SUPPLEMENTARY MATERIAL

Asymmetric autoregressive models: Statistical aspects and a financial application under COVID-19 pandemic

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ABSTRACT
In the present study, we provide a motivating example with a financial application under COVID-19 pandemic to investigate autoregressive (AR) modeling based on asymmetric distributions and its diagnostics. The objectives of this work are: (i) to formulate asymmetric AR models and their estimation and diagnostics; (ii) to assess the performance of the parameters estimators and of the local influence technique for these models; and (iii) to provide a tool to show how data following an asymmetric distribution under an AR structure should be analyzed. We take the advantages of the stochastic representation of the skew-normal distribution to estimate the parameters of the corresponding AR model efficiently with the expectation-maximization algorithm. Diagnostic analytics are conducted by using the local influence technique with four perturbation schemes. By employing Monte Carlo simulations, we evaluate the statistical behavior of the corresponding estimators and of the local influence technique. An illustration with financial data updated until 2020, analyzed using the methodology introduced in the present work, is presented as an example of effective applications, from where it is possible to explain atypical cases from the COVID-19 pandemic.

KEYWORDS
Expectation-maximization algorithm; Local influence; Maximum likelihood methods; Monte Carlo simulation; Non-normality; Times-series models.

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Supplementary material

Hessian matrix

The corresponding Hessian matrix $\ddot{Q}\,(\boldsymbol{\theta})$ evaluated at the maximum likelihood estimate $\widehat{\boldsymbol{\theta}}$ employing

$$\ell_c(\boldsymbol{\theta}, \boldsymbol{y}_c) = \sum_{t=p+1}^{T} \left(-\frac{1}{2}\log(\sigma^2) + \frac{1}{2}\log(1+\lambda^2) - \frac{1+\lambda^2}{2\sigma^2}\left(u_t - \frac{\lambda\sigma}{\sqrt{1+\lambda^2}}h_t\right)^2 \right). \quad (1)$$

Hence, by using (1), the Q function is stated as

$$\begin{aligned} Q(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} &= \mathrm{E}[\ell_c(\boldsymbol{\theta}, \boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} \\ &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{T-p}{2}\log(1+\lambda^2) \quad (2) \\ &\quad -\frac{(1+\lambda^2)}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2). \end{aligned}$$

In the case of the SNAR(p) model, the $(p+2)\times(p+2)$ Hessian matrix $\ddot{Q}(\widehat{\boldsymbol{\theta}})$ is established by

$$\ddot{Q}(\widehat{\boldsymbol{\theta}}) = \begin{pmatrix} \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\sigma^2} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\lambda} \\ \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\sigma^2\partial\boldsymbol{\beta}} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial(\sigma^2)^2} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\sigma^2\partial\lambda} \\ \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\lambda\partial\boldsymbol{\beta}} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\lambda\partial\sigma^2} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial\lambda^2} \end{pmatrix}\Bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}, \quad (3)$$

with $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\beta}}, \widehat{\sigma}^2, \widehat{\lambda})$ being the maximum likelihood estimate of $\boldsymbol{\theta}$. The expression for $\partial^2 Q\,(\boldsymbol{\theta})/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top$ and the other submatrices are presented below.

Based on the matrix differential calculus, we obtain the Q function defined in (2) expressed as

$$\begin{aligned} Q(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} =& \mathrm{E}[\ell_c(\boldsymbol{\theta}, \boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} \\ =& -\frac{T-p}{2}\log(\sigma^2) + \frac{T-p}{2}\log(1+\lambda^2) \\ & -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2), \end{aligned}$$

with

$$\begin{aligned}
\widehat{c}_t =& \mathrm{E}(H_t|\boldsymbol{Y}_o=\boldsymbol{y}_o)|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \tau_1 + \frac{\phi(\frac{\tau_1}{\tau_2})}{\Phi(\frac{\tau_1}{\tau_2})}\tau_2,\\
\widehat{c_t^2} =& \mathrm{E}({H_t}^2|\boldsymbol{Y}_o=\boldsymbol{y}_o)|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \tau_1^2+\tau_2^2 + \frac{\phi(\frac{\tau_1}{\tau_2})}{\Phi(\frac{\tau_1}{\tau_2})}\tau_1\tau_2,\\
\tau_1 =& \frac{\widehat{\lambda}}{\widehat{\sigma}\sqrt{1+\widehat{\lambda}^2}}\widehat{u}_t, \quad \tau_2 = \frac{1}{\sqrt{1+\widehat{\lambda}^2}}.
\end{aligned}$$

