the wave-packet energy, but do not otherwise change the form of the equations of motion (Eqs. **3**-**4**). We present here formulas derived in Ref. (2) and evaluate them for the Hamiltonian of a Weyl quasiparticle with chirality s,

$$H(\mathbf{p}) = sv\boldsymbol{\sigma}\cdot\mathbf{p}. \qquad [S1]$$

Let us denote the eigenstates $|u_n\rangle$ and eigenvalues ϵ_n, with n indicating the band. We label the upper band as 0 and the lower band as -1. We define the interband Berry connection as $\mathbf{A}_{nm} = -i\langle u_n|\partial_{\mathbf{p}}|u_m\rangle$, and the velocity matrix as $\mathbf{V}_{nm} = \langle u_n|\partial_{\mathbf{p}}H(\mathbf{p})|u_m\rangle$. We further define

$$G_{nl} = -\frac{1}{2}\mathbf{B}\cdot\left(\Sigma_{m\neq l}\mathbf{V}_{nm}\times\mathbf{A}_{ml} + \mathbf{V}_{ll}\times\mathbf{A}_{nl}\right), \qquad [S2]$$

which is then used in the computation of the correction to the intraband Berry connection

$$\mathbf{a}'_0 = e\hbar\sum_{n\neq 0}\left[\frac{G_{0n}\mathbf{A}_{n0}}{\epsilon_0-\epsilon_n}\right] + e\hbar\frac{1}{4}\partial_{p_i}\left[(\mathbf{B}\times\mathbf{A}_{0n})_i\,\mathbf{A}_{n0}\right] + \text{c.c.} \qquad [S3]$$

Because in our model $\mathbf{V}_{-1-1} = -\mathbf{V}_{00}$, $G_{0-1} = G_{-10} = 0$ and, consequently, the only contribution comes from the second term on the RHS of Eq. S3. The resulting correction to the Berry curvature reads

$$\boldsymbol{\Omega}' = \partial_{\mathbf{p}}\times\mathbf{a}'_0 = -\frac{e\hbar\mathbf{B}}{4|\mathbf{p}|^4} + \frac{e\hbar(\mathbf{B}\cdot\mathbf{p})\mathbf{p}}{2|\mathbf{p}|^6}. \qquad [S4]$$

This is taken into account in Eq. **10**.

Second-order corrections to energy are composed of several terms [see Eq. **4** in (2)]. Nonetheless, not all of them contribute to the final result. Firstly, there are two *geometrical* terms: one of them depends on the Berry curvature and the magnetic moment

$$\epsilon^{(1)} = \frac{e\hbar}{4}(\mathbf{B}\cdot\boldsymbol{\Omega})(\mathbf{B}\cdot\mathbf{m}) = \frac{e^2\hbar^2 v(\mathbf{B}\cdot\mathbf{p})^2}{16|\mathbf{p}|^5}, \qquad [S5]$$

while the other one depends on the quantum metric $g_{ij} = \text{Re}\langle\partial_{p_i}u_0|\partial_{p_j}u_0\rangle - A_{00,i}A_{00,j}$ and the inverse effective mass tensor $\alpha_{kl} = \partial_{p_k}\partial_{p_l}\epsilon_0$

$$\epsilon^{(2)} = -\frac{e^2\hbar^2}{8}\epsilon_{sik}\epsilon_{tjl}B_sB_tg_{ij}\alpha_{kl} = -\frac{e^2\hbar^2 v(\mathbf{B}\cdot\mathbf{p})^2}{16|\mathbf{p}|^5}, \qquad [S6]$$

and thus $\epsilon^{(1)} + \epsilon^{(2)} = 0$. All other second-order corrections to the energy derived in Ref. (2) vanish identically for the Hamiltonian in Eq. S1, except for

$$\epsilon^{(3)} = -e\hbar\mathbf{B}\cdot(\mathbf{a}'_0\times\mathbf{V}_{00}) = e^2\hbar^2 v\frac{|\mathbf{B}|^2|\mathbf{p}|^2 - (\mathbf{B}\cdot\mathbf{p})^2}{8|\mathbf{p}|^5}. \qquad [S7]$$

The total energy including the second order corrections is presented in Eq. **9**. We note that these corrections were also studied in (3).

