Supporting Information for "Principal Component Analysis of Hybrid Functional and Vector Data"

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Abstract

This Supporting Information contains the proofs of Theorems 1–7 (Appendix S1), specification of the vector part of the first 10 hybrid PCs used in simulation studies (Section 3) of the main text (Appendix S2), and a figure referenced in Section 1 of the main text (Figure S1).

Appendix S1: Proof of Theorems

- Abbreviations
  - "CS": Cauchy–Schwarz inequality
  - "Fubini": Fubini's Theorem
  - "HS": Hilbert-Schmidt Theorem
  - "DCT": Dominated Convergence Theorem
  - "Markov": Markov's Inequality

- **Proof of Theorem 1**

1. $\mathcal{K}$ is positive.

$$
\begin{aligned}
\langle \mathcal{K}\mathbf{h}, \mathbf{h}\rangle_{\mathcal{H}} &= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \sigma_y^{(uk)}(s_u, t_k) f^{(u)}(s_u) ds_u + \sum_{r=1}^{p} \sigma_{yx}^{(k)}(t_k, r) v_r \right\} f^{(k)}(t_k) dt_k \\
&\quad + \sum_{r=1}^{p} \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \sigma_{yx}^{(u)}(s_u, r) f^{(u)}(s_u) ds_u + \sum_{q=1}^{p} \sigma_x(q, r) v_q \right\} v_r \\
&= \sum_{k=1}^{K} \sum_{u=1}^{K} \int_{\mathcal{T}_k} \int_{\mathcal{T}_u} \sigma_y^{(uk)}(s_u, t_k) f^{(u)}(s_u) f^{(k)}(t_k) ds_u dt_k \\
&\quad + 2 \sum_{k=1}^{K} \sum_{r=1}^{p} \int_{\mathcal{T}_k} \sigma_{yx}^{(k)}(t_k, r) v_r f^{(k)}(t_k) dt_k + \sum_{r=1}^{p} \sum_{q=1}^{p} \sigma_x(q, r) v_q v_r \\
&= \sum_{k=1}^{K} \sum_{u=1}^{K} \int_{\mathcal{T}_k} \int_{\mathcal{T}_u} \mathrm{E}\{Y^{(u)}(s_u) Y^{(k)}(t_k)\} f^{(u)}(s_u) f^{(k)}(t_k) ds_u dt_k \\
&\quad + 2 \sum_{k=1}^{K} \sum_{r=1}^{p} \int_{\mathcal{T}_k} \mathrm{E}\{Y^{(k)}(t_k) X_r\} v_r f^{(k)}(t_k) dt_k + \sum_{r=1}^{p} \sum_{q=1}^{p} \mathrm{E}(X_q X_r) v_q v_r \\
&\overset{\text{Fubini}}{=} \mathrm{E}\Bigg[\left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} Y^{(k)}(t_k) f^{(k)}(t_k) dt_k \right\}^2 \\
&\quad + 2 \left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} Y^{(k)}(t_k) f^{(k)}(t_k) dt_k \right\} \cdot \left(\sum_{r=1}^{p} X_r v_r \right) + \left(\sum_{r=1}^{p} X_r v_r \right)^2 \Bigg] \\
&= \mathrm{E}\left[\left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} Y^{(k)}(t_k) f^{(k)}(t_k) dt_k + \sum_{r=1}^{p} X_r v_r \right\}^2 \right] \geq 0
\end{aligned}
$$

2. $\mathcal{K}$ is self-adjoint

$$
\begin{aligned}
\langle \mathcal{K}\mathbf{h}_1, \mathbf{h}_2\rangle_{\mathcal{H}} &= \sum_{k=1}^{K}\int_{\mathcal{T}_k}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_y^{(uk)}(s_u,t_k)f_1^{(u)}(s_u)ds_u + \sum_{r=1}^{p}\sigma_{yx}^{(k)}(t_k,r)v_{1r}\right\}f_2^{(k)}(t_k)dt_k \\
&+\sum_{r=1}^{p}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_{yx}^{(u)}(s_u,r)f_1^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\sigma_x(q,r)v_{1q}\right\}v_{2r} \\
&= \sum_{u=1}^{K}\int_{\mathcal{T}_u}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_y^{(uk)}(s_u,t_k)f_2^{(k)}(t_k)dt_k + \sum_{r=1}^{p}\sigma_{yx}^{(u)}(s_u,r)v_{2r}\right\}f_1^{(u)}(s_u)ds_u \\
&+\sum_{r=1}^{p}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_{yx}^{(k)}(t_k,r)f_2^{(k)}(t_k)dt_k + \sum_{q=1}^{p}\sigma_x(q,r)v_{2q}\right\}v_{1r} \\
&= \langle \mathbf{h}_1, \mathcal{K}\mathbf{h}_2\rangle_{\mathcal{H}}.
\end{aligned}
$$

3. $\mathcal{K}$ is compact

We will show that an image of a family of bounded functions under $\mathcal{K}$ is uniformly bounded and equicontinuous, and apply the Arzelá-Ascoli Theorem (Rudin, 1976) to show $\mathcal{K}$ is compact. Let $\mathcal{B} = \{\mathbf{h} \in \mathcal{H} : \|\mathbf{h}\|_{\mathcal{H}}^2 \le B < \infty\}$ denote a bounded family in $\mathcal{H}$. Clearly, $\mathbf{h} = (f, \mathbf{v}) \in \mathcal{B}$ implies $\|f\|_{\mathcal{F}}^2 \le B$ and $\|\mathbf{v}\|^2 \le B$. Define $\mathcal{I} = \{\mathcal{K}\mathbf{h} : \mathbf{h} \in \mathcal{B}\}$ as an image of $\mathcal{B}$ under $\mathcal{K}$. We first show that $\mathcal{I}$ is a family of uniformly bounded functions with respect to arguments on $\mathcal{T}$. Let $\mathbf{g} \in \mathcal{I}$ and $C = \min(C_1, C_2, C_3, C_4)$. Then, for all $\mathbf{t} \in \mathcal{T}$, we can establish:

$$
\begin{aligned}
\|\mathbf{g}[\mathbf{t}]\|^2 &= \sum_{k=1}^{K}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}}\sigma_y^{(uk)}(s_u,t_k)f^{(u)}(s_u)ds_u + \sum_{r=1}^{p}\sigma_{yx}^{(k)}(t_k,r)v_r\right\}^2 \\
&+\sum_{r=1}^{p}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_{yx}^{(u)}(s_u,r)f^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\sigma_x(q,r)v_q\right\}^2 \\
&\overset{\text{CS}}{\le} \sum_{k=1}^{K}\left[\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_y^{(uk)}(s_u,t_k)^2ds_u\right\}^{\frac{1}{2}}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}f^{(u)}(s_u)^2ds_u\right\}^{\frac{1}{2}} + \left\{\sum_{r=1}^{p}\sigma_{yx}^{(k)}(t_k,r)^2\right\}^{\frac{1}{2}}\left(\sum_{r=1}^{p}v_r^2\right)^{\frac{1}{2}}\right]^2 \\
&+\sum_{r=1}^{p}\left[\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_{yx}^{(u)}(s_u,r)^2ds_u\right\}^{\frac{1}{2}}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}f^{(u)}(s_u)^2ds_u\right\}^{\frac{1}{2}} + \left\{\sum_{q=1}^{p}\sigma_x(q,r)^2\right\}^{\frac{1}{2}}\left(\sum_{q=1}^{p}v_q^2\right)^{\frac{1}{2}}\right]^2
\end{aligned}
$$

$$= \sum_{k=1}^{K} \left\{ \|\sigma_y^{(k)}(\cdot, t_k)\|_{\mathcal{F}} \|f\|_{\mathcal{F}} + \|\sigma_{yx}^{(k)}(t_k, \cdot)\| \|\mathbf{v}\| \right\}^2 + \sum_{r=1}^{p} \{ \|\sigma_{yx}(\cdot, r)\|_{\mathcal{F}} \|f\|_{\mathcal{F}} + \|\sigma_x(\cdot, r)\| \|\mathbf{v}\| \}^2$$

$$\leq \sum_{k=1}^{K} (C_1^{1/2} B^{1/2} + C_2^{1/2} B^{1/2})^2 + \sum_{r=1}^{p} (C_3^{1/2} B^{1/2} + C_4^{1/2} B^{1/2})^2 \leq 4BC(K+p) < \infty.$$

We now show that $\mathcal{I}$ is equicontinuous. Denote the Lebesgue measure of $\mathcal{T}_k$ by $\mu(\mathcal{T}_k)$ and denote $T = \max_{k=1,\ldots,K} \mu(\mathcal{T}_k)$. Define

$$\tilde{\epsilon} = \frac{\epsilon^{1/2}}{(KB)^{1/2}(K^{1/2}T^{1/2} + p^{1/2})} \quad \text{and} \quad \delta = \min \left(\min_{u,k=1,\ldots,K} \delta^{(uk)}, \min_{\substack{k=1,\ldots,K \\ r=1,\ldots,p}} \delta_r^{(k)} \right),$$

for any $\epsilon > 0$. Then, for any $\mathbf{t}, \mathbf{t}^* \in \mathcal{T}$ such that $\|\mathbf{t} - \mathbf{t}^*\| < \delta$, we can establish:

$$\begin{aligned}
\|\mathbf{g}[\mathbf{t}] - \mathbf{g}[\mathbf{t}^*]\|^2 &= \sum_{k=1}^{K} \left[\sum_{u=1}^{K} \int_{\mathcal{T}_u} \left\{ \sigma_y^{(uk)}(s_u, t_k) - \sigma_y^{(uk)}(s_u, t_k^*) \right\} f^{(u)}(s_i) ds_u \right. \\
&\qquad \left. + \sum_{r=1}^{p} \left\{ \sigma_{yx}^{(k)}(t_k, r) - \sigma_{yx}^{(k)}(t_k^*, r) \right\} v_r \right]^2 \\
&\overset{\text{CS}}{\leq} \sum_{k=1}^{K} \left(\left[\sum_{u=1}^{K} \int_{\mathcal{T}_u} \{ \sigma_y^{(uk)}(s_u, t_k) - \sigma_y^{uk}(s_u, t_k^*) \}^2 ds_u \right]^{1/2} \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} f^{(u)}(s_u)^2 ds_u \right\}^{1/2} \right. \\
&\qquad \left. + \left[\sum_{r=1}^{p} \{ \sigma_{yx}^{(k)}(t_k, r) - \sigma_{yx}^{(k)}(t_k^*, r) \}^2 \right]^{1/2} \left(\sum_{r=1}^{p} v_r^2 \right)^{1/2} \right)^2 \\
&< \sum_{k=1}^{K} \left\{ \left(\sum_{u=1}^{K} \int_{\mathcal{T}_u} \tilde{\epsilon}^2 ds_u \right)^{1/2} \|f\|_{\mathcal{F}} + \left(\sum_{r=1}^{p} \tilde{\epsilon}^2 \right)^{1/2} \|\mathbf{v}\| \right\}^2 \\
&< \sum_{k=1}^{K} (\tilde{\epsilon} K^{1/2} T^{1/2} B^{1/2} + \tilde{\epsilon} p^{1/2} B^{1/2})^2 = \tilde{\epsilon}^2 KB(K^{1/2}T^{1/2} + p^{1/2})^2 = \epsilon.
\end{aligned}$$

Now, we can apply the Arzelá-Ascoli Theorem to conclude that, for any bounded sequence $\{\mathbf{h}_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$, the sequence $(\mathcal{K}\mathbf{h}_k)_{k\in\mathbb{N}}$ contains a converging subsequence. Thus $\mathcal{K}$ is a compact operator.