The first-order derivatives related to $\ddot{Q}(\widehat{\boldsymbol{\theta}})$ in (3) are given by

$$\begin{aligned}
\frac{\partial Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}} &= \sum_{t=p+1}^{\top} \frac{1+\lambda^2}{\sigma}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)\boldsymbol{x}_t,\\
\frac{\partial Q(\boldsymbol{\theta})}{\partial \sigma^2} &= -\frac{T-p}{2\sigma^2} + \sum_{t=p+1}^{\top} \frac{1+\lambda^2}{2\sigma^3}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)u_t,\\
\frac{\partial Q(\boldsymbol{\theta})}{\partial \lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{1+\lambda^2} - \left(\frac{u_t}{\sigma}\right)^2\lambda - \lambda\widehat{c_t^2} + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t}{\sigma}\widehat{c}_t\right).
\end{aligned}$$

The second-order derivatives corresponding to $\ddot{Q}(\widehat{\boldsymbol{\theta}})$ in (3) are stated by

$$\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}^\top} &= -\sum_{t=p+1}^{\top}\frac{1+\lambda^2}{\sigma^2}\boldsymbol{x}_t\boldsymbol{x}_t^\top,\\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \sigma^2} &= \sum_{t=p+1}^{\top}\frac{1+\lambda^2}{2\sigma^3}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t - \frac{2u_t}{\sigma}\right)\boldsymbol{x}_t,\\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \lambda} &= \sum_{t=p+1}^{\top}\left(\frac{2\lambda}{\sigma^2}u_t - \frac{2\lambda^2+1}{\sigma\sqrt{1+\lambda^2}}\widehat{c}_t\right)\boldsymbol{x}_t,\\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial (\sigma^2)^2} &= \frac{T-p}{2\sigma^4} - \sum_{t=p+1}^{\top}\frac{1+\lambda^2}{4\sigma^5}\left(\frac{4u_t}{\sigma} - \frac{3\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)u_t,\\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \sigma^2\partial \lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{\sigma^4}{u_t}^2 - \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t}{2\sigma^3}\widehat{c}_t\right),\\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \lambda^2} &= \frac{(T-p)\left(1-\lambda^2\right)}{(1+\lambda^2)^2} + \sum_{t=p+1}^{\top}\left(-\frac{{u_t}^2}{\sigma^2} - \widehat{c_t^2} + \frac{2\lambda^3+3\lambda}{\sqrt{(1+\lambda^2)^3}}\frac{u_t}{\sigma}\widehat{c}_t\right).
\end{aligned}$$

Matrices of perturbation

Based on the matrix differential calculus, we establish the derivatives involved in our calculations.

Case-weight perturbation Consider a perturbation is made on the residual of the SNAR(p) model, with $y_t = \boldsymbol{x}_t^\top \boldsymbol{\beta} + u_t$ being replaced by $\omega_t y_t = \omega_t \boldsymbol{x}_t^\top \boldsymbol{\beta} + \omega_t u_t$ and ω_t is the weight. Let $\boldsymbol{\omega} = (\omega_{p+1}, \ldots, \omega_T)^\top$ denote the $(T-p)\times 1$ vector and $\boldsymbol{\omega}_0 = (1, \ldots, 1)^\top$ denote the $(T-p)\times 1$ non-perturbation vector. Hence, the complete-data log-likelihood function of the perturbed model is stated by

$$\ell_c(\boldsymbol{\theta}, \boldsymbol{\omega}, \boldsymbol{y}_c) = \sum_{t=p+1}^{T} \Big(-\frac{1}{2}\log(\sigma^2) + \frac{1}{2}\log(1+\lambda^2) - \frac{1+\lambda^2}{2\sigma^2}\Big(\omega_t u_t - \frac{\lambda\sigma}{\sqrt{1+\lambda^2}} h_t\Big)^2\Big). \quad (4)$$

Thus, the perturbed Q function obtained from (4) is expressed as

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\omega})\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} &= \mathrm{E}[\ell_c(\boldsymbol{\theta}, \boldsymbol{\omega}, \boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} \\ &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) \\ &\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{\omega_t u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2). \quad (5) \end{aligned}$$