Eigenfunctions of the collision operator

The inner product of Eq. **19** can be rewritten in the spherical coordinates (p,θ,ϕ) in the momentum space as

$$\langle\eta|\zeta\rangle = \int\frac{d^3p}{(2\pi\hbar)^3}D(\mathbf{p})\delta\left[\epsilon_{\text{M}}(\mathbf{p})-\epsilon_{\text{F}}\right]\eta(\mathbf{p})^*\zeta(\mathbf{p}) = \int\frac{d(\cos\theta)d\phi}{(2\pi\hbar)^3}D(\hat{\mathbf{p}})\frac{p_{\text{F}}(\hat{\mathbf{p}})^2}{v_{\text{M}}(\hat{\mathbf{p}})}\eta(\hat{\mathbf{p}})^*\zeta(\hat{\mathbf{p}}), \qquad [S8]$$

where $v_{\text{M}}(\hat{\mathbf{p}}) = \mathbf{v}_{\text{M}}(\hat{\mathbf{p}})\cdot\hat{\mathbf{p}}$. Hereafter, we suppress the (s) labels on all quantities. We furthermore define

$$\mathcal{V}(\hat{\mathbf{p}}) = D(\hat{\mathbf{p}})\frac{p_{\text{F}}(\hat{\mathbf{p}})^2}{v_{\text{M}}(\hat{\mathbf{p}})}. \qquad [S9]$$

In the absence of spherical symmetry, we can find physically sensible expressions for the eigenfunctions of the single-species collision operator $\hat{C}$ defined below Eq. **18** by exploiting its algebraic properties and a few facts:

(1) The eigenfunctions, which we denote K_l^m, should be perturbations of spherical harmonics

$$K_l^m = Y_l^m + O(\alpha). \qquad [S10]$$

(2) The eigenfunctions are orthogonal with respect to the new inner product, which follows from the hermiticity of $\hat{C}[\eta]$

$$\int \frac{d^2\hat{p}}{(2\pi\hbar)^3}\mathcal{V}(\hat{\mathbf{p}})K_{l'}^{m'}(\hat{\mathbf{p}})^* K_l^m(\hat{\mathbf{p}}) = \delta_{l,l'}\delta_{m,m'}. \tag{S11}$$

(3) Let us expand the distribution function as $\eta = \sum_{l,m} X_l^m K_l^m(\hat{\mathbf{p}})$, where $X_l^m = X_l^m(\mathbf{x},t)$. The rate of particle number change is

$$\begin{aligned}\frac{d\langle n\rangle}{dt} &= \int \frac{d^3p}{(2\pi\hbar)^3}D(\hat{\mathbf{p}})\frac{df}{dt} = -\int \frac{d^3p}{(2\pi\hbar)^3}D(\hat{\mathbf{p}})\delta\left[\epsilon_{\mathrm{M}}(\mathbf{p})-\epsilon_{\mathrm{F}}\right]\hat{C}[\eta] \\ &= \sum_{l,m}\int \frac{d^2\hat{p}}{(2\pi\hbar)^3}\mathcal{V}(\hat{\mathbf{p}})\Gamma_{l,m}X_l^m(\mathbf{x})K_l^m(\hat{\mathbf{p}}).\end{aligned} \tag{S12}$$

For processes that conserve the particle number, there must be an eigenvector of $\hat{C}$ associated with the eigenvalue 0 and corresponding to the particle number. Because we are considering a perturbation to the no-field problem, in which the corresponding eigenfunction was Y_0^0, now the corresponding eigenfunction has to be K_0^0 and the eigenvalue associated to it is $\Gamma_{0,0} = 0$. Therefore, the RHS of the above equation is zero when

$$\int \frac{d^2\hat{p}}{(2\pi\hbar)^3}\mathcal{V}(\hat{\mathbf{p}})K_l^m(\hat{\mathbf{p}}) = 0, \qquad \forall l \geq 1 \tag{S13}$$

which, when taking Eq. S11 into account, implies that $K_0^0(\hat{\mathbf{p}}) \propto 1 \propto Y_0^0(\hat{\mathbf{p}})$.

