Supplemental Material

I. DERIVATION OF CAVITY SOLUTION

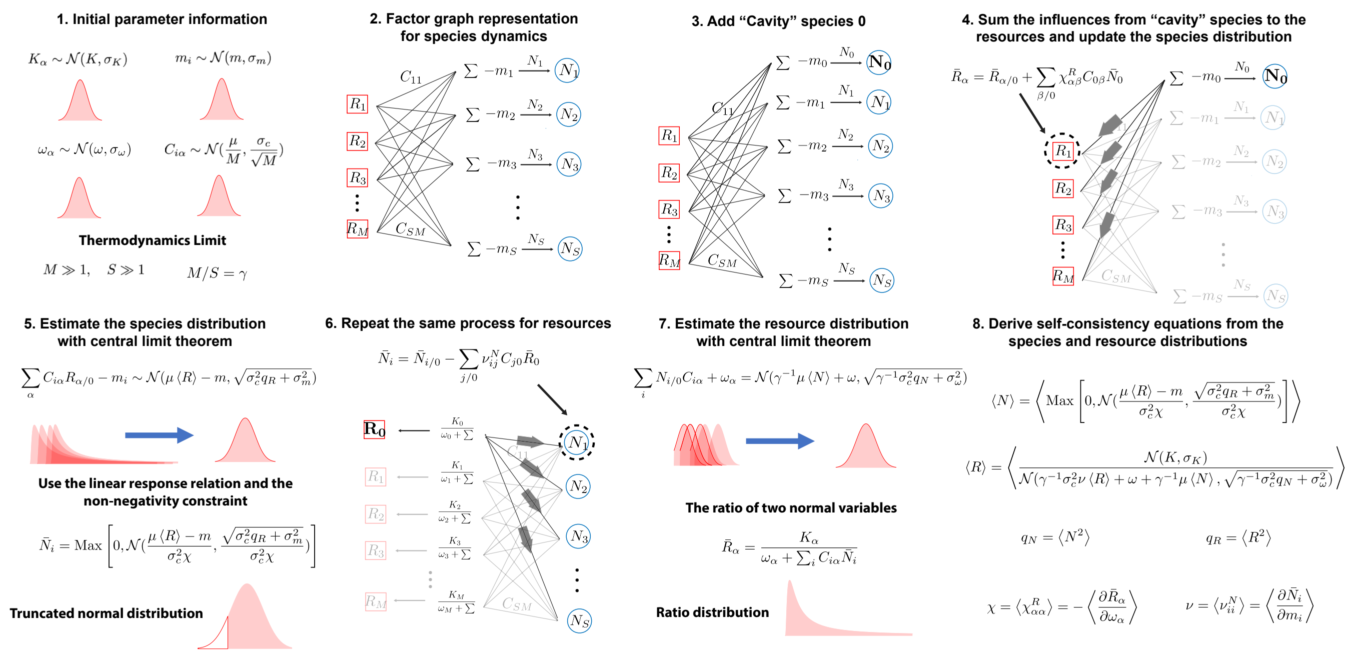

FIG. S1. Schematic outlining steps in cavity solution. **1.** The initial parameter information consists of the probability distributions for the mechanistic parameters: K_α, m_i and $C_{i\alpha}$. We assume they can be described by their first and second moments. **2.** The species dynamics $N_i(\sum_\alpha C_{i\alpha}R_\alpha - m_i)$ in eqs. (1) are expressed as a factor graph. **3.** Add the "Cavity" species 0 as the perturbation. **4.** Sum the resource abundance perturbations from the "Cavity" species 0 at steady state and update the species abundance distribution to reflect the new steady state. **5.** Employing the central limit theorem, the backreaction contribution from the "cavity" species 0 and the non-negativity constraint, the species distribution is expressed as a truncated normal distribution. **6.** Repeat **Step 2-4** for the resources. **7.** The resource distribution is the ratio distribution from the ratio of two normal variables K_α and $\omega_\alpha + \sum_i N_i C_{i\alpha}$. **8.** The self-consistency equations are obtained from the species and resource distributions. Note that $\gamma^{-1}\sigma_c^2\nu\langle R\rangle$ in the dominator of $\langle R\rangle$ is from the correlation between N_i and $C_{i\alpha}$ in $\sum_i N_i C_{i\alpha}$.

A. Model setup

In this section, we derive the cavity solution to the linear resource dynamics (eq. 1) in the main text)

$$\begin{cases} \frac{dN_i}{dt} = N_i\left(\sum_\beta C_{i\beta}R_\beta - m_i\right) \\ \\ \frac{dR_\alpha}{dt} = K_\alpha - \omega_\alpha R_\alpha - \sum_j N_j C_{j\alpha} R_\alpha \end{cases} \tag{1}$$

Note that here we follow closely our derivation in [1, 2]. The main difference is that here we consider linear resource dynamics, which as we will see below, makes the problem much more technically challenging.

Consumer preference $C_{i\alpha}$ are random variables drawn from a Gaussian distribution with mean μ/M and variance σ_c^2/M. They can be deposed into $C_{i\alpha} = \mu/M + \sigma_c d_{i\alpha}$, where the fluctuating part $d_{i\alpha}$ obeys

$$\langle d_{i\alpha}\rangle = 0 \tag{2}$$

$$\langle d_{i\alpha}d_{j\beta}\rangle = \frac{\delta_{ij}\delta_{\alpha\beta}}{M}. \tag{3}$$

We also assume that both the carrying capacity K_α and the minimum maintenance cost m_i are independent Gaussian random variables with mean and covariance given by

$$\langle K_\alpha \rangle = K \tag{4}$$
$$\mathrm{Cov}(K_\alpha, K_\beta) = \delta_{\alpha\beta}\sigma_K^2 \tag{5}$$
$$\langle m_i \rangle = m \tag{6}$$
$$\mathrm{Cov}(m_i, m_j) = \delta_{ij}\sigma_m^2 \tag{7}$$

Let $\langle R \rangle = \frac{1}{M}\sum_\beta R_\beta$ and $\langle N \rangle = \frac{1}{S}\sum_j N_j$ be the average resource and average species abundance, respectively. With all these defined, we can re-write eqs. (1) as

$$\frac{dN_i}{dt} = N_i \left\{ \mu \langle R \rangle - m + \sum_\beta \sigma_c d_{i\beta} R_\beta - \delta m_i \right\} \tag{8}$$
$$\frac{dR_\alpha}{dt} = K + \delta K_\alpha - \left[\omega_\alpha + \gamma^{-1}\mu \langle N \rangle + \sum_j \sigma_c d_{j\alpha} N_j \right] R_\alpha \tag{9}$$

where $\delta K_\alpha = K_\alpha - K, \delta m_i = m_i - m$ and $\gamma = M/S$. As noted in the main text, the basic idea of cavity method is to relate an ecosystem with $M+1$ resources (variables) and $S+1$ species (inequality constraints) to that with M resources and S species. Following eq. (8) and eq. (9), one can write down the ecological model for the $(M+1, S+1)$ system where resource R_0 and species N_0 are introduced to the (M, S) system as:

$$\frac{dN_0}{dt} = N_0 \left\{ \mu \langle R \rangle - m + \sum_\beta \sigma_c d_{0\beta} R_\beta - \delta m_0 \right\} \tag{10}$$
$$\frac{dR_0}{dt} = K + \delta K_0 - \left[\omega_0 + \gamma^{-1}\mu \langle N \rangle + \sum_j \sigma_c d_{j0} N_j \right] R_0 \tag{11}$$