In the case of the SNAR(p) model with case-weight perturbation, the $(p+2)\times(T-p)$ perturbation matrix $\boldsymbol{\Delta}$ must be evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ after taking derivatives, obtaining

$$\boldsymbol{\Delta} = \frac{\partial^2 Q(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\omega}}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}, \boldsymbol{\omega}=\boldsymbol{\omega}_0} = \begin{pmatrix} \frac{1+\widehat{\lambda}^2}{\widehat{\sigma}^2}\widehat{u}_t \boldsymbol{x}_t{}^\top \\ \frac{1+\widehat{\lambda}^2}{2\widehat{\sigma}^3}\left(\frac{2\widehat{u}_t^2}{\widehat{\sigma}} - \frac{\widehat{\lambda}\widehat{c}_t}{\sqrt{1+\widehat{\lambda}^2}}\widehat{u}_t\right) \\ -\frac{2\widehat{\lambda}}{\widehat{\sigma}^2}\widehat{u}_t^2 + \frac{2\widehat{\lambda}+1}{\sqrt{1+\widehat{\lambda}^2}}\frac{\widehat{c}_t}{\widehat{\sigma}}\widehat{u}_t \end{pmatrix}, \quad (6)$$

with $\widehat{u}_t = y_t - \boldsymbol{x}_t^\top\widehat{\boldsymbol{\beta}}$, $\widehat{c}_t = \mathrm{E}(H_t|Y_0, \widehat{\boldsymbol{\theta}}) = \tau_1 + \phi(\tau_1/\tau_2)/\Phi(\tau_1/\tau_2)\tau_2$, $\widehat{c_t^2} = \mathrm{E}({H_t}^2|Y_0, \widehat{\boldsymbol{\theta}}) = \tau_1^2 + \tau_2^2 + \phi(\tau_1/\tau_2)/\Phi(\tau_1/\tau_2)\tau_1\tau_2$, $\tau_1 = \widehat{\lambda}/\widehat{\sigma}\sqrt{1+\widehat{\lambda}^2}\widehat{u}_t$, and $\tau_2 = 1/\sqrt{1+\widehat{\lambda}^2}$.

With the function $Q(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \mathrm{E}[\ell_c(\boldsymbol{\theta}, \boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$, we obtain

$$\begin{aligned} Q(\boldsymbol{\theta}) &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) \\ &\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\left(\frac{u_t}{\sigma}\right)^2 + \frac{\lambda^2}{1+\lambda^2}\widehat{c_t^2} - \frac{u_t}{\sigma}\frac{2\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right) \\ &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) - \frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 \\ &\quad -\frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2), \end{aligned}$$

with

$$
\begin{aligned}
\widehat{c}_t &= \mathrm{E}(H_t|Y_0,\widehat{\boldsymbol{\theta}}) = \tau_1 + \frac{\phi(\frac{\tau_1}{\tau_2})}{\Phi(\frac{\tau_1}{\tau_2})}\tau_2, \\
\widehat{c_t^2} &= \mathrm{E}({H_t}^2|Y_0,\widehat{\boldsymbol{\theta}}) = \tau_1^2 + \tau_2^2 + \frac{\phi(\frac{\tau_1}{\tau_2})}{\Phi(\frac{\tau_1}{\tau_2})}\tau_1\tau_2, \\
\tau_1 &= \frac{\widehat{\lambda}}{\widehat{\sigma}\sqrt{1+\widehat{\lambda}^2}}\widehat{u}_t, \tau_2 = \frac{1}{\sqrt{1+\widehat{\lambda}^2}}.
\end{aligned}
$$

The first-order derivatives related to $\ddot{Q}(\widehat{\boldsymbol{\theta}})$ in (3) are stated as

$$
\begin{aligned}
\frac{\partial Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}} &= \sum_{t=p+1}^{\top} \frac{1+\lambda^2}{\sigma}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right){\boldsymbol{x}_t}^{\top}, \\
\frac{\partial Q(\boldsymbol{\theta})}{\partial \sigma^2} &= -\frac{(T-p)}{2\sigma^2} + \sum_{t=p+1}^{\top} \frac{1+\lambda^2}{2\sigma^3}\left(\frac{u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)u_t, \\
\frac{\partial Q(\boldsymbol{\theta})}{\partial \lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{1+\lambda^2} - \left(\frac{u_t}{\sigma}\right)^2\lambda - \lambda\widehat{c_t^2} + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t}{\sigma}\widehat{c}_t\right).
\end{aligned}
$$