(4) The rate of momentum change is

$$\begin{aligned}\frac{d\langle \mathbf{p}\rangle}{dt} &= \int \frac{d^3p}{(2\pi\hbar)^3}D(\hat{\mathbf{p}})\mathbf{p}\frac{df}{dt} \propto -\int \frac{d^3p}{(2\pi\hbar)^3}D(\hat{\mathbf{p}})\delta\left[\epsilon_{\mathrm{M}}(\mathbf{p})-\epsilon_{\mathrm{F}}\right]pY_1^M(\hat{\mathbf{p}})\hat{C}[\eta] \\ &= \frac{\epsilon_{\mathrm{F}}}{v}\sum_{l,m}\int \frac{d^2\hat{p}}{(2\pi\hbar)^3}\mathcal{V}(\hat{\mathbf{p}})\left\{\frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^M(\hat{\mathbf{p}})\right\}\times\Gamma_{l,m}X_l^mK_l^m(\hat{\mathbf{p}}),\end{aligned} \tag{S14}$$

where $M = -1, 0, +1$ for the different momentum components. A reasoning similar as in paragraph (3) dictates that for processes that conserve momentum and the particle number $\Gamma_{1,M} = 0$ and

$$\int \frac{d^3p}{(2\pi\hbar)^3}\left\{\frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^M(\hat{\mathbf{p}})\right\}\mathcal{V}(\hat{\mathbf{p}})K_l^m(\hat{\mathbf{p}}) = 0, \quad \forall l \geq 2 \tag{S15}$$

which, when taking Eq. S11 into account, implies that

$$\frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^M(\hat{\mathbf{p}}) = \sum \lambda_{M'}^M K_1^{M'}(\hat{\mathbf{p}}) + \mu^M K_0^0(\hat{\mathbf{p}}) \propto \sum \lambda_{M'}^M K_1^{M'}(\hat{\mathbf{p}}) + \nu^M Y_0^0(\hat{\mathbf{p}}), \tag{S16}$$

where $\lambda_{M'}^M$, μ^M, ν^M are some coefficients. It follows that K_1^M is a linear combination of $p_{\mathrm{F}}(\hat{\mathbf{p}})Y_1^{M'}(\hat{\mathbf{p}})$ and Y_0^0, with coefficients chosen such that the orthogonality relation is satisfied. (The matrix $\lambda_{M'}^M$ is of the form $1 - O(\alpha)$, 1 being the identity matrix, and therefore it is invertible.)

We take all modes with $L \geq 2$ to be eigenfunctions to the same eigenvalue. Hence, all modes with $L \geq 2$ constitute one eigenspace orthogonal to the eigenvectors with $L = 0, 1$. We can choose the basis of this eigenspace arbitrarily, as long as it satisfies the orthogonality condition (Eq. S11).

Let us orthonormalise the ordered set of basis vectors $\left(Y_0^0(\hat{\mathbf{p}}), \frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^0(\hat{\mathbf{p}}), \frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^1(\hat{\mathbf{p}}), \frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^{-1}(\hat{\mathbf{p}})\right)$ with regard to the inner product in Eq. S8 using the Gram-Schmidt process. This gives the vectors $K_0^0(\hat{\mathbf{p}})$, $K_1^0(\hat{\mathbf{p}})$, $K_1^1(\hat{\mathbf{p}})$, $K_1^{-1}(\hat{\mathbf{p}})$. The Gram-Schmidt process guarantees that $K_0^0 \propto Y_0^0$, which satisfies condition (3), and that K_1^M are combinations of $p_{\mathrm{F}}(\hat{\mathbf{p}})Y_1^{M'}$ and Y_0^0, which satisfies condition (4). The formulas for K_1^M depend on the order in which the basis vectors are orthonormalized, but the space spanned by them does not (as it is defined as the 3D subspace orthogonal to Y_0^0 in the space spanned by $\{Y_0^0, \frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^0, \frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^1, \frac{vp_{\mathrm{F}}(\hat{\mathbf{p}})}{\epsilon_{\mathrm{F}}}Y_1^{-1}\}$). Consequently, we can unambiguously define the projection operators