B. Perturbations in cavity solution

Following the same procedure as in [1], we introduce the following susceptibilities:

$$\chi^R_{\alpha\beta} = -\frac{\partial \bar{R}_\alpha}{\partial \omega_\beta} \tag{12}$$
$$\chi^N_{i\alpha} = -\frac{\partial \bar{N}_i}{\partial \omega_\alpha} \tag{13}$$
$$\nu^R_{\alpha i} = \frac{\partial \bar{R}_\alpha}{\partial m_i} \tag{14}$$
$$\nu^N_{ij} = \frac{\partial \bar{N}_i}{\partial m_j} \tag{15}$$

where we denote $\bar{X}$ as the steady-state value of X. Recall that the goal is to derive a set of self-consistency equations that relates the ecological system characterized by $M+1$ resources (variables) and $S+1$ species (constraints) to that with the new species and new resources removed: $(S+1, M+1) \to (S, M)$. To simplify notation, let $\bar{X}_{\backslash 0}$ denote the steady-state value of quantity X in the absence of the new resource and new species. Since the introduction of a new species and resource represents only a small (order $1/M$) perturbation to the original ecological system, we can express the steady-state species and resource abundances in the $(S+1, M+1)$ system with a first-order Taylor expansion around the (S, M) values. We note that the new terms $\sigma_c d_{i0} R_0$ in Eq. eq. (9) and $\sigma_c d_{0\alpha} N_0$ in eq. (8) can be treated as perturbations to m_i, and K_α, respectively, yielding:

$$\bar{N}_i = \bar{N}_{i/0} - \sigma_c \sum_{\beta/0} \chi^N_{i\beta} d_{0\beta} \bar{N}_0 - \sigma_c \sum_{j/0} \nu^N_{ij} d_{j0} \bar{R}_0 \tag{16}$$

$$\bar{R}_\alpha = \bar{R}_{\alpha/0} - \sigma_c \sum_{\beta/0} \chi^R_{\alpha\beta} d_{0\beta} \bar{N}_0 - \sigma_c \sum_{j/0} \nu^R_{\alpha j} d_{j0} \bar{R}_0 \tag{17}$$

Note $\sum_{j/0}$ and $\sum_{\beta/0}$ mean the sum excludes the new species 0 and the new resource 0. The next step is to plug eq. (16) and eq. (17) into eq. (10) and eq. (11) and solve for the steady-state value of N_0 and R_0.

C. Self-consistency equations for species

For the new cavity species, the steady equation takes the form

$$0 = \bar{N}_0 \left[\mu \langle R \rangle - m - \sigma_c^2 \bar{N}_0 \sum_{\alpha/0,\beta/0} \chi^R_{\alpha\beta} d_{0\alpha} d_{0\beta} - \sigma_c^2 \bar{R}_0 \sum_{\beta/0,j/0} \nu^R_{\beta j} d_{0\beta} d_{0j} + \sum_{\beta/0} \sigma_c d_{0\beta} \bar{R}_{\beta/0} + \sigma_c d_{00} \bar{R}_0 - \delta m_0 \right] \tag{18}$$

Notice that each of the sums in this equation is the sum over a large number of weak correlated random variables, and can therefore be well approximated by Gaussian random variables for large enough M and S. We can calculate the sum of the random variables:

$$\sum_{\beta/0,j/0} \nu^R_{\beta j} d_{0\beta} d_{0j} = \frac{1}{M} \sum_{\beta/0,j/0} \nu^R_{\beta j} \delta_{j0} \delta_{\beta 0} = 0 \tag{19}$$

$$\sum_{\alpha/0,\beta/0} \chi^R_{\alpha\beta} d_{0\alpha} d_{0\beta} = \frac{1}{M} \sum_{\alpha/0,\beta/0} \chi^R_{\alpha\beta} \delta_{\alpha\beta} = \frac{1}{M} \sum_{\alpha} \chi^R_{\alpha\alpha} = \frac{1}{M} \mathrm{Tr}(\chi^R_{\alpha\beta}) = \chi \tag{20}$$

where χ is the average susceptibility. Using these observations about above sums, we obtain

$$0 = \bar{N}_0 \left[\mu \langle R \rangle - m - \sigma_c^2 \chi \bar{N}_0 + \sum_{\beta/0} \sigma_c d_{0\beta} \bar{R}_{\beta/0} + \sigma_c d_{00} \bar{R}_0 - \delta m_0 \right] + \mathcal{O}(M^{-1/2}), \tag{21}$$

Employing the Central Limit Theorem, we introduce an auxiliary Gaussian variable z_N with zero mean and unit variance and rewrite this as

$$\sum_{\beta/0} \sigma_c d_{0\beta} \bar{R}_{\beta/0} + \sigma_c d_{0\beta} \bar{R}_0 - \delta m_0 = z_N \sqrt{\sigma_c^2 q_R + \sigma_m^2}, \tag{22}$$

where q_R is the second moment of the resource distribution,

$$q_R = \frac{1}{M} \sum_{\beta} R_\beta^2.$$

We can solve eq. (21) in terms of the quantities just defined:

$$\mu \langle R \rangle - m - \sigma_c^2 \chi \bar{N}_0 + \sqrt{\sigma_c^2 q_R + \sigma_m^2} z_N \leq 0 \tag{23}$$

Inverting this equation one gets the steady state of species

$$\bar{N}_0 = \max \left[0, \quad \frac{\mu \langle R \rangle - m + \sqrt{\sigma_c^2 q_R + \sigma_m^2} z_N}{\sigma_c^2 \chi} \right] \tag{24}$$

which is a truncated Gaussian.

Let $y = \max\left(0, \frac{a}{b} + \frac{c}{b} z\right)$, with z being a Gaussian random variable with zero mean and unit variance. Then its j-th moment is given by

$$\langle y^j \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\frac{a}{c}}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{c}{b} x + \frac{a}{b} \right)^j dx \tag{25}$$

$$= \left(\frac{c}{b} \right)^j \frac{1}{\sqrt{2\pi}} \int_{-\frac{a}{c}}^{\infty} e^{-\frac{x^2}{2}} \left(x + \frac{a}{c} \right)^j dx \tag{26}$$

$$= \left(\frac{c}{b} \right)^j w_j\left(\frac{a}{c}\right) \tag{27}$$

here we define $w_j(\frac{a}{c}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{a}{c}}^{\infty} e^{-\frac{x^2}{2}} \left(x + \frac{a}{c}\right)^j dx$

With this we can easily write down the self-consistency equations for the fraction of non-zero species and resources as well as the moments of their abundances at the steady state:

$$\phi_N = \frac{S^*}{S} = w_0 \left(\frac{\mu \langle R \rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}} \right) \tag{28}$$

$$\langle N \rangle = \frac{1}{S} \sum_j N_j = \left(\frac{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}{\sigma_c^2 \chi} \right) w_1 \left(\frac{\mu \langle R \rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}} \right) \tag{29}$$

$$q_N = \frac{1}{S} \sum_j N_j^2 = \left(\frac{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}{\sigma_c^2 \chi} \right)^2 w_2 \left(\frac{\mu \langle R \rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}} \right) \tag{30}$$

Note that S^* is the number of surviving species at the steady state.