□

· **Proof of Theorem 2**

We will extend the technique used in Happ and Greven (2018) to prove the theorem. From Spectral theorem for positive compact self-adjoint operators (Hsing & Eubank, 2015), it is known that

$$\mathcal{K}\mathbf{h} = \sum_{m=1}^{\infty} \lambda_m \langle \mathbf{h}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \boldsymbol{\xi}_m, \quad \text{for all } \mathbf{h} \in \mathcal{H},$$

where $\lambda_1 \geq \lambda_2, \dots, \geq 0$ and, $\boldsymbol{\xi}_m$'s are orthogonal in the sense that $\langle \boldsymbol{\xi}_m, \boldsymbol{\xi}_n \rangle_{\mathcal{H}} = \delta_{mn}$. For $M \in \mathbb{N}$, define:

$$\mathcal{K}_M \mathbf{h} = \sum_{m=1}^{M} \lambda_m \langle \mathbf{h}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \boldsymbol{\xi}_m.$$

Then, for all $\mathbf{h} \in \mathcal{H}$, we have $\langle \mathcal{K}\mathbf{h}, \mathbf{h} \rangle_{\mathcal{H}} - \langle \mathcal{K}_M \mathbf{h}, \mathbf{h} \rangle_{\mathcal{H}} = \sum_{m=M+1}^{\infty} \lambda_m \langle \mathbf{h}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}}^2 \geq 0$.

For the first result, define $\tilde{\mathbf{h}}^{(k)} = (0, \dots, 0, \tilde{f}^{(k)}, 0, \dots, 0) \in \mathcal{H}$, $k = 1, \dots, K$, where $\tilde{f}^{(k)} = \mu\{B_{t_k^*}(1/n)\}^{-1} I\{B_{t_k^*}(1/n)\} \in L^2(\mathcal{T}_k)$, and other functional and vector elements are equal to zero. $B_{t_k^*}(1/n)$ is a closed ball in $\mathcal{T}_k$ with center t_k^* and radius $1/n$, $\mu(\cdot)$ denotes a Lebesgue measure, $I(\cdot)$ is an indicator function, and $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
0 &\leq \langle \mathcal{K}\tilde{\mathbf{h}}^{(k)}, \tilde{\mathbf{h}}^{(k)} \rangle_{\mathcal{H}} - \langle \mathcal{K}_M \tilde{\mathbf{h}}^{(k)}, \tilde{\mathbf{h}}^{(k)} \rangle_{\mathcal{H}} \\
&= \int_{\mathcal{T}_k} \left\{ \int_{\mathcal{T}_k} \sigma_y^{(kk)}(s_k, t_k) \tilde{f}^{(k)}(s_k) ds_k \right\} \tilde{f}^{(k)}(t_k) dt_k - \sum_{m=1}^{M} \lambda_m \langle \tilde{\mathbf{h}}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}}^2 \\
&= \int_{\mathcal{T}_k} \int_{\mathcal{T}_k} \sigma_y^{(kk)}(s_k, t_k) \tilde{f}^{(k)}(s_k) \tilde{f}^{(k)}(t_k) ds_k dt_k \\
&\quad - \int_{\mathcal{T}_k} \int_{\mathcal{T}_k} \left\{ \sum_{m}^{M} \lambda_m \psi_m^{(k)}(s_k) \psi_m^{(k)}(t_k) \right\} \tilde{f}^{(k)}(s_k) \tilde{f}^{(k)}(t_k) ds_k dt_k \\
&= \mu \left\{ B_{t_k^*}\left(\frac{1}{n}\right) \right\}^{-2} \int_{B_{t_k^*}(\frac{1}{n})} \int_{B_{t_k^*}(\frac{1}{n})} \left\{ \sigma_y^{(kk)}(s_k, t_k) - \sum_{m=1}^{M} \lambda_m \psi_m^{(k)}(s_k) \psi_m^{(k)}(t_k) \right\} ds_k dt_k \\
&\longrightarrow \sigma_y^{(kk)}(t_k^*, t_k^*) - \sum_{m=1}^{M} \lambda_m \psi_m^{(k)}(t_k^*) \psi_m^{(k)}(t_k^*),
\end{aligned}$$

where the last convergence result follows from the Lebesgue Differentiation Theorem (Rudin, 1987). As t_k^* is an arbitrary value in $\mathcal{T}_k$, this implies that for all $M \in \mathbb{N}$ and $t_k \in \mathcal{T}_k$,

$$\sum_{m=1}^{M} \lambda_m \psi_m^{(k)}(t)^2 \leq \sigma_y^{(kk)}(t_k, t_k) \leq \left\|\sigma_y^{(kk)}\right\|_\infty < \infty,$$

where $\left\|\sigma_y^{(kk)}\right\|_\infty = \sup_{t_k \in \mathcal{T}_k} |\sigma_y^{(kk)}(t_k, t_k)|$, which is finite because $\sigma_y^{(kk)}$ is continuous, and $\mathcal{T}_k$ is compact. Then, by CS,

$$\begin{aligned} \sum_{m=1}^{\infty} |\lambda_m \psi_m^{(k)}(s_k) \psi_m^{(k)}(t_k)| &\leq \left\{\sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(s_k)^2\right\}^{1/2} \left\{\sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(t_k)^2\right\}^{1/2} \\ &< \sigma_y^{(kk)}(s_k, s_k)^{1/2} \sigma_y^{(kk)}(t_k, t_k)^{1/2} < \infty, \end{aligned}$$

which implies that the series $c^{(kk)}(s_k, t_k) = \sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(s_k) \psi_m^{(k)}(t_k)$ is absolutely convergent for all $s_k, t_k \in \mathcal{T}_k$. Now let $t_k \in \mathcal{T}_k$ be fixed in the following. For any $\epsilon > 0$, choose $M \in \mathbb{N}$ (may depend on ϵ) such that $\sum_{m=M+1}^{\infty} \lambda_m \psi_m^{(k)}(t_k)^2 < \epsilon^2$ for fixed $t_k \in \mathcal{T}_k$. Then again by CS,

$$\begin{aligned} \sum_{m=1}^{\infty} |\lambda_m \psi_m^{(k)}(s_k) \psi_m^{(k)}(t_k)| &\leq \left\{\sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(s_k)^2\right\}^{1/2} \left\{\sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(t_k)^2\right\}^{1/2} \\ &< \sigma_y^{(kk)}(s_k, s_k)^{1/2} \cdot \epsilon \\ &< \left\|\sigma_y^{(kk)}\right\|_\infty^{1/2} \cdot \epsilon. \end{aligned} \tag{1}$$

The upper bound in (1) does not depend on s_k, so the series $c^{(kk)}(s_k, t_k)$ converges uniformly in s_k for fixed t_k. Note that since $\psi_m^{(k)}(s_k)$ is continuous in s_k (cf. Happ and Greven (2018), Lemma 1), $c^{(kk)}(s_k, t_k)$ is continuous in s_k by the Uniform Limit Theorem (Munkres, 2000). Define $g^{(k)}(s_k) = \sigma_y^{(kk)}(s_k, t_k) - c^{(kk)}(s_k, t_k)$ for $s_k \in \mathcal{T}_k$. Define $\bar{\mathbf{h}}^{(k)} = (0, \ldots, 0, f^{(k)}, 0, \ldots, 0) \in \mathcal{H}$, where $f^{(k)}$ is any function in $L^2(\mathcal{T}_k)$, and other functional and vector elements are equal to zero. Then,

$$\begin{aligned}
\int_{\mathcal{T}_k} g^{(k)}(s_k) f^{(k)}(s_k) ds_k &= \int_{\mathcal{T}_k} \sigma_y^{(kk)}(s_k, t_k) f^{(k)}(s_k) ds_k - \int_{\mathcal{T}_k} c^{(kk)}(s_k, t_k) f^{(k)}(s_k) ds_k \\
&= \int_{\mathcal{T}_k} \sigma_y^{(kk)}(s_k, t_k) f^{(k)}(s_k) ds_k - \sum_{m=1}^{\infty} \lambda_m \left\{ \int_{\mathcal{T}_k} \psi_m^{(k)}(s_k) f^{(k)}(s_k) ds_k \right\} \psi_m^{(k)}(t_k) \\
&= (\mathcal{K}\bar{\mathbf{h}}^{(k)})^{(k)}(t_k) - \sum_{m=1}^{\infty} \lambda_m \left\langle \bar{\mathbf{h}}^{(k)}, \boldsymbol{\xi}_m \right\rangle_{\mathcal{H}} \psi_m^{(k)}(t_k) \\
&\overset{\text{HS}}{=} \sum_{m=1}^{\infty} \lambda_m \left\langle \bar{\mathbf{h}}^{(k)}, \boldsymbol{\xi}_m \right\rangle_{\mathcal{H}} \psi_m^{(k)}(t_k) - \sum_{m=1}^{\infty} \lambda_m \left\langle \bar{\mathbf{h}}^{(k)}, \boldsymbol{\xi}_m \right\rangle_{\mathcal{H}} \psi_m^{(k)}(t_k) = 0
\end{aligned}$$

Choosing $f^{(k)} = g^{(k)}$ implies $g^{(k)}(s_k) = 0$ for all $s_k \in \mathcal{T}_k$ as $g^{(k)}$ is continuous in s_k. Therefore,

$$c^{(kk)}(s_k, t_k) = \sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(s_k) \psi_m^{(k)}(t_k) = \sigma_y^{(kk)}(s_k, t_k), \quad \text{for all } s_k \in \mathcal{T}_k,$$

which implies that the series $\sigma_y^{(kk)}(t_k, t_k) = \sum_{m=1}^{\infty} \lambda_m \psi_m^{(k)}(t_k)^2$ converges uniformly by the Dini's Theorem (Rudin, 1976). Hence, M in (1) can be chosen to not depend on t_k, which implies that the convergence of $\sum_m \lambda_m \psi_m^{(k)}(s_k)\psi_m^{(k)}(t_k)$ to $\sigma_y^{(kk)}(s_k, t_k)$ is absolute and uniform in $s_k, t_k \in \mathcal{T}_k$.