$$P_0 = |K_0^0\rangle\langle K_0^0|, \tag{S17}$$

$$P_1 = \sum_{M=-1,0,1}|K_1^M\rangle\langle K_1^M|, \tag{S18}$$

$$P_{\mathrm{higher}} = 1 - P_0 - P_1. \tag{S19}$$

The eigenfunctions can be expanded to the appropriate order, e.g. up to α^2. Using the definition of the inner product in Eq. S8, it is possible to evaluate the action of the projection operators on any distribution function. In the main text we consider two coupled particle species that exchange particles, so that the (s) labels appear and the projections operators are defined as in Eq. **23**.

Table S1. Coefficients B_L^M with low L and $C_1^{\pm 1}$ obtained from the perturbative solution of the Boltzmann equation

Regime	Nonzero B_L^M coefficients up to $L = 2$	$C_1^{\pm 1}$ coefficients
Low-frequency normal	$B_0^0 = -evE\sqrt{\pi}\left[\frac{1}{\Gamma_{\text{inter}}} - \frac{4}{3\Gamma_{\text{mr}}}\right]$ $B_2^0 = evE\sqrt{\frac{\pi}{5}}\frac{2}{3\Gamma_{\text{mr}}}$ $B_2^{\pm 2} = -evE\sqrt{\frac{2\pi}{15}}\frac{1}{\Gamma_{\text{mr}}}$	$C_1^{\pm 1} = \mp evE\sqrt{\frac{2\pi}{3}}\frac{1}{\Gamma_{\text{mr}}}$
Anomaly-induced nonlocal	$B_0^0 = -evE\sqrt{\pi}\frac{6\Gamma_{\text{mr}}}{q^2v^2}$ $B_1^0 = -ievE\sqrt{3\pi}\frac{2}{qv}$ $B_2^0 = \sqrt{\frac{\pi}{5}}evE\left[\frac{4}{\Gamma_{\text{tot}}} + \frac{2}{3\Gamma_{\text{mr}}}\right]$ $B_2^{\pm 2} = -evE\sqrt{\frac{2\pi}{15}}\frac{1}{\Gamma_{\text{mr}}}$	$C_1^{\pm 1} = \mp evE\sqrt{\frac{2\pi}{3}}\frac{2(3\Gamma_{\text{mr}}+\Gamma_{\text{tot}})^2}{15\Gamma_{\text{mr}}^2\Gamma_{\text{tot}}}$
Hydrodynamic	$B_0^0 = evE\frac{11.3\Gamma_{\text{tot}}}{q^2v^2}$ $B_2^0 = evE\frac{2.4\Gamma_{\text{tot}}}{q^2v^2}$ $B_2^{\pm 2} = -evE\sqrt{\pi}\frac{2.9\Gamma_{\text{tot}}}{q^2v^2}$	$C_1^{\pm 1} = \mp evE\frac{10.2\Gamma_{\text{tot}}}{q^2v^2}$
Ballistic	$B_0^0 = -evE\frac{0.2}{qv}$	$C_1^{\pm 1} = \pm evE\frac{3.0}{qv}$
High-frequency normal	$B_0^0 = ievE\sqrt{\pi}\frac{2}{3\omega}$ $B_2^0 = -ievE\sqrt{\frac{\pi}{5}}\frac{2}{3\omega}$ $B_2^{\pm 2} = ievE\sqrt{\frac{2\pi}{15}}\frac{1}{\omega}$	$C_1^{\pm 1} = \pm ievE\sqrt{\frac{2\pi}{3}}\frac{1}{\omega}$

Only the leading-order contributions in the respective regimes are shown. The numerical coefficients in the hydrodynamic and ballistic regimes are rounded up to the first decimal place.