D. Self-consistency equations for resources

We now derive the equations for the steady-state of the resource dynamics. Inserting eq. (17) into eq. (11) gives:

$$0 = K + \delta K_0 - \bar{R}_0 \left[\omega + \gamma^{-1} \mu \langle N \rangle - \sigma_c^2 \bar{N}_0 \sum_{\beta/0, j/0} \chi_{j\beta}^N d_{j0} d_{0\beta} - \sigma_c^2 \bar{R}_0 \sum_{i/0, j/0} \nu_{ij}^N d_{0i} d_{0j} + \sum_{j/0} \sigma_c d_{j0} \bar{N}_{j/0} + \sigma_c d_{00} \bar{N}_0 + \delta\omega_0 \right] \tag{31}$$

We can simplify the sums by averaging over the random variables:

$$\sum_{\beta/0, j/0} \chi_{j\beta}^N d_{j0} d_{0\beta} = \frac{1}{M} \sum_{\beta/0, j/0} \chi_{j\beta}^N \delta_{j0} \delta_{\beta 0} = 0 \tag{32}$$

$$\sum_{i/0, j/0} \nu_{ij}^N d_{0i} d_{0j} = \frac{1}{M} \sum_{i/0, j/0} \nu_{ij}^N \delta_{ij} = \frac{1}{M} \sum_i \nu_{ii}^N = \frac{1}{M} \mathrm{Tr}(\nu_{ij}^N) = \gamma^{-1} \nu \tag{33}$$

where ν is the average susceptibility. Finally, note that we can write

$$\delta\omega_0 + \sum_j \sigma_c d_{j0} N_j = z_R \sqrt{\gamma^{-1} \sigma_c^2 q_N + \sigma_\omega^2}, \tag{34}$$

where we have introduced another auxiliary Gaussian variable z_R with zero mean and unit variance and q_N is the second moment of the resource distribution defined in eq. (56), Using these observations, we obtain a quadratic expression for the resource.

$$K + \delta K_0 - (\omega_0 + \gamma^{-1} \mu \langle N \rangle + \sqrt{\gamma^{-1} \sigma_c^2 q_N + \sigma_\omega^2} z_R) \bar{R}_0 + \gamma^{-1} \sigma_c^2 \nu \bar{R}_0^2 = 0 \tag{35}$$

1. *Cavity solution: without backreaction*

As discussed in the main text, we cannot solve the full resource equations exactly. For this reason, we perform an expansion, as a start, we calculate this equation by setting $\nu = 0$ in the resource equation. This is equivalent in the TAP language of ignoring the backreaction term.

Under this assumption, the quadratic equation for the resource, simply becomes a linear equation that can be re-arranged to give

$$\bar{R}_\alpha = \frac{K + \delta K_\alpha}{\omega + \gamma^{-1} \mu \langle N \rangle + z_R \sqrt{\gamma^{-1} \sigma_c^2 q_N + \sigma_\omega^2}} \tag{36}$$

Assuming the fluctuations in the denominator is small, *i.e.* $\sqrt{\gamma^{-1} \sigma_c^2 q_N + \sigma_\omega^2} \ll \omega + \gamma^{-1} \mu \langle N \rangle$, we can do a first-order Taylor expansion around the mean value and also ignore the coupling term between δK_α and z_R:

$$\bar{R}_\alpha = \frac{K + \delta K_\alpha}{\omega + \gamma^{-1} \mu \langle N \rangle} - \frac{K \sqrt{\gamma^{-1} \sigma_c^2 q_N + \sigma_\omega^2}}{(\omega + \gamma^{-1} \mu \langle N \rangle)^2} z_R \tag{37}$$

With all these approximations, we get the first two moments of the steady-state resource abundance distribution:

$$\langle R\rangle = \frac{K}{\omega+\gamma^{-1}\mu\langle N\rangle} \tag{38}$$

$$q_R = \langle R\rangle^2 + \frac{\sigma_K^2}{(\omega+\gamma^{-1}\mu\langle N\rangle)^2} + \frac{K^2(\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)}{(\omega+\gamma^{-1}\mu\langle N\rangle)^4} \tag{39}$$

The susceptibility is given by:

$$\chi = -\left\langle\frac{\partial\bar{R}_\alpha}{\partial w_\alpha}\right\rangle = \left\langle\frac{K_\alpha}{(\omega_\alpha+\sum_j c_{j\alpha}\bar{N}_j)^2} + \frac{2K\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}}{(\omega+\gamma^{-1}\mu\langle N\rangle)^3}z_R\right\rangle = \frac{K}{(\omega+\gamma^{-1}\mu\langle N\rangle)^2} \tag{40}$$

Combined with self-consistency equations for species, we get the full set of :

$$\phi_N = w_0\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R+\sigma_m^2}}\right), \quad \chi = \frac{K}{(\omega+\gamma^{-1}\mu\langle N\rangle)^2} \tag{41}$$

$$\langle N\rangle = \left(\frac{\sqrt{\sigma_c^2 q_R+\sigma_m^2}}{\sigma_c^2\chi}\right) w_1\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R+\sigma_m^2}}\right), \quad \langle R\rangle = \frac{K}{\omega+\gamma^{-1}\mu\langle N\rangle} \tag{42}$$

$$q_N = \left(\frac{\sqrt{\sigma_c^2 q_R+\sigma_m^2}}{\sigma_c^2\chi}\right)^2 w_2\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R+\sigma_m^2}}\right), \quad q_R = \langle R\rangle^2 + \frac{\sigma_K^2}{(\omega+\gamma^{-1}\mu\langle N\rangle)^2} + \frac{K^2(\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)}{(\omega+\gamma^{-1}\mu\langle N\rangle)^4} \tag{43}$$

2. *Cavity solution: with backreaction correction*

We start again with the full resource equation:

$$K+\delta K_0 - (\omega_0+\gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R)\bar{R}_0 + \gamma^{-1}\sigma_c^2\nu\bar{R}_0^2 = 0 \tag{44}$$

Since $R_0>0$ and $\nu<0$, the solution of eq. (44) gives:

$$R_0 = \frac{\omega+\gamma^{-1}\mu\langle N\rangle+\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R}{2\gamma^{-1}\sigma_c^2\nu} - \frac{\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle+\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K+\delta K_0)}}{2\gamma^{-1}\sigma_c^2\nu} \tag{45}$$