The second-order derivatives corresponding to $\ddot{Q}(\widehat{\boldsymbol{\theta}})$ in (3) are defined as

$$
\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}^{\top}} &= -\sum_{t=p+1}^{\top} \frac{1+\lambda^2}{\sigma^2}{\boldsymbol{x}_t}^{\top}\boldsymbol{x}_t, \\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \sigma^2} &= \sum_{t=p+1}^{\top} \frac{1+\lambda^2}{2\sigma^3}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t - \frac{2u_t}{\sigma}\right){\boldsymbol{x}_t}^{\top}, \\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \lambda} &= \sum_{t=p+1}^{\top}\left(\frac{2\lambda}{\sigma^2}u_t - \frac{2\lambda^2+1}{\sigma\sqrt{1+\lambda^2}}\widehat{c}_t\right){\boldsymbol{x}_t}^{\top}, \\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial (\sigma^2)^2} &= \frac{(T-p)}{2\sigma^4} - \sum_{t=p+1}^{\top} \frac{1+\lambda^2}{4\sigma^5}\left(\frac{4u_t}{\sigma} - \frac{3\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)u_t, \\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \sigma^2\partial \lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{\sigma^4}{u_t}^2 - \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t}{2\sigma^3}\widehat{c}_t\right), \\
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \lambda^2} &= \frac{(T-p)\left(1-\lambda^2\right)}{\left(1+\lambda^2\right)^2} + \sum_{t=p+1}^{\top}\left(-\frac{{u_t}^2}{\sigma^2} - \widehat{c_t^2} + \frac{2\lambda^3+3\lambda}{\sqrt{\left(1+\lambda^2\right)^3}}\frac{u_t}{\sigma}\widehat{c}_t\right).
\end{aligned}
$$

Data perturbation Consider $y_t(\omega) = \omega_t + y_t$. Let $\boldsymbol{\omega} = (\omega_{p+1}, \ldots, \omega_T)^{\top}$ be a $(T-p)\times 1$ vector and $\boldsymbol{\omega}_0 = (0,\ldots,0)^{\top}$ be the $(T-p)\times 1$ non-perturbation vector. The perturbed AR(p) model is $y_t + \omega_t = \beta_1\left(y_{t-1}+\omega_{t-1}\right) + \cdots + \beta_p\left(y_{t-p}+\omega_{t-p}\right) + u_t$, with $u_t = y_t - \boldsymbol{x}_t^{\top}\boldsymbol{\beta} + \mu(\omega_t)$ and $\mu(\omega_t) = \omega_t - \beta_1\omega_{t-1} - \cdots - \beta_p\omega_{t-p}$. Hence, the perturbed complete-data log-likelihood function model is established by

$$\ell_c(\boldsymbol{\theta}, \boldsymbol{\omega}, \boldsymbol{y}_c) = \sum_{t=p+1}^{T} \Big(-\frac{1}{2}\log(\sigma^2) + \frac{1}{2}\log(1+\lambda^2) - \frac{1+\lambda^2}{2\sigma^2}\Big(u_t + \mu(\omega_t) - \frac{\lambda\sigma}{\sqrt{1+\lambda^2}}h_t\Big)^2\Big).$$

Thus, the perturbed Q function is expressed as

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) \\ &\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t+\mu(\omega_t)}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2)(7) \end{aligned}$$

In the case of the SNAR(p) model with data perturbation, we get

$$\boldsymbol{\Delta} = \frac{\partial^2 Q(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\omega}}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}, \boldsymbol{\omega}=\boldsymbol{\omega}_0} = \begin{pmatrix} \frac{1+\widehat{\lambda}^2}{\widehat{\sigma}^2}\boldsymbol{x}_t{}^{\top} \\ \frac{1+\widehat{\lambda}^2}{2\widehat{\sigma}^3}\left(\frac{2\widehat{u}_t^2}{\widehat{\sigma}} - \frac{\widehat{\lambda}}{\sqrt{1+\widehat{\lambda}^2}}\widehat{c}_t\right) \\ -\frac{2\widehat{\lambda}}{\widehat{\sigma}^2}\widehat{u}_t^2 + \frac{2\widehat{\lambda}+1}{\sqrt{1+\widehat{\lambda}^2}}\frac{\widehat{c}_t}{\widehat{\sigma}} \end{pmatrix}, \tag{8}$$

with $\widehat{u}_t = y_t - \boldsymbol{x}_t^{\top}\widehat{\boldsymbol{\beta}}$, and $\widehat{h}_t$ being as in the data perturbation.