Perturbative solution of the Boltzmann equation

Before presenting the full perturbative solution, let us first explain a subtlety in the evaluation of the collision integral. The operator $P_0^{(s)}$ in Eq. **23** acts on both distribution functions $|\eta^{(\pm s)}\rangle$, therefore coupling the systems of equations for (s) and $(-s)$, which would suggest that the number of equations that need to be solved has doubled. However, we can use the following observation: the factor of s in the equations of motion always appears in the combination $s\,\alpha$ (see the formulas **3-4** and **16-17**). Therefore, in the expansions of $\langle K_0^{0,(s)}|\eta^{(s)}\rangle$ and $\langle K_0^{0,(-s)}|\eta^{(-s)}\rangle$ in α, the even-order terms have to be equal, and the odd-order terms have to be opposite. This in turn means that, as we are interested in the expansion up to the second order in α,

$$P_0^{(s)}[\eta^{(s)},\eta^{(-s)}] = 2|K_0^{0,(s)}\rangle \left\{\langle K_0^{0,(s)}|\eta^{(s)}\rangle\right\}_\alpha \tag{S20}$$

where $\{\dots\}_\alpha$ denotes evaluating only the linear order in α. This way the doubling of the number of equations that need to be solved is avoided.

The solution at the 0-th order in α was presented in the main text, except for one subtlety. The actual form of Eq. **37** reads

$$i\omega A_L^M - iqv\sqrt{\frac{(2L-1)}{(2L+1)}}C^{L0}_{10,(L-1)0}C^{LM}_{10,(L-1)M}A^M_{L-1} - iqv\sqrt{\frac{(2L+3)}{(2L+1)}}C^{L0}_{10,(L+1)0}C^{LM}_{10,(L+1)M}A^M_{L+1} = -\Gamma_{\text{tot}}A_L^M. \tag{S21}$$

The Clebsch-Gordan coefficients can be expressed explicitly to obtain

$$\sqrt{\frac{(2L-1)}{(2L+1)}}C^{L0}_{10,(L-1)0}C^{LM}_{10,(L-1)M} = \sqrt{\frac{L^2-M^2}{4L^2-1}}, \tag{S22}$$

$$\sqrt{\frac{(2L+3)}{(2L+1)}}C^{L0}_{10,(L+1)0}C^{LM}_{10,(L+1)M} = \sqrt{\frac{(L+1)^2-M^2}{4(L+1)^2-1}}, \tag{S23}$$

so that for $L \gg M$ the coefficients above all tend to a constant value $1/2$. In fact, already when taking $L=2$, $M=1$, the exact values $\sqrt{3/15}\approx 0.45$ and $\sqrt{8/35}\approx 0.48$ are very close to $1/2$, so approximating these coefficients as $1/2$ in Eq. **37** does not produce a noticeable error.

Let us turn our attention to the system of equations at the 1st order in α. The exact forms of the equations at this order are too lengthy to present. Let us however make a few remarks. At linear order in α, the electric field enters the equations for $L=0$, $M=0$ and $L=2$, $M=-2,0,2$. The equation for $L=0$, $M=0$ contains a term proportional to Γ_{inter} in agreement with Eq. S20. The equations for $L\geq 3$ asymptotically tend to the recurrence relation which at this level reads

$$(i\omega+\Gamma_{\text{tot}})\,B_L^M - iqv\frac{1}{2}\left[B^M_{L-1}+B^M_{L+1}\right] + \frac{1}{4}r^{L-2}(1-r^2)\left(A_1^{M-1}-A_1^{M+1}\right) = 0 \tag{S24}$$

and its general solution is

$$B_L^M = B_2^M r^{L-2} + X^M(L-2)r^{L-3}, \tag{S25}$$

where

$$X^M = -\frac{1}{4}(r+r^3)\left(A_1^{M-1}-A_1^{M+1}\right). \tag{S26}$$

These formulas are then plugged into the equations for $L=0,1,2$, and the system of equations is solved. Because the exact solution is too complicated, in Table S1 we present only the formulas obtained in the limiting cases corresponding to the different transport regimes, showing the leading-order terms and neglecting the coefficients that evaluate to zero.