For the second result, define $\tilde{\mathbf{h}}_r = (0, \ldots, 0, v_r, 0, \ldots, 0) \in \mathcal{H}$ where all functional and vector components equal zero except the rth element of the vector, which can take any real value. Then,

$$\left\langle \mathcal{K}\tilde{\mathbf{h}}_r, \tilde{\mathbf{h}}_r \right\rangle_{\mathcal{H}} = \sigma_x(r, r) v_r^2$$

and

$$\left\langle \mathcal{K}_M \tilde{\mathbf{h}}_r, \tilde{\mathbf{h}}_r \right\rangle_{\mathcal{H}} = \left\langle \sum_{m=1}^{M} \lambda_m \langle \tilde{\mathbf{h}}_r, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \, \boldsymbol{\xi}_m, \tilde{\mathbf{h}}_r \right\rangle_{\mathcal{H}} = \sum_{m=1}^{M} \lambda_m \left\langle \tilde{\mathbf{h}}_r, \boldsymbol{\xi}_m \right\rangle_{\mathcal{H}}^2 = \sum_{m=1}^{M} \lambda_m v_r^2 \theta_{mr}^2,$$

which imply

$$0 \le \left\langle \mathcal{K}\tilde{\mathbf{h}}_r, \tilde{\mathbf{h}}_r \right\rangle_{\mathcal{H}} - \left\langle \mathcal{K}_M\tilde{\mathbf{h}}_r, \tilde{\mathbf{h}}_r \right\rangle_{\mathcal{H}} = \sigma_x(r,r)v_r^2 - \sum_{m=1}^{M} \lambda_m \theta_{mr}^2 v_r^2$$
$$\implies \sum_{m=1}^{M} \lambda_m \theta_{mr}^2 \le \sigma_x(r,r) < \infty, \quad \text{for all } m \in \mathbb{N}$$
$$\implies \sum_{m=1}^{M} |\lambda_m \theta_{mr}^2| \le \sigma_x(r,r) < \infty, \quad \text{for all } m \in \mathbb{N}.$$

Thus, the series $c_r = \sum_{m=1}^{\infty} \lambda_m \theta_{mr}^2$ is absolutely convergent. Now set $g_r = \sigma_x(r,r) - c_r$. Then,

$$\begin{aligned} g_r v_r &= \sigma_x(r,r)v_r - c_r v_r \\ &= \sigma_x(r,r)v_r - \sum_{m=1}^{\infty} \lambda_m (v_r\theta_{mr})\theta_{mr} \\ &= (\mathcal{K}\tilde{\mathbf{h}}_r)_r - \sum_{m=1}^{\infty} \lambda_m \langle \boldsymbol{\xi}_m, \tilde{\mathbf{h}}_r \rangle_{\mathcal{H}} \theta_{mr} \\ &\overset{\text{HS}}{=} \sum_{m=1}^{\infty} \lambda_m \langle \boldsymbol{\xi}_m, \tilde{\mathbf{h}}_r \rangle_{\mathcal{H}} \theta_{mr} - \sum_{m=1}^{\infty} \lambda_m \langle \boldsymbol{\xi}_m, \tilde{\mathbf{h}}_r \rangle_{\mathcal{H}} \theta_{mr} = 0 \end{aligned}$$

Set $v_r = g_r$, which implies $g_r = 0$. Therefore $c_r = \sum_{m=1}^{\infty} \lambda_m \theta_{mr}^2 = \sigma_x(r,r)$, and the convergence is absolute.

□

· **Proof of Theorem 3**

Firstly, assuming $\mathrm{E}(\mathbf{Z}) = \mathbf{0}$, we can show that: $\mathrm{E}(\rho_m) = \mathrm{E}\langle \mathbf{Z}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} = \langle \mathrm{E}\mathbf{Z}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} = 0$.

Secondly, we can show that

$$\begin{aligned} \mathrm{Cov}(\rho_m, \rho_n) &= \mathrm{E}\left\{ \langle \mathbf{Z}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \langle \mathbf{Z}, \boldsymbol{\xi}_n \rangle_{\mathcal{H}} \right\} \\ &= \mathrm{E}\left[\left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} Y^{(k)}(t_k)\psi_m^{(k)}(t_k)dt_k + \sum_{r=1}^{p} X_r\theta_{mr} \right\} \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_n^{(u)}(s_u)ds_u + \sum_{q=1}^{p} X_q\theta_{nq} \right\} \right] \\ &= \mathrm{E}\left[\left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} Y^{(k)}(t_k)\psi_m^{(k)}(t_k)dt_k + \sum_{r=1}^{p} X_r\theta_{mr} \right\} \cdot \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_n^{(u)}(s_u)ds_u \right\} \right] \end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{E}\left[\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k} Y^{(k)}(t_k)\psi_m^{(k)}(t_k)dt_k + \sum_{r=1}^{p} X_r\theta_{mr}\right\}\cdot\left\{\sum_{q=1}^{p} X_q\theta_{nq}\right\}\right]\\
&\overset{\text{Fubini}}{=} \sum_{u=1}^{K}\int_{\mathcal{T}_u}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_y^{(uk)}(s_u,t_k)\psi_m^{(k)}(t_k)dt_k + \sum_{r=1}^{p}\sigma_{yx}^{(u)}(s_u,r)\theta_{mr}\right\}\psi_n^{(u)}(s_u)ds_u\\
&+ \sum_{q=1}^{p}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_{yx}^{(k)}(t_k,q)\psi_m^{(k)}(t_k)dt_k + \sum_{r=1}^{p}\sigma_x(q,r)\theta_{mr}\right\}\theta_{nq}\\
&\overset{\text{HS}}{=} \sum_{u=1}^{K}\int_{\mathcal{T}_u}\lambda_m\psi_m^{(u)}(s_u)\psi_n^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\lambda_m\theta_{mq}\theta_{nq}\\
&= \lambda_m\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\psi_m^{(u)}(s_u)\psi_n^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\theta_{mq}\theta_{nq}\right\} = \lambda_m\delta_{mn}
\end{aligned}
$$

Finally, noting that $\rho_m = \langle \mathbf{Z}, \boldsymbol{\xi}_m\rangle_{\mathcal{H}} = \sum_{u=1}^{K}\int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p} X_q\theta_{mq}$, we can show that

$$
\begin{aligned}
&\mathrm{E}\left\|\mathbf{Z}[t] - \sum_{m=1}^{M}\rho_m\boldsymbol{\xi}_m[t]\right\|^2 = \mathrm{E}\left[\sum_{k=1}^{K}\left\{Y^{(k)}(t_k) - \sum_{m=1}^{M}\rho_m\psi_m^{(k)}(t_k)\right\}^2 + \sum_{r=1}^{p}\left(X_r - \sum_{m=1}^{M}\rho_m\theta_{mr}\right)^2\right]\\
&= \sum_{k=1}^{K}\mathrm{E}\{Y^{(k)}(t_k)^2\} - \sum_{k=1}^{K}\mathrm{E}\left[2Y^{(k)}(t_k)\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}X_q\theta_{mq}\right\}\psi_m^{(k)}(t_k)\right]\\
&+ \sum_{k=1}^{K}\mathrm{E}\left(\left[\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}X_q\theta_{mq}\right\}\psi_m^{(k)}(t_k)\right]^2\right)\\
&+ \sum_{r=1}^{p}\mathrm{E}(X_r^2) - \sum_{r=1}^{p}\mathrm{E}\left[2X_r\sum_{m=1}^{M}\left\{\int_{\mathcal{T}_u}Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}X_q\theta_{mq}\right\}\theta_{mr}\right]\\
&+ \sum_{r=1}^{p}\mathrm{E}\left(\left[\sum_{m=1}^{M}\left\{\int_{\mathcal{T}_u}Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}X_q\theta_{mq}\right\}\theta_{mr}\right]^2\right)\\
&= \sum_{k=1}^{K}\sigma_y^{(kk)}(t_k,t_k) - ① + ② + \sum_{r=1}^{p}\sigma_x(r,r) - ③ + ④,
\end{aligned}
$$

where

$$
\begin{aligned}
① &= \sum_{k=1}^{K}\mathrm{E}\left[2Y^{(k)}(t_k)\sum_{m=1}^{M}\left\{\int_{\mathcal{T}_u}Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}X_q\theta_{mq}\right\}\psi_m^{(k)}(t_k)\right]\\
&\overset{\text{Fubini}}{=} 2\sum_{k=1}^{K}\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_y^{(uk)}(s_u,t_k)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\sigma_{yx}^{(k)}(t_k,q)\theta_{mq}\right\}\psi_m^{(k)}(t_k)
\end{aligned}
$$

$$\overset{\text{HS}}{=} 2\sum_{k=1}^{K}\sum_{m=1}^{M}\lambda_m \psi_m^{(k)}(t_k)^2,$$

$$
\begin{aligned}
② &= \sum_{k=1}^{K}\mathrm{E}\left(\left[\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p} X_q\theta_{mq}\right\}\psi_m^{(k)}(t_k)\right]^2\right) \\
&= \sum_{k=1}^{K}\mathrm{E}\Bigg(\left[\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p} X_q\theta_{mq}\right\}\psi_m^{(k)}(t_k)\right] \\
&\quad \times \left[\sum_{n=1}^{M}\left\{\sum_{j=1}^{K}\int_{\mathcal{T}_j} Y^{(j)}(s_j)\psi_n^{(j)}(s_j)ds_j + \sum_{i=1}^{p} X_i\theta_{ni}\right\}\psi_n^{(k)}(t_k)\right]\Bigg) \\
&= \sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{n=1}^{M}\mathrm{E}\Bigg[\Bigg\{\sum_{u=1}^{K}\sum_{j=1}^{K}\int_{\mathcal{T}_u}\int_{\mathcal{T}_j} Y^{(u)}(s_u)Y^{(j)}(s_j)\psi_m^{(u)}(s_u)\psi_n^{(j)}(s_j)ds_u ds_j \\
&\quad + \sum_{u=1}^{K}\int_{\mathcal{T}_u}\sum_{i=1}^{p} Y^{(u)}(s_u)X_i\psi_m^{(u)}(s_u)\theta_{ni}ds_u + \sum_{j=1}^{K}\int_{\mathcal{T}_j}\sum_{q=1}^{p} Y^{(j)}(s_j)X_q\psi_n^{(j)}(s_j)\theta_{mq}ds_j \\
&\quad + \sum_{q=1}^{p}\sum_{i=1}^{p} X_q X_i\theta_{mq}\theta_{ni}\Bigg\}\psi_m^{(k)}(t_k)\psi_n^{(k)}(t_k)\Bigg] \\
&\overset{\text{Fubini}}{=} \sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{n=1}^{M}\Bigg[\sum_{u=1}^{K}\int_{\mathcal{T}_u}\left\{\int_{\mathcal{T}_j}\sigma_y^{(uj)}(s_u,s_j)\psi_n^{(j)}(s_j)ds_j + \sum_{i=1}^{p}\sigma_{yx}^{(u)}(s_u,i)\theta_{ni}\right\}\psi_m^{(u)}(s_u)ds_u \\
&\quad + \sum_{q=1}^{p}\left\{\sum_{j=1}^{K}\int_{\mathcal{T}_j}\sigma_{yx}^{(j)}(s_j,q)\psi_n^{(j)}(s_j)ds_j + \sum_{i=1}^{p}\sigma_x(q,i)\theta_{ni}\right\}\theta_{mq}\Bigg]\psi_m^{(k)}(t_k)\psi_n^{(k)}(t_k) \\
&\overset{\text{HS}}{=} \sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{n=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\lambda_n\psi_n^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\lambda_n\theta_{nq}\theta_{mq}\right\}\psi_m^{(k)}(t_k)\psi_n^{(k)}(t_k) \\
&= \sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{n=1}^{M}\lambda_n\langle\boldsymbol{\xi}_m,\boldsymbol{\xi}_n\rangle_{\mathcal{H}}\psi_m^{(k)}(t_k)\psi_n^{(k)}(t_k) \\
&= \sum_{k=1}^{K}\sum_{m=1}^{M}\lambda_m\psi_m^{(k)}(t_k)^2,
\end{aligned}
$$