With the function $Q(\boldsymbol{\theta}, \boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \mathrm{E}[\ell_c(\boldsymbol{\theta}, \boldsymbol{\omega}, \boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$ given in (5), we obtain

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\omega}) &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) \\ &\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{\omega_t u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2). \end{aligned}$$

The first-order derivatives related to $\boldsymbol{\Delta}$ in (6) are stated as

$$\begin{aligned} \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial\boldsymbol{\beta}} &= \sum_{t=p+1}^{\top}\frac{1+\lambda^2}{\sigma}\left(\frac{\omega_t u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)\boldsymbol{x}_t{}^{\top}, \\ \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial\sigma^2} &= -\frac{(T-p)}{2\sigma^2} + \sum_{t=p+1}^{\top}\frac{1+\lambda^2}{2\sigma^3}\left(\frac{\omega_t u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)\omega_t u_t, \\ \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial\lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{1+\lambda^2} - \Big(\frac{\omega_t u_t}{\sigma}\Big)^2\lambda - \lambda\widehat{c_t^2} + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{\omega_t u_t}{\sigma}\widehat{c}_t\right), \end{aligned}$$

The second-order derivatives corresponding to $\mathbf{\Delta}$ in (6) are given by

$$\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}\partial\omega_t} &= \frac{1+\lambda^2}{\sigma^2}u_t{\boldsymbol{x}_t}^\top, \\
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2\partial\omega_t} &= \frac{1+\lambda^2}{2\sigma^3}\left(\frac{2{u_t}^2}{\sigma}\omega_t - \frac{\lambda\widehat{c}_t}{\sqrt{1+\lambda^2}}u_t\right), \\
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda\partial\omega_t} &= -\frac{2\lambda}{\sigma}{u_t}^2\omega_t + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t}{\sigma}\widehat{c}_t.
\end{aligned}$$

Noting that $\omega_0 = (1,\dots,1)^\top$, we get the expression established in (6).

Variance parameter perturbation Consider that the variance parameter σ^2 in the model is replaced by σ^2/ω_t, meaning that $u_t \sim \mathrm{SN}(0, {\omega_t}^{-1}\sigma^2, \lambda)$. Let $\boldsymbol{\omega} = (\omega_{p+1},\dots,\omega_T)^\top$ denote the $(T-p)\times 1$ vector and $\boldsymbol{\omega}_0 = (1,\dots,1)^\top$ denote the $(T-p)\times 1$ non-perturbation vector. Hence, the complete-data log-likelihood function of the perturbed model is expressed as

$$\ell_c(\boldsymbol{\theta},\boldsymbol{\omega},\boldsymbol{y}_c) = \sum_{t=p+1}^{T}\Big(-\frac{1}{2}\log(\sigma^2)+\frac{1}{2}\log(\omega_t)+\frac{1}{2}\log(1+\lambda^2)-\frac{\omega_t(1+\lambda^2)}{2\sigma^2}\Big(u_t - \frac{\lambda\sigma}{\sqrt{\omega_t(1+\lambda^2)}}h_t\Big)^2\Big).$$

Thus, the perturbed Q function is established by

$$\begin{aligned}
Q(\boldsymbol{\theta},\boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} &= -\frac{(T-p)}{2}\log(\sigma^2) + \sum_{t=p+1}^{\top}\frac{1}{2}\log(\omega_t) + \frac{(T-p)}{2}\log(1+\lambda^2) \\
&\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{\sqrt{\omega_t}u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2). \quad (9)
\end{aligned}$$

In the case of the SNAR(p) model with variance parameter perturbation, we get

$$\mathbf{\Delta} = \frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\omega}}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}},\boldsymbol{\omega}=\boldsymbol{\omega}_0} = \begin{pmatrix} \frac{1+\widehat{\lambda}^2}{\widehat{\sigma}}\left(\frac{\widehat{u}_t}{\widehat{\sigma}} - \frac{\widehat{\lambda}}{2\sqrt{1+\widehat{\lambda}^2}}\widehat{c}_t\right){\boldsymbol{x}_t}^\top \\ \frac{(1+\widehat{\lambda}^2)\widehat{u}_t^2}{2\widehat{\sigma}^4} - \frac{\widehat{\lambda}\sqrt{1+\widehat{\lambda}^2}}{4\widehat{\sigma}^3}\widehat{c}_t\widehat{u}_t \\ -\frac{\widehat{\lambda}}{\widehat{\sigma}^2}\widehat{u}_t^2 + \frac{2\widehat{\lambda}^2+1}{\sqrt{1+\widehat{\lambda}^2}}\frac{\widehat{c}_t}{2\widehat{\sigma}}\widehat{u}_t \end{pmatrix}, \quad (10)$$

with $\widehat{u}_t = y_t - \boldsymbol{x}_t^\top\widehat{\boldsymbol{\beta}}$, and $\widehat{h}_t$ being as in data perturbation.