For the 1st order expansion, we assume $4\gamma^{-1}\nu\sigma_c^2\delta K_0 + 2\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R + (\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)z_R^2 \ll (\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K$ and do a 1st order expansion around the mean of the form:

$$\begin{aligned}&\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle+\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K+\delta K_0)}\\ =&\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K} + \frac{(\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)z_R^2 + 2(\omega+\gamma^{-1}\mu\langle N\rangle)\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R - 4\gamma^{-1}\nu\sigma_c^2\delta K_0}{2\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}}\end{aligned} \tag{46}$$

Using these expressions, the moments of their abundances at steady state can be calculated yielding:

$$\langle R\rangle = \frac{\omega+\gamma^{-1}\mu\langle N\rangle}{2\gamma^{-1}\sigma_c^2\nu} - \frac{\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}}{2\gamma^{-1}\sigma_c^2\nu} - \frac{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}{4\gamma^{-1}\sigma_c^2\nu\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}} \tag{47}$$

$$\begin{aligned}q_R = \langle R\rangle^2 &+ \frac{(\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)^2 + 8(\gamma^{-1}\nu\sigma_c^2\sigma_K)^2}{2(2\gamma^{-1}\sigma_c^2\nu)^2[(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K]}\\ &+ \frac{(\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)[\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K} - (\omega+\gamma^{-1}\mu\langle N\rangle)]^2}{(2\gamma^{-1}\sigma_c^2\nu)^2[(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K]}\end{aligned} \tag{48}$$

From eq. (45),

$$\frac{\partial R_0}{\partial \omega} = \frac{1}{2\gamma^{-1}\sigma_c^2\nu}\left\{1 - \frac{\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}z_R}{\sqrt{(\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K+\delta K_0)}}\right\} \tag{49}$$

The term inside the bracket can be expanded as:

$$\begin{aligned}&\frac{\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}z_R}{\sqrt{(\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K+\delta K_0)}} \\ &\approx \frac{\omega+\gamma^{-1}\mu\langle N\rangle+\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R}{\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2-4\gamma^{-1}\nu\sigma_c^2 K}}\left[1 - \frac{(\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2)z_R^2+2(\omega+\gamma^{-1}\mu\langle N\rangle)\sqrt{\gamma^{-1}\sigma_c^2 q_N+\sigma_\omega^2}z_R-4\gamma^{-1}\nu\sigma_c^2\delta K_0}{2(\omega+\gamma^{-1}\mu\langle N\rangle)^2-4\gamma^{-1}\nu\sigma_c^2 K}\right]\end{aligned} \tag{50}$$

The susceptibilities are given by averaging eq. (49)

$$\chi = -\left\langle\frac{\partial R}{\partial \omega}\right\rangle \tag{51}$$

$$= -\frac{1}{2\gamma^{-1}\nu\sigma_c^2}\left\{1 - \frac{\omega+\gamma^{-1}\mu\langle N\rangle}{\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}} + \frac{3(\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2)(\omega+\gamma^{-1}\mu\langle N\rangle)}{2[(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K]^{3/2}}\right\} \tag{52}$$

$$\nu = \left\langle\frac{\partial N}{\partial m}\right\rangle = -\frac{\phi_N}{\sigma_c^2\chi} \tag{53}$$

Combined with self-consistency equations for species, get the full set of 1^{st} order self-consistency equations:

$$\phi_N = w_0\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}\right) \tag{54}$$

$$\langle N\rangle = \left(\frac{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}{\sigma_c^2\chi}\right) w_1\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}\right) \tag{55}$$

$$q_N = \left(\frac{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}{\sigma_c^2\chi}\right)^2 w_2\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}\right) \tag{56}$$

$$\langle R\rangle = \frac{\omega+\gamma^{-1}\mu\langle N\rangle}{2\gamma^{-1}\sigma_c^2\nu} - \frac{\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}}{2\gamma^{-1}\sigma_c^2\nu} - \frac{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}{4\gamma^{-1}\sigma_c^2\nu\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}} \tag{57}$$

$$\begin{aligned}q_R = \langle R\rangle^2 &+ \frac{(\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2)^2 + 8(\gamma^{-1}\nu\sigma_c^2\sigma_K)^2}{2(2\gamma^{-1}\sigma_c^2\nu)^2[(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K]} \\ &+ \frac{(\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2)[\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K} - (\omega+\gamma^{-1}\mu\langle N\rangle)]^2}{(2\gamma^{-1}\sigma_c^2\nu)^2[(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K]}\end{aligned} \tag{58}$$

$$\chi = -\frac{1}{2\gamma^{-1}\nu\sigma_c^2}\left\{1 - \frac{\omega+\gamma^{-1}\mu\langle N\rangle}{\sqrt{(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K}} + \frac{3(\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2)(\omega+\gamma^{-1}\mu\langle N\rangle)}{2[(\omega+\gamma^{-1}\mu\langle N\rangle)^2 - 4\gamma^{-1}\nu\sigma_c^2 K]^{3/2}}\right\} \tag{59}$$

$$\nu = -\frac{\phi_N}{\sigma_c^2\chi} \tag{60}$$

II. COMPARISON BETWEEN WITH AND WITHOUT BACKREACTION

We can reduce the cavity solution with backreaction to the simpler one when σ_c is large. In fact all the complexity of cavity solution with backreaction comes from the expression for eq. (45):

$$R_0 = \frac{\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}z_R}{2\gamma^{-1}\sigma_c^2\nu} - \frac{\sqrt{(\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2}z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K+\delta K_0)}}{2\gamma^{-1}\sigma_c^2\nu} \tag{61}$$

However, if we assume $(\omega + \gamma^{-1}\mu \langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2 z_R})^2 \gg -4\gamma^{-1}\nu\sigma_c^2(K + \delta K_0)$, we can expand the second term following $\sqrt{1-x} \approx 1 - \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3)$.

$$R_0 = \frac{K + \delta K_0}{\omega + \gamma^{-1}\mu \langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2 z_R}} + \frac{\gamma^{-1}\sigma_c^2 \nu (K + \delta K_0)^2}{(\omega + \gamma^{-1}\mu \langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2 z_R})^3} \tag{62}$$

The first term of above equation is the cavity solution without backreaction.