$$
\begin{aligned}
③ &= \sum_{r=1}^{p}\mathrm{E}\left[2X_r\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}X_q\theta_{mq}\right\}\theta_{mr}\right] \\
&\overset{\text{Fubini}}{=} 2\sum_{r=1}^{p}\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\sigma_{yx}^{(u)}(s_u,r)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\sigma_x(r,q)\theta_{mq}\right\}\theta_{mr} \\
&\overset{\text{HS}}{=} 2\sum_{r=1}^{p}\sum_{m=1}^{M}\lambda_m\theta_{mr}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
④ &= \sum_{r=1}^{p} \mathrm{E}\left(\left[\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p} X_q\theta_{mq}\right\}\theta_{mr}\right]^2\right) \\
&= \sum_{r=1}^{p} \mathrm{E}\Bigg(\left[\sum_{m=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} Y^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p} X_q\theta_{mq}\right\}\theta_{mr}\right] \\
&\quad \times \left[\sum_{n=1}^{M}\left\{\sum_{j=1}^{K}\int_{\mathcal{T}_j} Y^{(j)}(s_j)\psi_n^{(j)}(s_j)ds_u + \sum_{i=1}^{p} X_i\theta_{ni}\right\}\theta_{nr}\right]\Bigg) \\
&= \sum_{r=1}^{p}\sum_{m=1}^{M}\sum_{n=1}^{M} \mathrm{E}\Bigg[\Bigg\{\sum_{u=1}^{K}\sum_{j=1}^{K}\int_{\mathcal{T}_u}\int_{\mathcal{T}_j} Y^{(u)}(s_u)Y^{(j)}(s_j)\psi_m^{(u)}(s_u)\psi_n^{(j)}(s_j)ds_u ds_j \\
&\quad + \sum_{u=1}^{K}\int_{\mathcal{T}_u}\sum_{i=1}^{p} Y^{(u)}(s_u)X_i\psi_m^{(u)}(s_u)\theta_{ni}ds_u + \sum_{j=1}^{K}\int_{\mathcal{T}_j}\sum_{q=1}^{p} Y^{(j)}(s_j)X_q\psi_n^{(j)}(s_j)\theta_{mq}ds_j \\
&\quad + \sum_{q=1}^{p}\sum_{i=1}^{p} X_q X_i\theta_{mq}\theta_{ni}\Bigg\}\theta_{mr}\theta_{nr}\Bigg] \\
&\overset{\text{Fubini}}{=} \sum_{r=1}^{p}\sum_{m=1}^{M}\sum_{n=1}^{M}\Bigg[\sum_{u=1}^{K}\int_{\mathcal{T}_u}\left\{\sum_{j=1}^{K}\int_{\mathcal{T}_j}\sigma_y^{(uj)}(s_u,s_j)\psi_n^{(j)}(s_j)ds_j + \sum_{i=1}^{p}\sigma_{yx}^{(u)}(s_u,i)\theta_{ni}\right\}\psi_m^{(u)}(s_u)ds_u \\
&\quad + \sum_{q=1}^{p}\left\{\sum_{j=1}^{K}\int_{\mathcal{T}_j}\sigma_{yx}^{(j)}(s_j,q)\psi_n^{(j)}(s_j)ds_j + \sum_{i=1}^{p}\sigma_x(q,i)\theta_{ni}\right\}\theta_{mq}\Bigg]\theta_{mr}\theta_{nr} \\
&\overset{\text{HS}}{=} \sum_{r=1}^{p}\sum_{m=1}^{M}\sum_{n=1}^{M}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\lambda_n\psi_n^{(u)}(s_u)\psi_m^{(u)}(s_u)ds_u + \sum_{q=1}^{p}\lambda_n\theta_{nq}\theta_{mq}\right\}\theta_{mr}\theta_{nr} \\
&= \sum_{r=1}^{p}\sum_{m=1}^{M}\sum_{n=1}^{M}\lambda_n\langle\boldsymbol{\xi}_n,\boldsymbol{\xi}_m\rangle_{\mathcal{H}}\theta_{mr}\theta_{nr} \\
&= \sum_{r=1}^{p}\sum_{m=1}^{M}\lambda_m\theta_{mr}^2.
\end{aligned}
$$

Combining these results, we can establish that

$$
\begin{aligned}
\mathrm{E}\left\|\mathbf{Z}[t] - \sum_{m=1}^{M}\rho_m\boldsymbol{\xi}_m[t]\right\|^2 &= \sum_{k=1}^{K}\sigma_y^{(kk)}(t_k,t_k) - ① + ② + \sum_{r=1}^{p}\sigma_x(r,r) - ③ + ④ \\
&= \sum_{k=1}^{K}\sigma_y^{(kk)}(t_k,t_k) - 2\sum_{k=1}^{K}\sum_{m=1}^{M}\lambda_m\psi_m^{(k)}(t_k)^2 + \sum_{k=1}^{K}\sum_{m=1}^{M}\lambda_m\psi_m^{(k)}(t_k)^2 \\
&\quad + \sum_{r=1}^{p}\sigma_x(r,r) - 2\sum_{r=1}^{p}\sum_{m=1}^{M}\lambda_m\theta_{mr}^2 + \sum_{r=1}^{p}\sum_{m=1}^{M}\lambda_m\theta_{mr}^2
\end{aligned}
$$

$$= \sum_{k=1}^{K} \left\{ \sigma_y^{(kk)}(t_k, t_k) - \sum_{m=1}^{M} \lambda_m \psi_m^{(k)}(t_k)^2 \right\} + \sum_{r=1}^{p} \left\{ \sigma_x(r, r) - \sum_{m=1}^{M} \lambda_m \theta_{mr}^2 \right\}$$

$$\longrightarrow 0 \quad \text{as} \quad M \longrightarrow \infty,$$

where the convergence follows from Theorem 2.

□

· Proof of Theorem 4

For notational simplicity, we drop the "tilde" ($\sim$) throughout the proof. By the Hilbert-Schmidt theorem, elements of the functional part of $(\mathcal{K}\boldsymbol{\xi})[\mathbf{t}]$—i.e., $(\mathcal{K}\boldsymbol{\xi})^{(k)}(t_k)$, $k = 1, \ldots, K$—can be expressed as

$$\begin{aligned}
(\mathcal{K}\boldsymbol{\xi})^{(k)}(t_k) &= \sum_{u=1}^{K} \int_{\mathcal{T}_u} \sigma_y^{(uk)}(s_u, t_k)\psi^{(u)}(s_u)ds_u + \sum_{r=1}^{p} \sigma_{yx}^{(k)}(t_k, r)\theta_r \\
&= \sum_{u=1}^{K} \int_{\mathcal{T}_u} \text{Cov}\{Y^{(u)}(s_u), Y^{(k)}(t_k)\}\psi^{(u)}(s_u)ds_u + \sum_{r=1}^{p} \text{Cov}\{Y^{(k)}(t_k), X_r\}\theta_r \\
&= \sum_{u=1}^{K} \int_{\mathcal{T}_u} \text{Cov}\left\{ \sum_{l=1}^{L} \eta_l \phi_l^{(u)}(s_u), \sum_{g=1}^{L} \eta_g \phi_g^{(k)}(t_k) \right\} \psi^{(u)}(s_u)ds_u \\
&\quad + \sum_{r=1}^{p} \text{Cov}\left\{ \sum_{g=1}^{L} \eta_g \phi_g^{(k)}(t_k), \sum_{j=1}^{J} \gamma_j w_{jr} \right\} \theta_r \\
&= \sum_{u=1}^{K} \int_{\mathcal{T}_u} \sum_{l=1}^{L} \sum_{g=1}^{L} \text{Cov}(\eta_l, \eta_g)\phi_l^{(u)}(s_u)\phi_g^{(k)}(t_k)\psi^{(u)}(s_u)ds_u \\
&\quad + \sum_{r=1}^{p} \sum_{g=1}^{L} \sum_{j=1}^{J} \text{Cov}(\eta_g, \gamma_j)\phi_g^{(k)}(t_k)w_{jr}\theta_r \\
&= \sum_{g=1}^{L} \phi_g^{(k)}(t_k) \left[\sum_{l=1}^{L} \text{Cov}(\eta_l, \eta_g) \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \phi_l^{(u)}(s_u)\psi^{(u)}(s_u)ds_u \right\} \right. \\
&\quad \left. + \sum_{j=1}^{J} \text{Cov}(\eta_g, \gamma_j) \sum_{r=1}^{p} w_{jr}\theta_r \right] \\
&= \sum_{g=1}^{L} \phi_g^{(k)}(t_k) \left\{ \sum_{l=1}^{L} \text{Cov}(\eta_l, \eta_g)c_l + \sum_{j=1}^{J} \text{Cov}(\eta_g, \gamma_j)d_j \right\} \\
&\overset{\text{HS}}{=} \lambda\psi^{(k)}(t_k) \ \cdots\cdots \ \text{Ⓐ}
\end{aligned}$$

where $c_l = \sum_{u=1}^{K} \int_{\mathcal{T}_u} \phi_l^{(u)}(s_u)\psi^{(u)}(s_u)ds_u$ and $d_j = \mathbf{w}_j^T \boldsymbol{\theta} = \sum_{r=1}^{p} w_{jr}\theta_r$.

Then, generalizing the augmentation approach of Zemyan (2012) and Happ and Greven (2018) to a hybrid setting, we multiply both sides of Ⓐ by $\phi_h^{(k)}(t_k)$, integrate with respect to $t_k \in \mathcal{T}_k$ and sum over $k = 1, \ldots, K$, so that:

$$\begin{aligned}
&\sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_h^{(k)}(t_k) \left[\sum_{g=1}^{L} \phi_g^{(k)}(t_k) \left\{ \sum_{l=1}^{L} \mathrm{Cov}(\eta_l, \eta_g) c_l + \sum_{j=1}^{J} \mathrm{Cov}(\eta_g, \gamma_j) d_j \right\} \right] dt_k \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_h^{(k)}(t_k) \lambda \psi^{(k)}(t_k) dt_k \\
&\Longrightarrow \sum_{g=1}^{L} \left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_h^{(k)}(t_k)\phi_g^{(k)}(t_k) dt_k \right\} \left\{ \sum_{l=1}^{L} \mathrm{Cov}(\eta_l, \eta_g) c_l + \sum_{j=1}^{J} \mathrm{Cov}(\eta_g, \gamma_j) d_j \right\} \\
&\qquad = \lambda \sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_h^{(k)}(t_k) \psi^{(k)}(t_k) dt_k \\
&\Longrightarrow \sum_{l=1}^{L} \mathrm{Cov}(\eta_l, \eta_h) c_l + \sum_{j=1}^{J} \mathrm{Cov}(\eta_h, \gamma_j) d_j = \lambda c_h \\
&\Longrightarrow \sum_{l=1}^{L} V_y^{(hl)} c_l + \sum_{j=1}^{J} V_{yx}^{(hj)} d_j = \lambda c_h \quad \cdots\cdots \quad ①
\end{aligned}$$

where $V_y^{(hl)} = \mathrm{Cov}(\eta_h, \eta_l)$ and $V_{yx}^{(hj)} = \mathrm{Cov}(\eta_h, \gamma_j)$.