With the function $Q(\boldsymbol{\theta},\boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \mathrm{E}[\ell_c(\boldsymbol{\theta},\boldsymbol{\omega},\boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$ defined in (7), we obtain

$$
\begin{aligned}
Q(\boldsymbol{\theta},\boldsymbol{\omega}) &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) \\
&\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\left(\frac{u_t+\mu(\omega_t)}{\sigma}\right)^2 + \frac{\lambda^2}{1+\lambda^2}\widehat{c_t^2} - \frac{u_t+\mu(\omega_t)}{\sigma}\frac{2\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right) \\
&= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\lambda^2) \\
&\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t+\mu(\omega_t)}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2).
\end{aligned}
$$

The first-order derivatives related to $\boldsymbol{\Delta}$ in (8) are stated as

$$
\begin{aligned}
\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}} &= \sum_{t=p+1}^{\top}\frac{1+\lambda^2}{\sigma}\left(\frac{u_t+\mu(\omega_t)}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)\left(\boldsymbol{x}_t{}^{\top} + (\omega_{t-1},\ldots,\omega_{t-p})^{\top}\right) \\
\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2} &= -\frac{(T-p)}{2\sigma^2} + \sum_{t=p+1}^{\top}\frac{1+\lambda^2}{2\sigma^3}\left(\frac{(u_t+\mu(\omega_t))^2}{\sigma} - \frac{\lambda(u_t+\mu(\omega_t))}{\sqrt{1+\lambda^2}}\widehat{c}_t\right) \\
\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{1+\lambda^2} - \left(\frac{u_t+\mu(\omega_t)}{\sigma}\right)^2\lambda - \lambda\widehat{c_t^2} + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t+\mu(\omega_t)}{\sigma}\widehat{c}_t\right)
\end{aligned}
$$

The second-order derivatives corresponding to $\boldsymbol{\Delta}$ in (8) are given by

$$
\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}\partial\omega_t} &= \frac{1+\lambda^2}{\sigma^2}\left(\boldsymbol{x}_t{}^{\top} + (\omega_{t-1},\ldots,\omega_{t-p})^{\top}\right) \\
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2\partial\omega_t} &= \frac{1+\lambda^2}{2\sigma^3}\left(\frac{2(u_t+\mu(\omega_t))}{\sigma} - \frac{\lambda\widehat{c}_t}{\sqrt{1+\lambda^2}}\right) \\
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda\partial\omega_t} &= \left(-\frac{2\lambda}{\sigma^2}(u_t+\mu(\omega_t)) + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{\widehat{c}_t}{\sigma}\right)
\end{aligned}
$$

Noting that $\omega_0 = (0,\ldots,0)^{\top}$, we obtain the expression established in (8).

With the function $Q(\boldsymbol{\theta},\boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \mathrm{E}[\ell_c(\boldsymbol{\theta},\boldsymbol{\omega},\boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$ given in (11),we get

$$
\begin{aligned}
Q(\boldsymbol{\theta},\boldsymbol{\omega}) &= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\omega_t\lambda^2) \\
&\quad -\frac{1+\omega_t\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t{}^2}{\sigma^2} + \frac{\omega_t\lambda^2}{1+\omega_t\lambda^2}\widehat{c_t^2} - \frac{u_t}{\sigma}\frac{2\sqrt{\omega_t}\lambda}{\sqrt{1+\omega_t\lambda^2}}\widehat{c}_t\right) \\
&= -\frac{(T-p)}{2}\log(\sigma^2) + \frac{(T-p)}{2}\log(1+\omega_t\lambda^2) \\
&\quad -\frac{1+\omega_t\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{u_t}{\sigma} - \frac{\sqrt{\omega_t}\lambda}{\sqrt{1+\omega_t\lambda^2}}\widehat{c}_t\right)^2 - \frac{\omega_t\lambda^2}{2}\sum_{t=p+1}^{\top}(\widehat{c_t^2} - (\widehat{c}_t)^2).
\end{aligned}
$$

The first-order derivatives related to $\mathbf{\Delta}$ in (11) are written as

$$\begin{aligned}
\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}} &= \sum_{t=p+1}^{\top} \frac{1+\omega_t\lambda^2}{\sigma}\left(\frac{u_t}{\sigma}-\frac{\lambda\sqrt{\omega_t}}{\sqrt{1+\omega_t\lambda^2}}\widehat{c}_t\right)\boldsymbol{x}_t{}^{\top}\\
\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2} &= \sum_{t=p+1}^{\top}\left(-\frac{1}{2\sigma^2}+\frac{1+\omega_t\lambda^2}{2\sigma^3}\left(\frac{u_t}{\sigma}-\frac{\lambda\sqrt{\omega_t}}{\sqrt{1+\omega_t\lambda^2}}\widehat{c}_t\right)u_t\right)\\
\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\omega_t\lambda}{1+\omega_t\lambda^2}-\frac{u_t{}^2}{\sigma^2}\omega_t\lambda-\omega_t\lambda\widehat{c_t^2}+\frac{\sqrt{\omega_t}\left(2\omega_t\lambda^2+1\right)}{\sqrt{1+\omega_t\lambda^2}}\frac{u_t}{\sigma}\widehat{c}_t\right).
\end{aligned}$$