A. Comparing the cavity solutions to numerical simulations

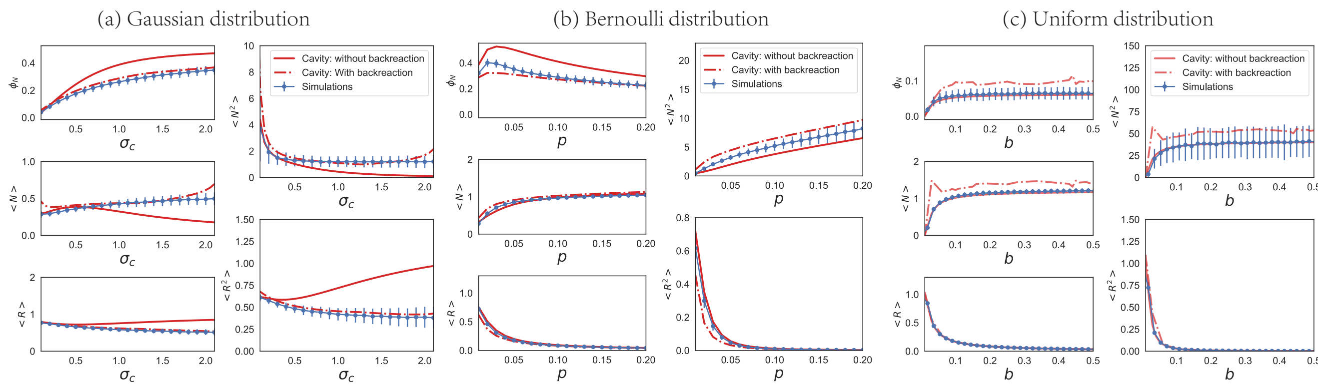

FIG. S2. Comparison of numerics and cavity solutions with and without the backreaction term as a function of σ_c. $\phi_N = \frac{S^*}{S}$ is the fraction of surviving species. $\langle N\rangle, \langle N^2\rangle, \langle R\rangle$ and $\langle R^2\rangle$ are the first and second moments of the species and resources distribution respectively. The simulations details can be found at the SM: III. **C** is sampled either from a Gaussian, Bernoulli, or uniform distribution as indicated.

We show a comparison between theoretical and numerical results for different choices of how to sample the consumption matrix in Fig. 2 in the main text and Fig. S2. These figures show that the cavity solution with backreaction performs better for the Gaussian and Bernoulli cases. However, in the uniform case, the cavity solution without backreaction matches with numerical simulations perfectly, while the cavity solution with backreaction performs worse than without backreaction. In the section II, we have shown the cavity solution with backreaction can be reduced to the cavity solution without backreaction and hence should be a more robust solution. So why does it perform badly in the uniform case? The reason is that in the uniform case $\mu = Mb/2 \gg 1$ when the system size M is large, leading to $|\chi| \sim \frac{1}{(\omega+\gamma^{-1}\mu\langle N\rangle)^2} \ll 1$. From eqs. (55, 56), we see that both $\langle N\rangle$ and $\langle N^2\rangle$ depends on $\frac{1}{\chi} \gg 1$ and the numerical solver becomes unstable.

III. SIMULATION DETAILS

A. Parameters

All simulations are done with the CVXPY package[3] in PYTHON 3. All codes are available on GitHub at https://github.com/Emergent-Behaviors-in-Biology/species-packing-bound.

- Fig. 2: the consumer matrix **C** is sampled from the Gaussian distribution $\mathcal{N}(\frac{\mu}{M}, \frac{\sigma_c}{\sqrt{M}})$. $S = 100$, $M = 100$, $\mu = 1$, $K = 1$, $\sigma_K = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\omega = 1$, $\sigma_\omega = 0$ and each data point is averaged from 1000 independent realizations. We only provide the cavity solution with backreaction here.
- Fig. 3, Fig. S3, Fig. S5, Fig. S6: the consumer matrix **C** is sampled from the Gaussian distribution $\mathcal{N}(\frac{\mu}{M}, \frac{\sigma_c}{\sqrt{M}})$. $S = 500$, $M = 100$, $\mu = 1$, $\sigma_\kappa = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\omega = 1$, $\sigma_\omega = 0$ for externally supplied resource dynamics and $S = 500$, $M = 100$, $\mu = 1$, $\sigma_\kappa = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\tau = 1$, $\sigma_\tau = 0$ for the self-renewing one. Each data point is averaged from 1000 independent realizations. For Fig. S3 , $K = 10$. For Fig. S5, $\sigma_c = 5$, K and κ are fixed at 4; For Fig. S6, $\sigma_c = 5$, $\kappa = 4$, S/M has a range from 1 to 100, and each data point is averaged from 100 independent realizations.

- Fig. 4: the consumer matrix $\mathbf{C}$ is sampled from the Bernoulli distribution $Bernoulli(p)$ and p are fixed to 0.1, 0.2 and 0.1. m_i follows metabolic tradeoffs Eq. (70) with $\sigma_\epsilon = 0$, $\tilde{m} = 1$. We also set $S = 500$, $M = 100$, $K = 10$, $\sigma_K = 0.1$. Each data point is averaged from 100 independent realizations.
- Fig. S2(a): the simulation is the same as Fig. 2. We show both the cavity solutions with and without reaction here.
- Fig. S2(b): the consumer matrix $\mathbf{C}$ is sampled from the Bernoulli distribution $Bernoulli(p)$. $S = 100$, $M = 100$, $K = 1$, $\sigma_K = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\omega = 1$, $\sigma_\omega = 0$ and each data point is averaged from 1000 independent realizations. The cavity solution is obtained by approximating the Bernoulli distribution to the corresponding Gaussian distribution *i.e.* $\mu = pM$, $\sigma_c = \sqrt{Mp(1-p)}$
- Fig. S2(c): the consumer matrix $\mathbf{C}$ is sampled from the uniform distribution $\mathcal{U}(0, b)$. $S = 100$, $M = 100$, $K = 1$, $\sigma_K = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\omega = 1$, $\sigma_\omega = 0$ and each data point is averaged from 1000 independent realizations. The cavity solution is obtained by approximating the uniform distribution to the corresponding Gaussian distribution, *i.e.* $\mu = bM/2$, $\sigma_c = b\sqrt{M/12}$.
- Fig. S4(a): the consumer matrix $\mathbf{C}$ is sampled from the Bernoulli distribution $Bernoulli(p)$. $S = 500$, $M = 100$, $K = 1$, $\sigma_K = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\omega = 1$, $\sigma_\omega = 0$ and each data point is averaged from 1000 independent realizations. The cavity solution is obtained by approximating the Bernoulli distribution to the corresponding Gaussian distribution *i.e.* $\mu = pM$, $\sigma_c = \sqrt{Mp(1-p)}$
- Fig. S4(b): the consumer matrix $\mathbf{C}$ is sampled from the uniform distribution $\mathcal{U}(0, b)$. $S = 500$, $M = 100$, $K = 1$, $\sigma_K = 0.1$, $m = 1$, $\sigma_m = 0.1$, $\omega = 1$, $\sigma_\omega = 0$ and each data point is averaged from 1000 independent realizations. The cavity solution is obtained by approximating the uniform distribution to the corresponding Gaussian distribution, *i.e.* $\mu = bM/2$, $\sigma_c = b\sqrt{M/12}$.

B. Distinction between extinct and surviving species

In the main text, we show that the value of species packing $\frac{S^*}{M}$ for the externally supplied resources must be smaller than 0.5. However, in numerical simulations, even for the extinct species the abundance is never exactly equal 0 due to numerical errors. As a result, we must choose a threshold to distinguish extinct and surviving species in order to calculate S^*. Since we are using the equivalence with convex optimization to solve the generalized consumer-resource models[2, 4], we can easily choose a reasonable threshold (e.g. 10^{-2} in Fig. S3) since the extinct and surviving species are well separated in two peaks (see Fig. S3).