The solutions at the 0-th and 1-st orders are then used to find the solutions at the 2nd order in α. Here the electric field enters equations for $L=1$, $M=\pm1$ and $L=3$, $M=\pm1,\pm3$. Again, the exact forms of these equations are too lengthy to present. The asymptotic form of the equations for $L\geq 4$ can be written as the recurrence relation

$$\begin{aligned} 8r^4\left[(1+r^2)C_L^M - rC^M_{L-1} - rC^M_{L+1}\right] + r^L\left[-13+10r^2-5r^4+2L-2Lr^4\right]\left(X^{M-1}-X^{M+1}\right) \\ + 2r^{L+1}(1-r^4)\left(B_2^{M-1}-B_2^{M+1}\right) + 4r^{L+3}(1+r^2)A_1^M = 0. \end{aligned} \tag{S27}$$

The general solution is

$$C_L^M = C_3^M r^{L-3} + Y^M(L-3)r^{L-4} + Z^M(L-3)^2 r^{L-4}, \tag{S28}$$

where

$$Y^M = \frac{(3-6r^2+5r^4)\left(X^{M-1}-X^{M+1}\right)}{4(1-r^2)} - \frac{r(1-r^4)\left(B_2^{M-1}-B_2^{M+1}\right)+2r^3(1+r^2)A_1^M}{4(1-r^2)} \tag{S29}$$

and

$$Z^M = -\frac{1}{8}(1+r^2)\left(X^{M-1}-X^{M+1}\right). \tag{S30}$$

These formulas are then plugged into the equations for $L=0,1,2,3$, and the system of equations is solved. Because at this level of approximation only the $C_1^{\pm1}$ components contribute to the final expression for the current, in Table S1 we present only the formulas for $C_1^{\pm1}$ in the different transport regimes.

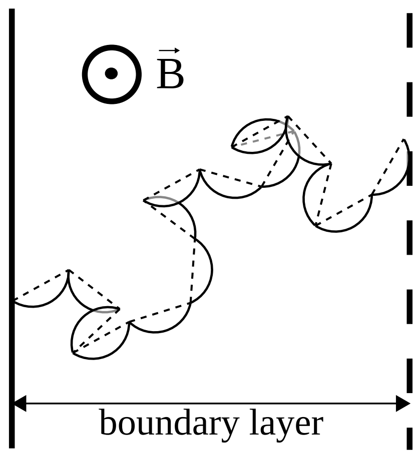

Fig. S1. The effect of the Lorentz force on transport in the hydrodynamic regime. The loss of momentum takes place due to the particle performing a random walk across the boundary layer. The full path traversed by a particle in the presence of magnetic field (solid line) is longer than in its absence (dashed line).

The classical Lorentz force and the magnetization current

In the main text we neglected the the classical Lorentz force, which shows up as the last term on the RHS of Eq. **26** and contributes $ev\left[\hat{\mathbf{p}}\times\mathbf{B}\right]\cdot\partial_{\mathbf{p}}\eta^{(s)}$ to the linearized Boltzmann equation at the classical level. Taking $\mathbf{B}=B\hat{\mathbf{x}}$, this term evaluates to:

$$evB\left(\hat{p}_z\partial_{p_y}-\hat{p}_y\partial_{p_z}\right)\eta^{(s)}. \qquad [S31]$$

This term does not change the conductivity in either the normal regimes or in the AIN regime when $\mathbf{E}$ is parallel to $\mathbf{B}$. The reason is that in all of these regimes, the distribution function at the classical level is proportional to $E\,\hat{p}_x$, and evaluating Eq. S31 gives 0.