The second-order derivatives corresponding to $\mathbf{\Delta}$ in (11) are defined by

$$\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}\partial\omega_t} &= \left(\frac{\lambda^2}{\sigma^2}u_t-\frac{\lambda\left(1+2\lambda^2\omega_t\right)}{2\sqrt{\omega_t\left(1+\lambda^2\omega_t\right)}}\frac{\widehat{c}_t}{\sigma}\right)\boldsymbol{x}_t{}^{\top}\\
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2\partial\omega_t} &= \left(\frac{\lambda^2 u_t}{2\sigma^4}-\frac{\lambda}{4\sigma^3}\frac{1+2\lambda^2\omega_t}{\sqrt{\omega_t\left(1+\lambda^2\omega_t\right)}}\widehat{c}_t\right)u_t\\
\frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda\partial\omega_t} &= \frac{\lambda}{\left(1+\lambda^2\omega_t\right)^2}-\frac{u_t{}^2\lambda}{\sigma^2}-\lambda\widehat{c_t^2}\\
&+ \frac{\left(6\lambda^2\omega_t+1\right)\left(\lambda^2\omega_t+1\right)-\lambda^2\omega_t\left(2\lambda^2\omega_t+1\right)}{2\sqrt{\omega_t}\left(1+\lambda^2\omega_t\right)^{3/2}}\frac{u_t}{\sigma}\widehat{c}_t
\end{aligned}$$

Noting that $\omega_0=(1,\ldots,1)^{\top}$, we obtain (11).

Skewness parameter perturbation Consider a particular skewed feature of the distribution. Then, we can investigate the effect on the model fit by making a change of the skewness parameter λ. In our perturbed model, we propose to replace λ by $\sqrt{\omega}_i\lambda$. Let $\boldsymbol{\omega}=(\omega_{p+1},\ldots,\omega_T)^{\top}$ denote the $(T-p)\times 1$ vector and $\omega_0=(1,\ldots,1)^{\top}$ denote the $(T-p)\times 1$ non-perturbation vector. Hence, the complete-data log-likelihood function of the perturbed model is stated by

$$\ell_c(\boldsymbol{\theta},\boldsymbol{\omega},\boldsymbol{y}_c)=\sum_{t=p+1}^{T}\Big(-\frac{1}{2}\log(\sigma^2)+\frac{1}{2}\log(1+\omega_t\lambda^2)-\frac{1+\omega_t\lambda^2}{2\sigma^2}\Big(u_t-\frac{\sqrt{\omega_t}\lambda\sigma}{\sqrt{1+\omega_t\lambda^2}}h_t\Big)^2\Big).$$

Thus, the perturbed Q function is expressed as

$$\begin{aligned}
Q(\boldsymbol{\theta},\boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} &= -\frac{(T-p)}{2}\log\left(\sigma^2\right)+\frac{(T-p)}{2}\log(1+\omega_t\lambda^2)\\
&-\sum_{t=p+1}^{\top}\frac{1+\omega_t\lambda^2}{2}\left(\frac{u_t}{\sigma}-\frac{\sqrt{\omega_t}\lambda}{\sqrt{1+\omega_t\lambda^2}}\widehat{c}_t\right)^2-\sum_{t=p+1}^{\top}\frac{\omega_t\lambda^2}{2}(\widehat{c_t^2}-(\widehat{c}_t)^2).
\end{aligned}$$

In the case of the SNAR(p) model with the skewness parameter perturbation, we get