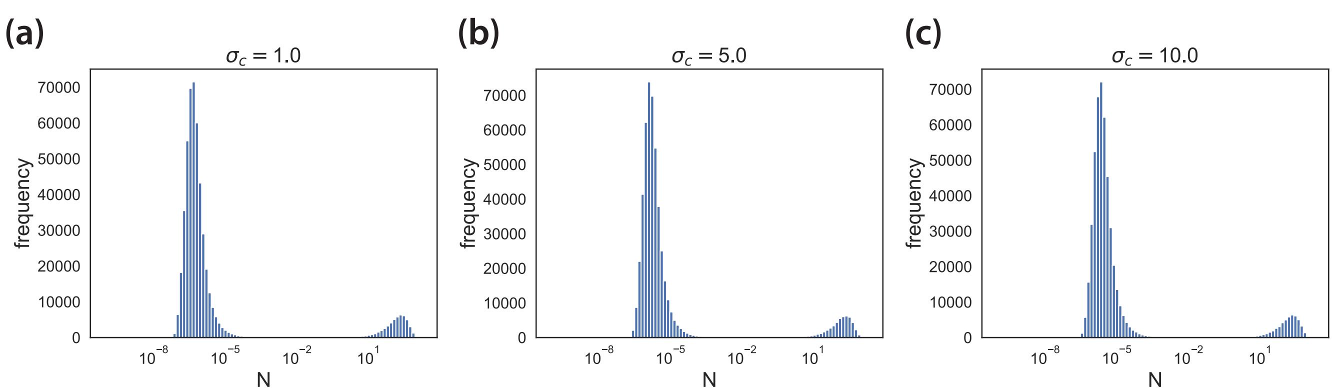

FIG. S3. Species abundance N in equilibrium at different σ_c for externally supplied resource dynamics at $K = 10$. The simulations parameters can be found at the SM: III.

IV. AN UPPER BOUND FOR SPECIES PACKING

By analyzing the susceptibilities in the full Cavity solutions, an upper bound for species packing can be derived for both resource dynamics in GCRMs. The derivations can also be extended to the case where metabolic tradeoffs impose hard or soft constraints on the parameter values.

A. Externally supplied resource dynamics

The response functions χ and ν can be written as:

$$\chi = -\frac{1}{2\gamma^{-1}\sigma_c^2\nu}\left\{1 - \left\langle \frac{\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2} z_R}{\sqrt{(\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2} z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K + \delta K_0)}} \right\rangle\right\} \tag{63}$$

$$\nu = -\frac{\phi_N}{\sigma_c^2\chi} \tag{64}$$

Substituting eq. (64) into eq. (63) and rearranging yields

$$\gamma^{-1}\phi_N = \frac{1}{2}\left\{1 - \left\langle \frac{\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2} z_R}{\sqrt{(\omega + \gamma^{-1}\mu\langle N\rangle + \sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_\omega^2} z_R)^2 - 4\gamma^{-1}\nu\sigma_c^2(K + \delta K_0)}} \right\rangle\right\}. \tag{65}$$

The numerator of the term in angle brackets is the total depletion rate for a given resource when it is first added to the system. Depletion rates are always positive in this model, so the right-hand side is always less than $1/2$. Noticing $\gamma = \frac{M}{S}$, $\phi_N = S^*/S$, $\chi > 0$, we immediately obtain an upper bound on $\frac{S^*}{M}$:

$$\frac{1}{2} > \frac{S^*}{M}. \tag{66}$$

B. Self-renewing(MacArthur's) resource dynamics

Using the analytical expressions χ, ν and self-consistent equations in ref. [5], we can derive the following expressions:

$$\langle N\rangle = \left(\frac{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}{\sigma_c^2(\phi_R - \gamma^{-1}\phi_N)}\right) w_1\left(\frac{\mu\langle R\rangle - m}{\sqrt{\sigma_c^2 q_R + \sigma_m^2}}\right), \quad \langle R\rangle = \left(\frac{\sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_K^2}}{\phi_R(\phi_R - \gamma^{-1}\phi_N)^{-1}}\right) w_1\left(\frac{\kappa - \gamma^{-1}\mu\langle N\rangle}{\sqrt{\gamma^{-1}\sigma_c^2 q_N + \sigma_K^2}}\right) \tag{67}$$

To derive bounds, we consider various limits of these expressions. First, consider the case were we put many species $S \to \infty$ into the ecosystem with fixed number of resources M, (i.e $\gamma = \frac{M}{S} \to 0$). In order to keep $\langle N\rangle$ positive, we must have $\phi_R - \gamma^{-1}\phi_N > 0$, giving an upper bound:

$$1 \geq \frac{M^*}{M} > \frac{S^*}{M} \tag{68}$$

C. Externally supplied resources with metabolic tradeoffs

Here we consider two kinds of constraints on the parameters, encoding metabolic tradeoffs. In the first, the maintenance cost $m_i = m$ is the same for all species, and the sum of the consumption preferences is constrained to equal some fixed "enzyme budget" E that is nearly the same for all species:

$$\sum_\alpha C_{i\alpha} = E + \delta E_i \tag{69}$$

where δE_i is a small random variable with mean zero and variance σ_E^2. A hard constraint can be generated by taking $\sigma_E = 0$.

The second kind of constraint does not make any assumptions about $C_{i\alpha}$, but assigns a cost $\tilde{m}$ to every unit of consumption capacity, so that

$$m_i = (1 + \epsilon_i)\tilde{m}\sum_\alpha C_{i\alpha} + \delta m_i \tag{70}$$

where ϵ_i and δm_i are small random variables with mean zero and variances σ_ϵ^2 and σ_m^2, respectively. A hard constraint can be generated by taking $\sigma_\epsilon = \sigma_m = 0$.

In the simplest way of setting up the first constraint, the equilibrium equations actually reduce to the same form as the second. Specifically, one usually generates a consumer preference matrix satisfying the constraint by first generating an i.i.d. matrix $\tilde{C}_{i\alpha}$, and then setting $C_{i\alpha} = (E + \delta E_i)\tilde{C}_{i\alpha}/\sum_\beta \tilde{C}_{i\beta}$. The resulting dynamics can be written as:

$$\frac{dN_i}{dt} = N_i \left[\sum_\alpha (E + \delta E_i) \frac{\tilde{C}_{i\alpha}}{\sum_\beta \tilde{C}_{i\beta}} R_\alpha - m_i \right] \tag{71}$$

$$= \frac{N_i(E + \delta E_i)}{\sum_\beta \tilde{C}_{i\beta}} \left[\sum_\alpha \tilde{C}_{i\alpha} R_\alpha - m \frac{\sum_\beta \tilde{C}_{i\beta}}{E + \delta E_i} \right]. \tag{72}$$

Dropping the tilde's, we can write the equilibrium condition in the same form that results from the second kind of constraint:

$$0 = N_i \{ \sum_\alpha C_{i\alpha} [R_\alpha - (1 + \epsilon_i)\tilde{m}] - \delta m_i \} \tag{73}$$

with

$$\tilde{m} = \frac{m}{E} \tag{74}$$

$$\epsilon_i = -\frac{\delta E_i}{E} \tag{75}$$

$$\delta m_i = 0. \tag{76}$$

Inspection of Equation 73 immediately reveals an important novelty: now when we add a new resource as part of the cavity protocol, the perturbation to the growth rate can either be positive or negative, depending on the sign of $[R_\alpha - (1 + \epsilon_i)\tilde{m}]$. This turns out to be the crucial factor that prevents the proof of the $S^*/M < 1/2$ bound from going through, regardless of the size of σ_ϵ or σ_m.