The situation is different, however, in the hydrodynamic and anomalous regimes. Let us start with the anomalous regime, where we can calculate the change of conductivity perturbatively in B. The linearized Boltzmann equation at the classical level is (neglecting the label (s)):

$$\left(i\omega+\frac{1}{\Gamma}\right)\eta-iqv\hat{p}_z\eta+evE\hat{p}_x+evB\left(\hat{p}_z\partial_{p_y}-\hat{p}_y\partial_{p_z}\right)\eta=0. \qquad [S32]$$

Here Γ is some relaxation time; it does not matter which one, as all relaxation times, as well as ω, are much smaller than qv. The leading order solution in the anomalous regime is

$$\eta^0=-i\frac{evE\hat{p}_x}{qv\left(\hat{p}_z+i\delta\right)} \qquad [S33]$$

with δ infinitesimal. Solving perturbatively in B at the next two orders produces

$$\eta^1=-\frac{e^2vEB}{q^2\epsilon_{\mathrm{F}}}\frac{\hat{p}_x\hat{p}_y}{\hat{p}_z^3+i\delta}, \qquad [S34]$$

$$\eta^2=i\frac{e^3v^2EB^2}{q^3\epsilon_{\mathrm{F}}^2}\left(\frac{\hat{p}_x}{\hat{p}_z^3+i\delta}+\frac{3\hat{p}_x\hat{p}_y^2}{\hat{p}_z^5+i\delta}\right). \qquad [S35]$$

Calculating the current and multiplying by 2 due to the two nodes (s) and $(-s)$ gives

$$\mathbf{J}=2e\int\frac{d^3p}{(2\pi\hbar)^3}v\hat{\mathbf{p}}\partial_{\epsilon_{\mathrm{M}}}f_0\eta=\left(1+\frac{e^2v^2B^2}{4q^2\epsilon_{\mathrm{F}}^2}\right)\frac{3\pi}{2}\frac{\epsilon\omega_{\mathrm{P}}^2}{qv}\mathbf{E}. \qquad [S36]$$

So, magnetic field causes positive magnetoconductivity. Using the parameters for WP_2 (4, 5) and taking $qv=5\Gamma_{\mathrm{tot}}$, we obtain $e^2v^2/q^2\epsilon_{\mathrm{F}}^2\approx10^{-3}\ \mathrm{T}^{-2}$, while for comparison the magnitude of quantum effects is $\alpha^2/|\mathbf{B}|^2\approx e^2\hbar^2v^4/\epsilon_{\mathrm{F}}^4\approx10^{-8}\ \mathrm{T}^{-2}$.

In the hydrodynamic regime, we estimate the classical magnetoconductivity by resorting to the following picture. The relaxation of momentum happens, when particles traverse the skin layer while performing a random walk with the step size l_{mc}. The presence of the magnetic field curves the path and makes the total path traversed between the two boundaries of the skin layer longer: see Fig. S1. For a particle whose x component of the momentum reads $\epsilon_{\mathrm{F}}\hat{p_x}/v$, the radius of the cyclotron orbit is $r_c=\frac{\epsilon_{\mathrm{F}}}{evB\sqrt{1-\hat{p}_x^2}}$. The length of the arc traversed by a particle between two collisions is roughly equal to $l_{\mathrm{mc}}\left(1+\frac{l_{\mathrm{mc}}^2}{8r_c^2}\right)$. Because conductivity is proportional to the relaxation time, we can hypothesise that the effective Γ_{tot} changes to $K(\hat{\mathbf{p}})\Gamma_{\mathrm{tot}}$ where

$$K(\hat{\mathbf{p}})=\left(1+\frac{e^2v^4B^2(1-\hat{p}_x^2)}{\Gamma_{\mathrm{tot}}^2\epsilon_{\mathrm{F}}^2}\right). \qquad [S37]$$

Then, the total current is of the order

$$\mathbf{J}\approx\int\frac{d^3p}{(2\pi\hbar)^3}\hat{\mathbf{p}}\partial_{\epsilon_{\mathrm{M}}}f_0\frac{e^2v^2E\Gamma_{\mathrm{tot}}\hat{p}_x}{q^2v^2}K(\hat{\mathbf{p}})\approx\left(1+\frac{e^2v^4B^2}{\Gamma_{\mathrm{tot}}^2\epsilon_{\mathrm{F}}^2}\right)\frac{\Gamma_{\mathrm{tot}}}{q^2v^2}\epsilon\omega_{\mathrm{P}}^2\mathbf{E}. \qquad [S38]$$