Also by the Hilbert-Schmidt theorem, the elements of the vector part of $(\mathcal{K}\boldsymbol{\xi})[\mathbf{t}]$—i.e., $(\mathcal{K}\boldsymbol{\xi})_r$, $r = 1, \ldots, p$—can be expressed as

$$\begin{aligned}
(\mathcal{K}\boldsymbol{\xi})_r &= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \sigma_{yx}^{(k)}(t_k, r)\psi^{(k)}(t_k)dt_k + \sum_{q=1}^{p} \sigma_x(q, r)\theta_q \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \mathrm{Cov}\{Y^{(k)}(t_k), X_r\}\psi^{(k)}(t_k)dt_k + \sum_{q=1}^{p} \mathrm{Cov}(X_q, X_r)\theta_q \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \mathrm{Cov}\left\{ \sum_{h=1}^{L} \eta_h \phi_h^{(k)}(t_k), \sum_{d=1}^{J} \gamma_d w_{dr} \right\} \psi^{(k)}(t_k)dt_k + \sum_{q=1}^{p} \mathrm{Cov}\left\{ \sum_{d=1}^{J} \gamma_d w_{dr}, \sum_{i=1}^{J} \gamma_i w_{iq} \right\} \theta_q \\
&= \sum_{h=1}^{L}\sum_{d=1}^{J} \mathrm{Cov}(\eta_h, \gamma_d) w_{dr} \left\{ \sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_h^{(k)}(t_k)\psi^{(k)}(t_k)dt_k \right\} + \sum_{d=1}^{J}\sum_{i=1}^{J} \mathrm{Cov}(\gamma_d, \gamma_i) w_{dr} (\mathbf{w}_i^T \boldsymbol{\theta}) \\
&= \sum_{d=1}^{J} w_{dr} \left\{ \sum_{h=1}^{L} \mathrm{Cov}(\eta_h, \gamma_d) c_h + \sum_{i=1}^{J} \mathrm{Cov}(\gamma_d, \gamma_i) d_i \right\} \overset{\mathrm{HS}}{=} \lambda \theta_r
\end{aligned}$$

Concatenating these p elements as $(\mathcal{K}\boldsymbol{\xi}) = [(\mathcal{K}\boldsymbol{\xi})_1, \ldots, (\mathcal{K}\boldsymbol{\xi})_p]^T$, we have

$$(\mathcal{K}\boldsymbol{\xi}) = \sum_{d=1}^{J} \mathbf{w}_d \left\{ \sum_{h=1}^{L} \mathrm{Cov}(\eta_h, \gamma_d) c_h + \sum_{i=1}^{J} \mathrm{Cov}(\gamma_d, \gamma_i) d_i \right\} = \lambda \boldsymbol{\theta} \quad \cdots\cdots \quad \text{Ⓑ}$$

Now again, based on the generalized augmentation approach, we multiply both sides of Ⓑ by $\mathbf{w}_j^T$ to obtain:

$$\begin{aligned} &\mathbf{w}_j^T \cdot \sum_{d=1}^{J} \mathbf{w}_d \left\{ \sum_{h=1}^{L} \mathrm{Cov}(\eta_h, \gamma_d) c_h + \sum_{i=1}^{J} \mathrm{Cov}(\gamma_d, \gamma_i) d_i \right\} = \mathbf{w}_j^T \cdot \lambda \boldsymbol{\theta} \\ \Longrightarrow\ & \sum_{h=1}^{L} \mathrm{Cov}(\eta_h, \gamma_j) c_h + \sum_{i=1}^{J} \mathrm{Cov}(\gamma_j, \gamma_i) d_i = \lambda d_j \\ \Longrightarrow\ & \sum_{h=1}^{L} V_{xy}^{(jh)} c_h + \sum_{i=1}^{J} V_x^{(ji)} d_i = \lambda d_j \quad \cdots\cdots \quad \text{②} \end{aligned}$$

where $V_{xy}^{(jh)} = \mathrm{Cov}(\gamma_j, \eta_h)$ and $V_x^{(ji)} = \mathrm{Cov}(\gamma_j, \gamma_i)$.

Define 4 matrices: $V_y = \{V_y^{(hl)}\}_{h=1,\ldots,L}^{l=1,\ldots,L} \in \mathbb{R}^{L\times L}$, $V_{yx} = \{V_{yx}^{(hj)}\}_{h=1,\ldots,L}^{j=1,\ldots,J} \in \mathbb{R}^{L\times J}$, $V_{xy} = V_{yx}^T$, and $V_x = \{V_x^{(ju)}\}_{j=1,\ldots,J}^{u=1,\ldots,J} \in \mathbb{R}^{J\times J}$. Define a $(L+J)\times(L+J)$ positive semi-definite matrix $\mathbf{V}$ that contains all these 4 matrices as blocks:

$$\mathbf{V} = \begin{bmatrix} V_y & V_{yx} \\ V_{xy} & V_x \end{bmatrix} \in \mathbb{R}^{(L+J)\times(L+J)}.$$

Let $\mathbf{c} = [c_1, \ldots, c_L]^T \in \mathbb{R}^L$ and $\mathbf{d} = [d_1, \ldots, d_J]^T \in \mathbb{R}^J$, and denote $\mathbf{e} = [\mathbf{c}, \mathbf{d}]^T \in \mathbb{R}^{L+J}$. Then, simultaneously solving for $\{c_1, \ldots, c_L\}$, $\{d_1, \ldots, d_J\}$ and λ from equations ① and ② is equivalent to solving the following eigen equation

$$\mathbf{V}\mathbf{e} = \lambda \mathbf{e}.$$

Therefore, the positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M > 0$ of $\mathbf{Z}$ $(M \leq L + J)$ are exactly the positive eigenvalues of the covariance operator $\mathcal{K}$.

Denote $\mathbf{e}_m = [\mathbf{c}_m, \mathbf{d}_m]^T$ as the mth eigenvector of $\mathbf{V}$, corresponding to the eigenvalue λ_m, $m = 1, \ldots, M$. Here, $\mathbf{c}_m = [c_{m1}, \ldots c_{mL}]^T$ denotes the first L elements of $\mathbf{e}_m$, and $\mathbf{d}_m = [d_{m1}, \ldots d_{mJ}]^T$ denotes the last J elements of $\mathbf{e}_m$. Elements of the functional part of the mth eigenfunction of $\mathcal{K}$ can be expressed as following:

$$\begin{aligned}
\lambda_m \psi_m^{(k)}(t_k) &\overset{\text{(A)}}{=} \sum_{h=1}^{L} \phi_h^{(k)}(t_k) \left\{ \sum_{l=1}^{L} \mathrm{Cov}(\eta_h, \eta_l) c_{ml} + \sum_{j=1}^{J} \mathrm{Cov}(\eta_h, \gamma_j) d_{mj} \right\} \\
&= \sum_{h=1}^{L} \phi_h^{(k)}(t_k) \left\{ \sum_{l=1}^{L} V_y^{(hl)} c_{ml} + \sum_{j=1}^{J} V_{yx}^{(hj)} d_{mj} \right\} \\
&\overset{\text{(1)}}{=} \sum_{h=1}^{L} \phi_h^{(k)}(t_k) \lambda_m c_{mh} \\
&\Longrightarrow \ \psi_m^{(k)}(t_k) = \sum_{h=1}^{L} c_{mh} \phi_h^{(k)}(t_k).
\end{aligned}$$

Similarly, the vector part of the mth eigenfunction of $\mathcal{K}$ can be expressed as following:

$$\begin{aligned}
\lambda_m \boldsymbol{\theta}_m &\overset{\text{(B)}}{=} \sum_{j=1}^{J} \mathbf{w}_j \left\{ \sum_{h=1}^{L} \mathrm{Cov}(\gamma_j, \eta_h) c_{mh} + \sum_{i=1}^{J} \mathrm{Cov}(\gamma_j, \gamma_i) d_{mi} \right\} \\
&= \sum_{j=1}^{J} \mathbf{w}_j \left\{ \sum_{h=1}^{L} V_{xy}^{(jh)} c_{mh} + \sum_{i=1}^{J} V_x^{(ji)} d_{mi} \right\} \\
&\overset{\text{(2)}}{=} \sum_{j=1}^{J} \mathbf{w}_j \lambda_m d_{mj} \\
&\Longrightarrow \ \boldsymbol{\theta}_m = \sum_{j=1}^{J} d_{mj} \mathbf{w}_j.
\end{aligned}$$

Finally, using the obtained forms of the eigenfunctions, the mth PC score is derived as:

$$\begin{aligned}
\rho_m &= \langle \mathbf{Z}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} Y^{(k)}(t_k) \psi_m^{(k)}(t_k) dt_k + \mathbf{X}^T \boldsymbol{\theta}_m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{ \sum_{l=1}^{L} \eta_l \phi_l^{(k)}(t_k) \right\} \left\{ \sum_{h=1}^{L} c_{mh} \phi_h^{(k)}(t_k) \right\} dt_k + \left(\sum_{q=1}^{J} \gamma_q \mathbf{w}_q \right)^T \left(\sum_{j=1}^{J} d_{mj} \mathbf{w}_j \right) \\
&= \sum_{l=1}^{L} \sum_{h=1}^{L} \eta_l c_{mh} \cdot \sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_l^{(k)}(t_k) \phi_h^{(k)}(t_k) dt_k + \sum_{q=1}^{J} \sum_{j=1}^{J} \gamma_q d_{mj} (\mathbf{w}_q^T \mathbf{w}_j) \\
&= \sum_{h=1}^{L} \eta_h c_{mh} + \sum_{j=1}^{J} \gamma_j d_{mj}.
\end{aligned}
$$

□

· **Proof of Theorem 5**

We first derive several preliminary results prior to the proof of this theorem.

(a) $\mathrm{Var}\{Y^{[L](k)}(t_k)\} = \mathrm{Var}\{\sum_{h=1}^{L} \eta_h \phi_h^{(k)}(t_k)\} = \sum_{h=1}^{L} \tau_h \phi_h^{(k)}(t_k)^2 \leq \sum_{h=1}^{\infty} \tau_h \phi_h^{(k)}(t_k)^2 = \sigma_y^{(kk)}(t_k, t_k)$ $\leq \|\sigma_y^{(kk)}\|_\infty < \infty$, where the third equality follows from Proposition 3 in Happ and Greven (2018), and the last two inequalities follow from the proof of Theorem 2.