Following the same steps as above, we arrive at the following set of equilibrium conditions for the new species N_0 and resource R_0:

$$0 = \bar{N}_0 \left[\mu \langle R \rangle - \mu \tilde{m} + \sigma_N z_N - \sigma_c^2 \chi \bar{N}_0 \right] \tag{77}$$

$$0 = K + \delta K_0 - (\omega + \gamma^{-1}\mu\langle N \rangle + \sigma_R z_R + \gamma^{-1}\sigma_c^2 \nu \tilde{m})\bar{R}_0 + \gamma^{-1}\sigma_c^2 \nu \bar{R}_0^2 \tag{78}$$

where

$$\sigma_N^2 = \sigma_m^2 + \sigma_c^2 [q_R - 2\tilde{m}\langle R \rangle + \tilde{m}^2(1 + \sigma_\epsilon^2)] \tag{79}$$

$$\sigma_R^2 = \sigma_\omega^2 + \gamma^{-1}\sigma_c^2 q_N + \gamma^{-2}\sigma_c^4 \nu^2 \tilde{m}^2 \sigma_\epsilon^2. \tag{80}$$

These are nearly identical to the equations we had before. The two key changes are the presence of a term with a negative sign inside the coefficient σ_N of the random variable z_N, and the $\gamma^{-1}\sigma_c^2\nu\tilde{m}$ term inside the parentheses in the equation for the resources.

We can now proceed in the same way as before, solving for $\bar{N}_0$ and $\bar{R}_0$ and taking derivatives to compute the susceptibilities. We find:

$$\chi = -\frac{1}{2\gamma^{-1}\sigma_c^2\nu} \left\{ 1 - \left\langle \frac{\omega + \gamma^{-1}\mu\langle N \rangle + \sigma_R z_R + \gamma^{-1}\sigma_c^2\nu\tilde{m}}{\sqrt{(\omega + \gamma^{-1}\mu\langle N \rangle + \sigma_R z_R + \gamma^{-1}\sigma_c^2\nu\tilde{m})^2 - 4\gamma^{-1}\sigma_c^2\nu(K + \delta K_0)}} \right\rangle \right\} \tag{81}$$

$$\nu = -\frac{\phi_N}{\sigma_c^2 \chi} \tag{82}$$

This is almost the same as the expression in Equation (63) obtained in the absence of constraints, except for the extra term $\gamma^{-1}\sigma_c^2\nu\tilde{m}$ in the numerator and denominator. This term is significant because ν is a negative number, and if its absolute value is large enough, it can make the whole term in angle brackets negative. Inserting the second equation into the first, we obtain a formula for S^*/M:

$$\frac{S^*}{M} = \gamma^{-1}\phi_N = \tfrac{1}{2} \left\{ 1 - \left\langle \frac{\omega + \gamma^{-1}\mu\langle N \rangle + \sigma_R z_R + \gamma^{-1}\sigma_c^2\nu\tilde{m}}{\sqrt{(\omega + \gamma^{-1}\mu\langle N \rangle + \sigma_R z_R + \gamma^{-1}\sigma_c^2\nu\tilde{m})^2 - 4\gamma^{-1}\sigma_c^2\nu(K + \delta K_0)}} \right\rangle \right\} \tag{83}$$

The term in brackets can now be negative, but is always greater than -1. We thus obtain the bound:

$$\frac{S^*}{M} < 1. \tag{84}$$

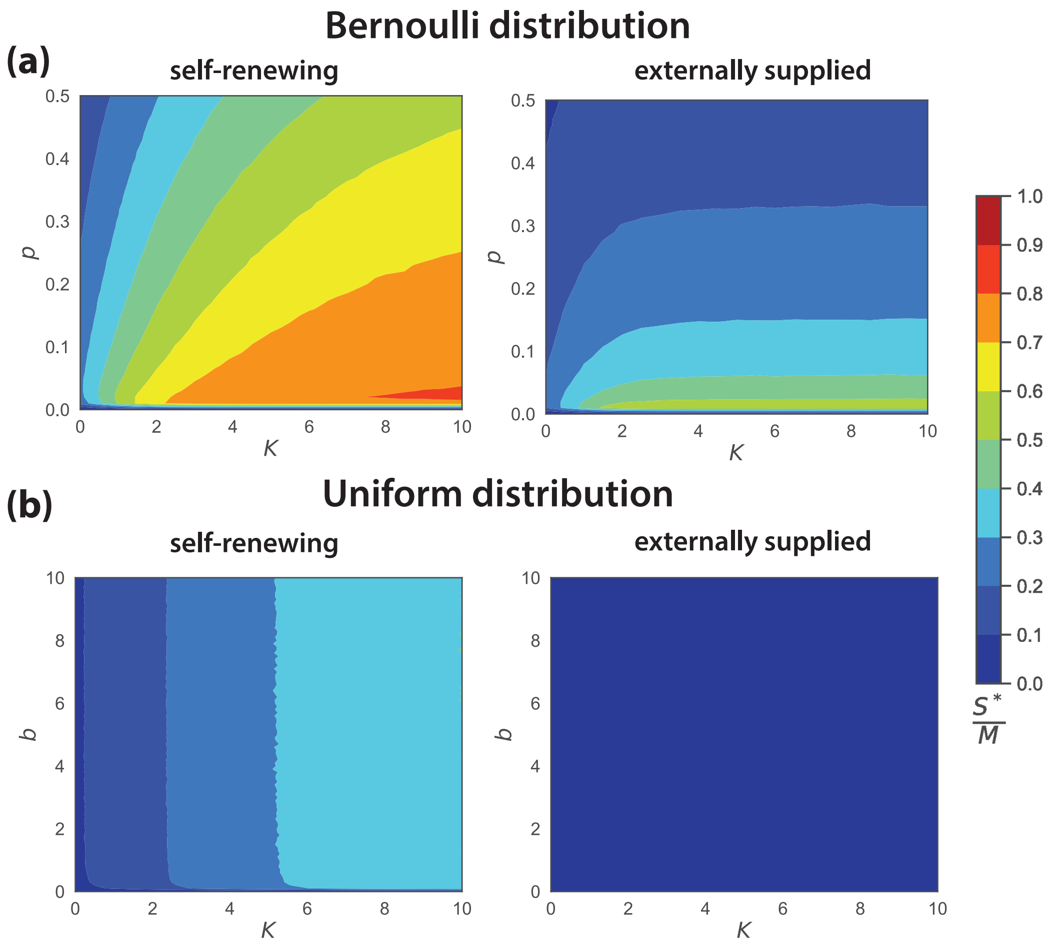

FIG. S4. Comparison of species packing $\frac{S^*}{M}$ for different distributions of consumption matrices **C** with self-renewing and externally-supplied resource dynamics. The simulations represent averages from 1000 independent realizations with the system size $M = 100$, $S = 500$ and parameters at the SM: III.