So in this regime the classical magnetoconductivity is also positive. Using the parameters for WP_2 (4, 5), we obtain $e^2v^4/\Gamma_{\mathrm{tot}}^2\epsilon_{\mathrm{F}}^2\approx10^{-2}\ \mathrm{T}^{-2}$, while for comparison the magnitude of quantum effects is $\alpha^2/|\mathbf{B}|^2\approx e^2\hbar^2v^4/\epsilon_{\mathrm{F}}^4\approx10^{-8}\ \mathrm{T}^{-2}$.

Let us now show that the magnetization current (Eq. **29**) is zero. It can be written as

$$\mathbf{J}_{\mathrm{magn}}=\nabla\times\mathbf{M}, \qquad [S39]$$

where

$$\mathbf{M}=\sum_{s=\pm1}\int\frac{d^3p}{(2\pi)^3}D(\mathbf{p})^{(s)}\frac{se\hbar v}{2|\mathbf{p}|}\hat{\mathbf{p}}f^{(s)}(q,\mathbf{p}) \qquad [S40]$$

is the total magnetization. Because the distribution functions only change along z, we have

$$\mathbf{J}_{\mathrm{magn}}=-iq\left(\hat{\mathbf{y}}M_x-\hat{\mathbf{x}}M_y\right). \qquad [S41]$$

Due to the symmetry of the problem, $M_y = 0$. However, also the total magnetization in the $\hat{x}$ direction is identically null, which can be inferred as follows. The expression S40 can be expanded in α in the same way as the current in Eq. **31**:

$$M_x = -\frac{e\epsilon_{\mathrm{F}}}{v(2\pi\hbar)^3} \times \sum_{s=\pm 1} s \int d(\cos\theta)d\phi \left[1 + s\alpha\hat{p}_x\right] \hat{p}_x \eta^{(s)}(q, \hat{\mathbf{p}}). \qquad \text{[S42]}$$

The sum over s selects contributions on the order α. There are two such contributions. One comes from evaluating the integral of $\alpha\hat{p}_x^2 \left\{\eta^{(s)}\right\}_{\alpha=0}$, where $\left\{\eta^{(s)}\right\}_{\alpha=0}$ is the distribution function at the classical order, and the result of the integration is a combination of A_0^0, A_2^0 and $A_2^{\pm 2}$ (which correspond to modes even in $\hat{p}_x$). However, these coefficients are zero as seen in Eq. **39**. The second contribution comes from evaluating the integral of $\hat{p}_x \left\{\eta^{(s)}\right\}_{\alpha}$, where $\left\{\eta^{(s)}\right\}_{\alpha}$ is the distribution function at the linear order in α, and the result of the integration is a combination of $B_1^{\pm 1}$ (which correspond to modes odd in $\hat{p}_x$). These coefficients, however, are also zero, as seen in Table S1. Therefore, the total M_x summed over the two valleys is zero, and thus $\mathbf{J}_{\mathrm{magn}} = 0$.

References

1. Y Gao, SA Yang, Q Niu, Field induced positional shift of bloch electrons and its dynamical implications. *Phys. Rev. Lett.* **112**, 166601 (2014).
2. Y Gao, SA Yang, Q Niu, Geometrical effects in orbital magnetic susceptibility. *Phys. Rev. B* **91**, 214405 (2015).
3. EV Gorbar, VA Miransky, IA Shovkovy, PO Sukhachov, Second-order chiral kinetic theory: Chiral magnetic and pseudomagnetic waves. *Phys. Rev. B* **95**, 205141 (2017).
4. N Kumar, et al., Extremely high magnetoresistance and conductivity in the type-ii weyl semimetals wp2 and mop2. *Nat. Commun.* **8**, 1642 (2017).
5. J Gooth, et al., Thermal and electrical signatures of a hydrodynamic electron fluid in tungsten diphosphide. *Nat. Commun.* **9**, 4093 (2018).