(b) $\mathrm{Var}\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\} = \mathrm{E}\{|Y^{(k)}(t_k) - \sum_{h=1}^{L} \eta_h \phi_h^{(k)}(t_k)|^2\}$ converges to 0 as $L \to \infty$ by Proposition 4 in Happ and Greven (2018). Note that the first equality follows from the fact that $\mathrm{E}\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\} = \mathrm{E}\{Y^{(k)}(t_k)\} - \sum_{h=1}^{L} \mathrm{E}(\eta_h) \phi_h^{(k)}(t_k) = 0$.

(c) $\mathrm{Var}(X_r^{[J]}) = \mathrm{Var}(\sum_{j=1}^{J} \gamma_j w_{jr}) = \sum_{j=1}^{J} \kappa_j w_{jr}^2 \leq \sum_{j=1}^{p} \kappa_j w_{jr}^2 = \sigma_x(r, r) \leq \|\sigma_x\|_\infty < \infty$, where the last equality follows from eigendecomposition of a $p \times p$ covariance matrix $\sigma_x(\cdot, \cdot)$, and $\|\sigma_x\|_\infty = \sup_r |\sigma_x(r, r)|$, which is bounded by the finite second moment assumption.

(d) $\mathrm{E}\|\mathbf{X} - \mathbf{X}^{[J]}\|^2 = \sum_{j=J+1}^{p} \kappa_j$ (Grim, 1986), which converges to 0 as $J \to p$. Then, by the continuous mapping theorem (CMT), $\mathrm{E}\|X_r - X_r^{[J]}\|^2 \to 0$ as $J \to p$, which in turn implies $\mathrm{Var}(X_r - X_r^{[J]}) \to 0$ as $J \to p$ given that $E(X_r - X_r^{[J]}) = 0$.

(e) Boundedness of $\|\sigma_y^{(k)}(\cdot, t_k) - \tilde{\sigma}_y^{(k)}(\cdot, t_k)\|_{\mathcal{F}}$

$$\begin{aligned}
|\sigma_y^{(uk)}(s_u, t_k) - \tilde{\sigma}_y^{(uk)}(s_u, t_k)| &\leq \left|\text{Cov}\{Y^{(u)}(s_u), Y^{(k)}(t_k)\}\right| + \left|\text{Cov}\{Y^{[L](u)}(s_u), Y^{[L](k)}(t_k)\}\right| \\
&\overset{CS}{\leq} \sqrt{\text{Var}\{Y^{(u)}(s_u)\}}\sqrt{\text{Var}\{Y^{(k)}(t_k)\}} \\
&\quad + \sqrt{\text{Var}\{Y^{[L](u)}(s_u)\}}\sqrt{\text{Var}\{Y^{[L](k)}(t_k)\}} \\
&\leq \sqrt{\left\|\sigma_y^{(uu)}\right\|_\infty}\sqrt{\left\|\sigma_y^{(kk)}\right\|_\infty} + \sqrt{\left\|\sigma_y^{(uu)}\right\|_\infty}\sqrt{\left\|\sigma_y^{(kk)}\right\|_\infty} \quad [\because (a)] \\
&= 2\sqrt{\left\|\sigma_y^{(uu)}\right\|_\infty}\sqrt{\left\|\sigma_y^{(kk)}\right\|_\infty}
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\sigma_y^{(k)}(\cdot, t_k) - \tilde{\sigma}_y^{(k)}(\cdot, t_k)\|_{\mathcal{F}} &= \left[\sum_{u=1}^{K} \int_{\mathcal{T}_u} \left\{\sigma_y^{(uk)}(s_u, t_k) - \tilde{\sigma}_y^{(uk)}(s_u, t_k)\right\}^2 ds_u\right]^{1/2} \\
&= \left[\sum_{u=1}^{K} \int_{\mathcal{T}_u} 4\left\|\sigma_y^{(uu)}\right\|_\infty \left\|\sigma_y^{(kk)}\right\|_\infty ds_u\right]^{1/2} \quad [\because (a)] \\
&= 2\|\sigma_y\|_\infty (KT)^{1/2} < \infty,
\end{aligned}$$

where $\|\sigma_y\|_\infty = \sup_{k=1,\ldots,K} \|\sigma_y^{(kk)}\|_\infty$, and $T = \max_{k=1,\ldots,K} \mu(\mathcal{T}_k)$.

(f) Boundedness of $\|\sigma_x(q, \cdot) - \tilde{\sigma}_x(q, \cdot)\|$.

$$\begin{aligned}
|\sigma_x(q, r) - \tilde{\sigma}_x(q, r)| &\leq |\text{Cov}(X_q, X_r)| + \left|\text{Cov}(X_q^{[J]}, X_r^{[J]})\right| \\
&\overset{CS}{\leq} \sqrt{\text{Var}(X_q)}\sqrt{\text{Var}(X_r)} + \sqrt{\text{Var}(X_q^{[J]})}\sqrt{\text{Var}(X_r^{[J]})} \\
&\leq 2\|\sigma_x\|_\infty \quad [\because (c)].
\end{aligned}$$

This implies that $\|\sigma_x(q, \cdot) - \tilde{\sigma}_x(q, \cdot)\| = [\sum_{r=1}^{p}\{\sigma_x(q, r) - \tilde{\sigma}_x(q, r)\}^2]^{1/2} \leq 2\|\sigma_x\|_\infty p^{1/2}$

(g) Boundedness of $\|\sigma_{yx}^{(k)}(t_k,\cdot) - \tilde{\sigma}_{yx}^{(k)}(t_k,\cdot)\|$ and $\|\sigma_{yx}(\cdot,r) - \tilde{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}}$.

$$
\begin{aligned}
|\sigma_{yx}^{(k)}(t_k,r) - \tilde{\sigma}_{yx}^{(k)}(t_k,r)| &= \left|\mathrm{Cov}\{Y^{(k)}(t_k), X_r\}\right| + \left|\mathrm{Cov}\{Y^{[L](k)}(t_k), X_r^{[J]}\}\right| \\
&\overset{CS}{\leq} \sqrt{\mathrm{Var}\{Y^{(k)}(t_k)\}}\sqrt{\mathrm{Var}(X_r)} + \sqrt{\mathrm{Var}\{Y^{[L](k)}(t_k)\}}\sqrt{\mathrm{Var}(X_r^{[J]})} \\
&\leq \sqrt{\left\|\sigma_y^{(kk)}\right\|_\infty}\sqrt{\|\sigma_x\|_\infty} + \sqrt{\left\|\sigma_y^{(kk)}\right\|_\infty}\sqrt{\|\sigma_x\|_\infty} \quad [\because (a),(c)] \\
&\leq 2\|\sigma_y\|_\infty^{1/2}\|\sigma_x\|_\infty^{1/2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\sigma_{yx}^{(k)}(t_k,\cdot) - \tilde{\sigma}_{yx}^{(k)}(t_k,\cdot)\| &= \left[\sum_{r=1}^{p}\left\{\sigma_{yx}^{(k)}(t_k,r) - \tilde{\sigma}_{yx}^{(k)}(t_k,r)\right\}^2\right]^{1/2} \\
&\leq 2(\|\sigma_y\|_\infty\|\sigma_x\|_\infty p)^{1/2} < \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\sigma_{yx}(\cdot,r) - \tilde{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}} &= \left[\sum_{k=1}^{K}\int_{\mathcal{T}_k}\left\{\sigma_{yx}^{(k)}(t_k,r) - \tilde{\sigma}_{yx}^{(k)}(t_k,r)dt_k\right\}^2\right]^{1/2} \\
&\leq 2(\|\sigma_y\|_\infty\|\sigma_x\|_\infty KT)^{1/2} < \infty.
\end{aligned}
$$

(h) Convergence of $|\sigma_y^{(uk)}(s_u,t_k) - \tilde{\sigma}_y^{(uk)}(s_u,t_k)|$

$$
\begin{aligned}
|\sigma_y^{(uk)}(s_u,t_k) - \tilde{\sigma}_y^{(uk)}(s_u,t_k)| &= \left|\mathrm{Cov}\left\{Y^{(u)}(s_u), Y^{(k)}(t_k)\right\} - \mathrm{Cov}\left\{Y^{[L](u)}(s_u), Y^{[L](k)}(t_k)\right\}\right| \\
&\leq \left|\mathrm{Cov}\left\{Y^{(u)}(s_u) - Y^{[L](u)}(s_u), Y^{(k)}(t_k)\right\}\right| \\
&\quad + \left|\mathrm{Cov}\left\{Y^{[L](u)}(s_u), Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\right\}\right| \\
&\overset{\mathrm{CS}}{\leq} \sqrt{\mathrm{Var}\left\{Y^{(u)}(s_u) - Y^{[L](u)}(s_u)\right\}}\sqrt{\mathrm{Var}\left\{Y^{(k)}(t_k)\right\}} \\
&\quad + \sqrt{\mathrm{Var}\left\{Y^{[L](u)}(s_u)\right\}}\sqrt{\mathrm{Var}\left\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\right\}}
\end{aligned}
$$

$$\longrightarrow 0 \quad \text{as } L \to \infty \quad [\because (a), (b)]$$

(i) $|\sigma_x(q,r) - \tilde{\sigma}_x(q,r)| = |\sum_{j=1}^{p} \kappa_j w_{jq} w_{jr} - \sum_{j=1}^{J} \kappa_j w_{jq} w_{jr}| \longrightarrow 0$ as $J \to P$.

(j) Convergence of $|\sigma_{yx}^{(k)}(t_k, r) - \tilde{\sigma}_{yx}^{(k)}(t_k, r)|$

$$\begin{aligned}
\left|\sigma_{yx}^{(k)}(t_k, r) - \tilde{\sigma}_{yx}^{(k)}(t_k, r)\right| &= \left|\mathrm{Cov}\left\{Y^{(k)}(t_k), X_r\right\} - \mathrm{Cov}\left\{Y^{[L](k)}(t_k), X_r^{[J]}\right\}\right| \\
&\leq \left|\mathrm{Cov}\left\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k), X_r\right\}\right| \\
&\quad + \left|\mathrm{Cov}\left\{Y^{[L](k)}(t_k), X_r - X_r^{[J]}\right\}\right| \\
&\overset{CS}{\leq} \sqrt{\mathrm{Var}\left\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\right\}}\sqrt{\mathrm{Var}(X_r)} \\
&\quad + \sqrt{\mathrm{Var}\left\{Y^{[L](k)}(t_k)\right\}}\sqrt{\mathrm{Var}(X_r - X_r^{[J]})} \\
&\longrightarrow 0 \quad \text{as } L \to \infty, J \to p \quad [\because (a), (b), (c), (d)]
\end{aligned}$$

The sketch of the proof is as follows. We first show that $\tilde{\mathcal{K}}$ is a bounded operator and converges in norm to $\mathcal{K}$ as $L \to \infty$ and $J \to p$. This will imply that eigenvalues of $\mathcal{K}$ of finite multiplicity and the corresponding eigenprojections are exactly the limits of those of $\tilde{\mathcal{K}}$ (Weidman, 1997). Once these are established, we can show that $\|\tilde{\mathbf{Z}} - \mathbf{Z}\|_{\mathcal{H}}$ converges to zero, and the eigenvectors and PC scores of $\tilde{\mathcal{K}}$ of multiplicity 1 converge to those of $\mathcal{K}$ as $L \to \infty$ and $J \to p$. The detailed proof consists of the following steps.