$$\boldsymbol{\Delta} = \frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\omega}}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}},\boldsymbol{\omega}=\boldsymbol{\omega}_0} = \begin{pmatrix} \left(\frac{\widehat{u}_t}{\widehat{\sigma}^2}\widehat{\lambda}^2 - \frac{\widehat{\lambda}\left(2\widehat{\lambda}^2+1\right)}{2\sqrt{1+\widehat{\lambda}^2}}\frac{\widehat{c}_t}{\widehat{\sigma}}\right)\boldsymbol{x}_t{}^\top \\ \frac{\widehat{\lambda}^2\widehat{u}_t^2}{2\widehat{\sigma}^4} - \frac{\widehat{\lambda}}{2\widehat{\sigma}^3}\frac{1+2\widehat{\lambda}^2}{2\sqrt{1+\widehat{\lambda}^2}}\widehat{c}_t\widehat{u}_t \\ \frac{\widehat{\lambda}}{\left(\widehat{\lambda}^2+1\right)^2} - \frac{\widehat{\lambda}}{\widehat{\sigma}^2}\widehat{u}_t^2 - \widehat{\lambda}\widehat{c_t^2} + \frac{\left(6\widehat{\lambda}^2+1\right)\left(\widehat{\lambda}^2+1\right)-\widehat{\lambda}^2\left(2\widehat{\lambda}^2+1\right)}{2\left(\widehat{\lambda}^2+1\right)^{3/2}}\frac{\widehat{c}_t}{\widehat{\sigma}}\widehat{u}_t \end{pmatrix}, \tag{11}$$

with $\widehat{h}_t$, $\widehat{c_t^2}$ being as in the data perturbation and $\widehat{u}_t = y_t - \boldsymbol{x}_t^\top\widehat{\boldsymbol{\beta}}$.

With the function $Q(\boldsymbol{\theta},\boldsymbol{\omega})|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \mathrm{E}[\ell_c(\boldsymbol{\theta},\boldsymbol{\omega},\boldsymbol{Y}_c)|\boldsymbol{Y}_o = \boldsymbol{y}_o]|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$ given in (9), we obtain

$$\begin{aligned} Q(\boldsymbol{\theta},\boldsymbol{\omega}) &= -\frac{(T-p)}{2}\log(\sigma^2) + \sum_{t=p+1}^{\top}\frac{1}{2}\log(\omega_t) + \frac{(T-p)}{2}\log(1+\lambda^2) \\ &\quad -\frac{1+\lambda^2}{2}\sum_{t=p+1}^{\top}\left(\frac{\sqrt{\omega_t}u_t}{\sigma} - \frac{\lambda}{\sqrt{1+\lambda^2}}\widehat{c}_t\right)^2 - \frac{\lambda^2}{2}\sum_{t=p+1}^{T}(\widehat{c_t^2} - (\widehat{c}_t)^2). \end{aligned}$$

The first-order derivatives related to $\boldsymbol{\Delta}$ in (10) are stated as

$$\begin{aligned} \frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}} &= \sum_{t=p+1}^{\top}\left(\frac{(1+\lambda^2)\omega_t u_t}{\sigma^2} - \frac{\lambda\sqrt{\omega_t\,(1+\lambda^2)}}{\sigma}\widehat{c}_t\right)\boldsymbol{x}_t^\top \\ \frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2} &= \sum_{t=p+1}^{\top}\left(-\frac{1}{2\sigma^2} + \frac{\left(1+\lambda^2\right)\omega_t u_t^2}{2\sigma^4} - \frac{\lambda\sqrt{\omega_t\,(1+\lambda^2)}}{2\sigma^3}\widehat{c}_t u_t\right) \\ \frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda} &= \sum_{t=p+1}^{\top}\left(\frac{\lambda}{1+\lambda^2} - \frac{\omega_t {u_t}^2}{\sigma^2}\lambda - \lambda\widehat{c_t^2} + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{\sqrt{\omega_t}u_t}{\sigma}\widehat{c}_t\right) \end{aligned}$$

The second-order derivatives corresponding to $\boldsymbol{\Delta}$ in (10) are written as

$$\begin{aligned} \frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\boldsymbol{\beta}\partial\omega_t} &= \left(\frac{\left(1+\lambda^2\right)}{\sigma^2}u_t - \frac{\lambda\sqrt{1+\lambda^2}}{\sigma}\frac{1}{2\sqrt{\omega_t}}\widehat{c}_t\right)\boldsymbol{x}_t{}^\top \\ \frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\sigma^2\partial\omega_t} &= \frac{\left(1+\lambda^2\right)u_t^2}{2\sigma^4} - \frac{\lambda\sqrt{1+\lambda^2}}{4\sigma^3\sqrt{\omega_t}}\widehat{c}_t u_t \\ \frac{\partial^2 Q(\boldsymbol{\theta},\boldsymbol{\omega})}{\partial\lambda\partial\omega_t} &= -\frac{{u_t}^2\lambda}{\sigma^2} + \frac{2\lambda^2+1}{\sqrt{1+\lambda^2}}\frac{u_t}{2\sqrt{\omega_t}}\frac{\widehat{c}_t}{\sigma} \end{aligned}$$

Noting that $\omega_0 = (1,\ldots,1)^\top$, we get the expression established in (10).