The term approaches -1 in the limit $\nu \to -\infty$, which is the same limit required to saturate the bound in the model with self-renewing resources. As in that case, the limit cannot actually be achieved, because $\nu \to -\infty$ implies $\chi \to 0$ (Equation (82)), and χ appears in the denominator of the final expression for $\bar{N}_0$ (Equation (24)), while the numerator always remains finite.

The only way to achieve the limit $\frac{S^*}{M} = 1$ is to make the numerator vanish in the same way as the denominator, which can only happen in the presence of hard constraints $\sigma_m = \sigma_\epsilon = 0$. In this case, it is easy to see that setting $R_\alpha = \tilde{m}$ for all α and $\chi \to 0, \nu \to -\infty$ solves both the steady state equations, regardless of the value of $\tilde{N}_0$. In Equation (77) for $\tilde{N}_0$, the mean and the fluctuating part inside the brackets both vanish individually ($\mu\langle R\rangle - \mu\tilde{m} = 0$, $\sigma_N = 0$), and the back-reaction term also vanishes ($\sigma_c^2 \chi \bar{N}_0 = 0$), leaving the equation trivially satisfied. In Equation (78) only the terms with ν are significant in this limit, and they cancel each other perfectly. This is the "shielded phase" discussed in [6].

Note also that if we take the $\chi \to 0$, $\nu \to -\infty$ limit first, before performing any substitutions, Equations (81) and (82) are satisfied independently of the choice of ϕ_N. This means that $\gamma^{-1}\phi_N = S^*/M$ can be greater than 1, as observed in the simulations of [7].

D. Numerical evidence

We show a comparison between the cavity solution and numerical results in Fig. 3 and Fig. S4 for three different distributions of the consumption matrix **C**. For the Gaussian and Bernoulli distributions, $\frac{S^*}{M}$ can reach the upper bound we derived for two different resource dynamics. For externally supplied resource dynamics, $\frac{S^*}{M}$ never exceeds

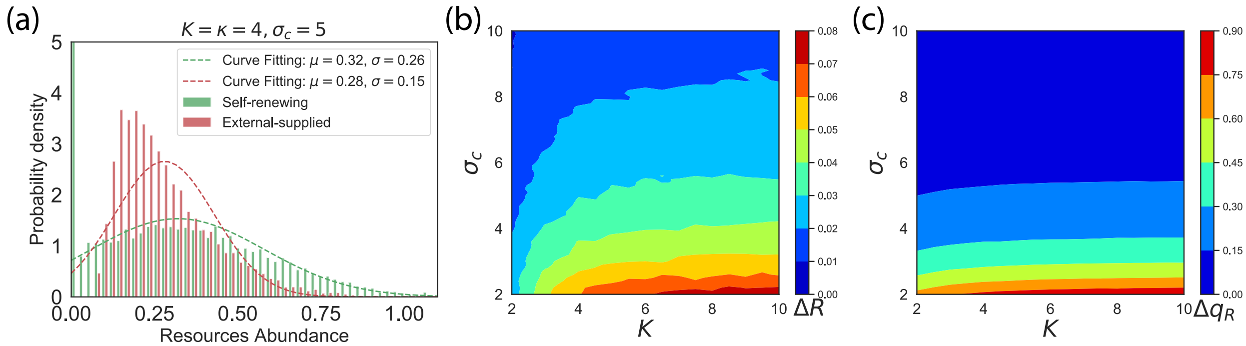

FIG. S5. (a) Comparison of resource abundance for self-renewing and externally-supplied resource dynamics at $K = \kappa = 4$ and $\sigma_c = 5$. The dashed lines are the gaussian curve fitting about the abundances with mean μ and variance σ^2. (b, c) the difference of the first and second moment of the resource abundance between self-renewing and externally-supplied resource dynamics with the same $K = \kappa$ and σ_c, $\Delta R = R^s - R^e$, $\Delta q_R = q_R^s - q_R^e$, where the upper label e and s represents self-renewing and externally-supplied, respectively. All simulations are the same as Fig. 3 in the main text.

0.5. For the uniform case, since the fluctuation of consumption matrix is small, the niche overlap is large and there is fierce competitions among species and theses ecosystems live very far from the upper bounds we derive. However, even for the uniform case, the species packing fraction is significantly larger for self-renewing resource dynamics than externally supplied resource dynamics. For the Bernoulli case, when the binomial probability $p \sim 1/M$, the bound can be slightly above 0.5, as shown in Fig. S4. In this regime, the consumer matrix is sparse. Each species only consumes one or two different resources and species rarely compete with each other making it is possible to pack more species.

E. Numerical analysis

Eq. (28) shows the fraction of surviving species S^*/S is determined by the first moment ($R = \langle R \rangle$) and second moment($q_R = \langle R^2 \rangle$) of the resource abundance. In Fig. S5 (a), the simulation shows the two dynamics have similar means but quite different variance for $K = \kappa = 4$ and $\sigma_c = 5$. And the external-supplied resource dynamics with a larger q_R (sharper distribution) have a smaller fraction of surviving species S^*/S.

Fig. S5 (b, c) shows the first and second moment differences between self-renewing and externally-supplied resource dynamics, ΔR and Δq_R are always positive, which means the self-renewing resource dynamics always has larger R and q_R across the whole heat map. And thus, it is generally true that external-supplied dynamics has sharper resource distribution and can explain the lower diversity (in high σ_c regime, it looks ΔR and Δq_R are close to zero but considering there is σ_c in the dominator of eq. (28), a slight difference in q_R can have a huge difference.). However, note that S^*/S (the fraction of species in the regional species pool that survive) is not the same as species packing S^*/M and it cannot explain why the species packing bound is exactly at 0.5.

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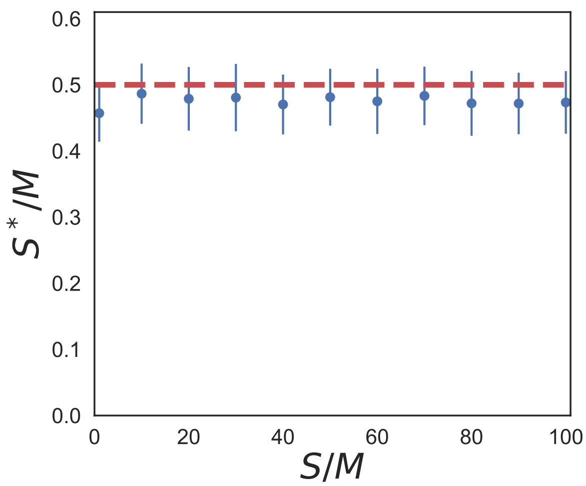

FIG. S6. The species packing ratio $\frac{S^*}{M}$ at various S/M for externally supplied resource dynamics. Other parameters are the same as Fig. 3 in the main text.