Step-I. We show that $\tilde{\mathcal{K}}$ is a bounded operator. Let $\|\sigma\|_\infty = \max(\|\sigma_y\|_\infty, \|\sigma_x\|_\infty)$, and note that $\|f\|_{\mathcal{F}}, \|\mathbf{v}\| < \|\mathbf{h}\|_{\mathcal{H}}$. Then, we have:

$$\begin{aligned}
\|\tilde{\mathcal{K}}\mathbf{h}\|_H^2 = &\sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \tilde{\sigma}_y^{(uk)}(s_u, t_u) f^{(u)}(s_u) ds_u + \sum_{q=1}^{p} \tilde{\sigma}_{yx}^{(k)}(t_k, q) v_q \right\}^2 dt_k \\
&+ \sum_{r=1}^{p} \left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \tilde{\sigma}_{yx}^{(u)}(s_u, r) f^{(u)}(s_u) ds_u + \sum_{q=1}^{p} \tilde{\sigma}_x(q, r) v_q \right\}^2
\end{aligned}$$

$$
\begin{aligned}
&\overset{CS}{\leq} \sum_{k=1}^{K}\int_{\mathcal{T}_k}\left[\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\tilde{\sigma}_y^{(uk)}(s_u,t_k)^2 ds_u\right\}^{\frac{1}{2}}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} f^{(u)}(s_u)^2 ds_u\right\}^{\frac{1}{2}} + \left\{\sum_{q=1}^{p}\tilde{\sigma}_{yx}^{(k)}(t_k,q)^2\right\}^{\frac{1}{2}}\left\{\sum_{q=1}^{p} v_q^2\right\}^{\frac{1}{2}}\right]^2 dt_k \\
&\quad + \sum_{r=1}^{p}\left[\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u}\tilde{\sigma}_{yx}^{(u)}(s_u,r)^2 ds_u\right\}^{\frac{1}{2}}\left\{\sum_{u=1}^{K}\int_{\mathcal{T}_u} f^{(u)}(s_u)^2 ds_u\right\}^{\frac{1}{2}} + \left\{\sum_{q=1}^{p}\tilde{\sigma}_x(r,q)^2\right\}^{\frac{1}{2}}\left\{\sum_{q=1}^{p} v_q^2\right\}^{\frac{1}{2}}\right]^2 \\
&= \sum_{k=1}^{K}\int_{\mathcal{T}_k}\left(\left[\sum_{u=1}^{K}\int_{\mathcal{T}_u}\mathrm{Cov}\{Y^{[L](u)}(s_u),Y^{[L](k)}(t_k)\}^2 ds_u\right]^{\frac{1}{2}}\|f\|_{\mathcal{F}} + \left[\sum_{q=1}^{p}\mathrm{Cov}\{Y^{[L](k)}(t_k),X_q^{[J]}\}^2\right]^{\frac{1}{2}}\|\mathbf{v}\|\right)^2 dt_k \\
&\quad + \sum_{r=1}^{p}\left(\left[\sum_{u=1}^{K}\int_{\mathcal{T}_u}\mathrm{Cov}\{Y^{[L](u)}(s_u),X_r^{[J]}\}^2 ds_u\right]^{\frac{1}{2}}\|f\|_{\mathcal{F}} + \left\{\sum_{q=1}^{p}\mathrm{Cov}(X_q^{[J]},X_r^{[J]})^2\right\}^{\frac{1}{2}}\|\mathbf{v}\|\right)^2 \\
&\overset{CS}{\leq} \sum_{k=1}^{K}\int_{\mathcal{T}_k}\left(\left[\sum_{u=1}^{K}\int_{\mathcal{T}_u}\mathrm{Var}\{Y^{[L](u)}(s_u)\}\mathrm{Var}\{Y^{[L](k)}(t_k)\}ds_u\right]^{\frac{1}{2}}\|f\|_{\mathcal{F}} + \left[\sum_{q=1}^{p}\mathrm{Var}\{Y^{[L](k)}(t_k)\}\mathrm{Var}(X_q^{[J]})\right]^{\frac{1}{2}}\|\mathbf{v}\|\right)^2 dt_k \\
&\quad + \sum_{r=1}^{p}\left(\left[\sum_{u=1}^{K}\int_{\mathcal{T}_u}\mathrm{Var}\{Y^{[L](u)}(s_u)\}\mathrm{Var}(X_r^{[J]})ds_u\right]^{\frac{1}{2}}\|f\|_{\mathcal{F}} + \left\{\sum_{q=1}^{p}\mathrm{Var}(X_q^{[J]})\mathrm{Var}(X_r^{[J]})\right\}^{\frac{1}{2}}\|\mathbf{v}\|\right)^2 \\
&\leq \sum_{k=1}^{K}\int_{\mathcal{T}_k}\left\{\left(\sum_{u=1}^{K}\int_{\mathcal{T}_u}\left\|\sigma_y^{(uu)}\right\|_\infty\left\|\sigma_y^{(kk)}\right\|_\infty ds_u\right)^{1/2}\|f\|_{\mathcal{F}} + \left(\sum_{r=1}^{p}\left\|\sigma_y^{(kk)}\right\|_\infty\|\sigma_x\|_\infty\right)^{1/2}\|\mathbf{v}\|\right\}^2 dt_k \\
&\quad + \sum_{r=1}^{p}\left\{\left(\sum_{u=1}^{K}\int_{\mathcal{T}_u}\left\|\sigma_y^{(uu)}\right\|_\infty\|\sigma_x\|_\infty ds_u\right)^{1/2}\|f\|_{\mathcal{F}} + \left(\sum_{q=1}^{p}\|\sigma_x\|_\infty^2\right)^{1/2}\|\mathbf{v}\|\right\}^2 \quad [\because (a),(c)] \\
&\leq \left\{\|\sigma_y\|_\infty(KT)^{1/2}\|f\|_{\mathcal{F}} + (\|\sigma_y\|_\infty\|\sigma_x\|_\infty p)^{1/2}\|\mathbf{v}\|\right\}^2 KT \\
&\quad + \left\{(\|\sigma_y\|_\infty\|\sigma_x\|_\infty KT)^{1/2}\|f\|_{\mathcal{F}} + \|\sigma_x\|_\infty p^{1/2}\|\mathbf{v}\|\right\}^2 p \\
&\leq \|\mathbf{h}\|_{\mathcal{H}}^2\left\{\|\sigma_y\|_\infty(KT)^{1/2} + (\|\sigma_y\|_\infty\|\sigma_x\|_\infty p)^{1/2}\right\}^2 KT \\
&\quad + \|\mathbf{h}\|_{\mathcal{H}}^2\left\{(\|\sigma_y\|_\infty\|\sigma_x\|_\infty KT)^{1/2} + \|\sigma_x\|_\infty p^{1/2}\right\}^2 p \\
&= 2\|\mathbf{h}\|_{\mathcal{H}}^2\|\sigma\|_\infty^2\{KT + (pKT)^{1/2}\}^2\{(pKT)^{1/2} + p\}^2
\end{aligned}
$$

The last term is constant and finite, hence $\tilde{\mathcal{K}}$ is a bounded operator.

Step-II. We show that $\tilde{\mathcal{K}}$ converges in norm to $\mathcal{K}$. Let $\|\cdot\|_{op}$ denote an operator norm induced by $\|\cdot\|_{\mathcal{H}}$. Let $\breve{\sigma}_y(\cdot,\cdot) = \sigma_y(\cdot,\cdot) - \tilde{\sigma}_y(\cdot,\cdot)$, $\breve{\sigma}_{yx}(\cdot,\cdot) = \sigma_{yx}(\cdot,\cdot) - \tilde{\sigma}_{yx}(\cdot,\cdot)$, and $\breve{\sigma}_x(\cdot,\cdot) =$

$\sigma_x(\cdot,\cdot)-\tilde{\sigma}_x(\cdot,\cdot)$. Then

$$\|\mathcal{K}-\tilde{\mathcal{K}}\|_{op}^2 = \sup_{\|\mathbf{h}\|_{\mathcal{H}}=1} \|(\mathcal{K}-\tilde{\mathcal{K}})\mathbf{h}\|_{\mathcal{H}}^2$$

$$= \sup_{\|\mathbf{h}\|_{\mathcal{H}}=1} \left(\sum_{k=1}^K \int_{\mathcal{T}_k} \left[\sum_{u=1}^K \int_{\mathcal{T}_u} \check{\sigma}_y^{(uk)}(s_u,t_k) f^{(u)}(s_u) ds_u + \sum_{q=1}^p \check{\sigma}_{yx}^{(k)}(t_k,q) v_q \right]^2 dt_k \right.$$

$$\left. + \sum_{r=1}^p \left[\sum_{u=1}^K \int_{\mathcal{T}_u} \check{\sigma}_{yx}^{(u)}(s_u,r) f^{(u)}(s_u) ds_u + \sum_{q=1}^p \check{\sigma}_x(q,r) v_r \right]^2 \right)$$

$$\overset{CS}{\leq} \sup_{\|\mathbf{h}\|_{\mathcal{H}}=1} \left(\sum_{k=1}^K \int_{\mathcal{T}_k} \left[\left\{ \sum_{u=1}^K \int_{\mathcal{T}_u} |\check{\sigma}_y^{(uk)}(s_u,t_k)|^2 ds_u \right\}^{\frac{1}{2}} \left\{ \sum_{u=1}^K \int_{\mathcal{T}_u} |f^{(u)}(s_u)|^2 ds_u \right\}^{\frac{1}{2}} + \left\{ \sum_{q=1}^p \check{\sigma}_{yx}^{(k)}(t_k,q)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{q=1}^p v_q^2 \right\}^{\frac{1}{2}} \right]^2 dt_k \right.$$

$$\left. + \sum_{r=1}^p \left[\left\{ \sum_{u=1}^K \int_{\mathcal{T}_u} |\check{\sigma}_{yx}^{(u)}(s_u,r)|^2 ds_u \right\}^{\frac{1}{2}} \left\{ \sum_{u=1}^K \int_{\mathcal{T}_u} |f^{(u)}(s_u)|^2 ds_u \right\}^{\frac{1}{2}} + \left\{ \sum_{q=1}^p \check{\sigma}_x(q,r)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{q=1}^p v_q^2 \right\}^{\frac{1}{2}} \right]^2 \right)$$

$$= \sup_{\|\mathbf{h}\|_{\mathcal{H}}=1} \left[\sum_{k=1}^K \int_{\mathcal{T}_k} \left\{ \|\check{\sigma}_y^{(k)}(\cdot,t_k)\|_{\mathcal{F}} \, \|f\|_{\mathcal{F}} + \|\check{\sigma}_{yx}^{(k)}(t_k,\cdot)\| \, \|\mathbf{v}\| \right\}^2 dt_k \right.$$

$$\left. + \sum_{r=1}^p \{ \|\check{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}} \, \|f\|_{\mathcal{F}} + \|\check{\sigma}_x(\cdot,r)\| \|\mathbf{v}\| \}^2 \right]$$

$$\leq \sum_{k=1}^K \int_{\mathcal{T}_k} \left\{ \|\check{\sigma}_y^{(k)}(\cdot,t_k)\|_{\mathcal{F}} + \|\check{\sigma}_{yx}^{(k)}(t_k,\cdot)\| \right\}^2 dt_k + \sum_{r=1}^p \{ \|\check{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}} + \|\check{\sigma}_x(\cdot,r)\| \}^2,$$

where the last inequality holds since $\|f\|_{\mathcal{F}}, \|\mathbf{v}\| \leq 1$ for all $\mathbf{h} \in \mathcal{F}$ when $\|\mathbf{h}\|_{\mathcal{H}} = 1$. By (e) and (g), the upper bound for the integrand of the first term is:

$$\left\{ \|\check{\sigma}_y^{(k)}(\cdot,t_k)\|_{\mathcal{F}} + \|\check{\sigma}_{yx}^{(k)}(t_k,\cdot)\| \right\}^2 \leq 4\{ \|\sigma_y\|_\infty (KT)^{1/2} + (\|\sigma_y\|_\infty \|\sigma_x\|_\infty p)^{1/2} \}^2,$$

which is a constant and hence integrable over $\mathcal{T}$. Thus, we can apply the DCT:

$$\lim_{L\to\infty, J\to p} \|\mathcal{K}-\tilde{\mathcal{K}}\|_{op}^2 \leq \lim_{L\to\infty, J\to p} \sum_{k=1}^K \int_{\mathcal{T}_k} \left\{ \|\check{\sigma}_y^{(k)}(\cdot,t_k)\|_{\mathcal{F}} + \|\check{\sigma}_{yx}^{(k)}(t_k,\cdot)\| \right\}^2 dt_k$$

$$+ \lim_{L\to\infty, J\to p} \sum_{r=1}^p \{ \|\check{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}} + \|\check{\sigma}_x(\cdot,r)\| \}^2$$

$$\overset{DCT}{=} \sum_{k=1}^K \int_{\mathcal{T}_k} \lim_{L\to\infty, J\to p} \left\{ \|\check{\sigma}_y^{(k)}(\cdot,t_k)\|_{\mathcal{F}} + \|\check{\sigma}_{yx}^{(k)}(t_k,\cdot)\| \right\}^2 dt_k$$

$$
\begin{aligned}
&+\sum_{r=1}^{p} \lim_{L\to\infty, J\to p} \{\|\check{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}} + \|\check{\sigma}_{x}(\cdot,r)\|\}^2 \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{ \lim_{L\to\infty, J\to p} \|\check{\sigma}_y^{(k)}(\cdot,t_k)\|_{\mathcal{F}} + \lim_{L\to\infty, J\to p} \|\check{\sigma}_{yx}^{(k)}(t_k,\cdot)\| \right\}^2 dt_k \\
&+\sum_{r=1}^{p} \left\{ \lim_{L\to\infty, J\to p} \|\check{\sigma}_{yx}(\cdot,r)\|_{\mathcal{F}} + \lim_{L\to\infty, J\to p} \|\check{\sigma}_{x}(\cdot,r)\| \right\}^2 \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \left[\left\{ \lim_{\substack{L\to\infty \\ J\to p}} \sum_{u=1}^{K} \int_{\mathcal{T}_u} \check{\sigma}_y^{(uk)}(s_u,t_k)^2 ds_u \right\}^{\frac{1}{2}} + \left\{ \lim_{\substack{L\to\infty \\ J\to p}} \sum_{q=1}^{p} \check{\sigma}_{yx}^{(k)}(t_k,q)^2 \right\}^{\frac{1}{2}} \right]^2 dt_k \\
&+\sum_{r=1}^{p} \left[\left\{ \lim_{\substack{L\to\infty \\ J\to p}} \sum_{u=1}^{K} \int_{\mathcal{T}_u} \check{\sigma}_{yx}^{(u)}(s_u,r)^2 ds_u \right\}^{\frac{1}{2}} + \left\{ \lim_{\substack{L\to\infty \\ J\to p}} \sum_{q=1}^{p} \check{\sigma}_{x}(q,r)^2 \right\}^{1/2} \right]^2 .
\end{aligned}
$$

Now noting that $\check{\sigma}_y^{(uk)}(s_u,t_k)^2 \leq 4\|\sigma_y\|_\infty^2$ and $\sigma_{yx}^{(k)}(t_k,q)^2 \leq 4\|\sigma_y\|_\infty\|\sigma_x\|_\infty$ by (e) and (g), respectively, we can apply the DCT to interchange the integral and limit in the last equation and derive that

$$
\begin{aligned}
\lim_{\substack{L\to\infty \\ J\to p}} \|\mathcal{K}-\tilde{\mathcal{K}}\|_{op}^2 \leq \sum_{k=1}^{K} \int_{\mathcal{T}_k} &\left[\left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \lim_{\substack{L\to\infty \\ J\to p}} \check{\sigma}_y^{(uk)}(s_u,t_k)^2 ds_u \right\}^{\frac{1}{2}} + \left\{ \sum_{q=1}^{p} \lim_{\substack{L\to\infty \\ J\to p}} \check{\sigma}_{yx}^{(k)}(t_k,q)^2 \right\}^{\frac{1}{2}} \right]^2 dt_k \\
+\sum_{r=1}^{p} &\left[\left\{ \sum_{u=1}^{K} \int_{\mathcal{T}_u} \lim_{\substack{L\to\infty \\ J\to p}} \check{\sigma}_{yx}^{(u)}(s_u,r)^2 ds_u \right\}^{\frac{1}{2}} + \left\{ \sum_{q=1}^{p} \lim_{\substack{L\to\infty \\ J\to p}} \check{\sigma}_{x}(q,r)^2 \right\}^{\frac{1}{2}} \right]^2 ,
\end{aligned}
$$

which equals zero by (h), (i) and (j). Therefore, $\tilde{\mathcal{K}}$ converges in norm to $\mathcal{K}$ as $L \to \infty$ and $J \to p$.

Step-III. Let $\tilde{\lambda}_m$ and λ_m ($m \in \mathbb{N}$) be finite-multiplicity eigenvalues of $\tilde{\mathcal{K}}$ and $\mathcal{K}$, respectively. Denote $\tilde{\mathcal{P}}$ and $\mathcal{P}$ as the corresponding eigenprojections. Since both $\tilde{\mathcal{K}}$ and $\mathcal{K}$ are bounded, and $\tilde{\mathcal{K}}$ converges in norm to $\mathcal{K}$, we can establish that $\tilde{\lambda}_m \to \lambda_m$ and $\|\mathcal{P}_m - \tilde{\mathcal{P}_m}\|_{op} \to 0$ including multiplicity as $L \to \infty$ and $J \to p$ (Weidman, 1997).

Step-IV. We show that if $\tilde{\lambda}_m$ and λ_m have multiplicity 1, $\tilde{\boldsymbol{\xi}}_m$ converges to $\boldsymbol{\xi}_m$. Note that $\tilde{\mathcal{P}}_m$ and $\mathcal{P}_m$ project $\mathbf{h} \in \mathcal{H}$ onto eigenspaces spanned by $\tilde{\boldsymbol{\xi}}_m$ and $\boldsymbol{\xi}_m$, respectively, and can be expressed as $\tilde{\mathcal{P}}_m\mathbf{h} = \langle \mathbf{h}, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} \tilde{\boldsymbol{\xi}}_m$ and $\mathcal{P}_m\mathbf{h} = \langle \mathbf{h}, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \boldsymbol{\xi}_m$. Without loss of generality, assume that $\langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} \geq 0$.

$$
\begin{aligned}
\|\mathcal{P}_m - \tilde{\mathcal{P}}_m\|_{op}^2 &= \sup_{\|\mathbf{h}\|_{\mathcal{H}}=1} \|(\mathcal{P}_m - \tilde{\mathcal{P}}_m)\mathbf{h}\|_{\mathcal{H}}^2 \\
&\geq \|\mathcal{P}_m \boldsymbol{\xi}_m - \tilde{\mathcal{P}}_m \boldsymbol{\xi}_m\|_{\mathcal{H}}^2 \quad (\because \|\boldsymbol{\xi}_m\|_{\mathcal{H}} = 1) \\
&= \|\langle \boldsymbol{\xi}_m, \boldsymbol{\xi}_m \rangle_{\mathcal{H}} \boldsymbol{\xi}_m - \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} \tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}}^2 \\
&= \|\boldsymbol{\xi}_m - \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} \tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}}^2 \\
&= \|\boldsymbol{\xi}_m\|_{\mathcal{H}}^2 - 2\langle \boldsymbol{\xi}_m, \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} + \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}^2 \|\tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}}^2 \\
&= 1 - \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}^2 \\
&= (1 + \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}})(1 - \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}) \\
&\geq (1 - \langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}) \\
&= \frac{1}{2}(\|\boldsymbol{\xi}_m\|_{\mathcal{H}}^2 - 2\langle \boldsymbol{\xi}_m, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} + \|\tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}}^2) \\
&= \frac{1}{2}\|\boldsymbol{\xi}_m - \tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}}^2
\end{aligned}
$$

$\|\mathcal{P}_m - \tilde{\mathcal{P}}_m\|_{op} \to 0$ hence implies $\|\tilde{\boldsymbol{\xi}}_m - \boldsymbol{\xi}_m\|_{\mathcal{H}} \to 0$ for $L \to \infty$ and $J \to p$.

Step-V. We show that $\tilde{\mathbf{Z}}$ converges in probability to $\mathbf{Z}$.

$$
\begin{aligned}
\mathrm{E}\|\mathbf{Z} - \tilde{\mathbf{Z}}\|_{\mathcal{H}}^2 &= \mathrm{E}\left[\sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{Y^{(K)}(t_k) - Y^{[L](k)}(t_k)\right\}^2 dt_k + \sum_{r=1}^{p} (X_r - X_r^{[J]})^2\right] \\
&\overset{Fubini}{=} \sum_{k=1}^{K} \int_{\mathcal{T}_k} \mathrm{E}\left[\left\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\right\}^2\right] dt_k + \mathrm{E}\|\mathbf{X} - \mathbf{X}^{[J]}\|^2.
\end{aligned}
$$

$\mathrm{E}[\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\}^2]$ converges to 0 as $L \to \infty$ by (b) and is thus bounded. Since $\mu(\mathcal{T}) < \infty$, we can apply the DCT to conclude that $\sum_{k=1}^{K} \int_{\mathcal{T}_k} \mathrm{E}[\{Y^{(k)}(t_k) - Y^{[L](k)}(t_k)\}^2] dt_k$ converges to 0 as $L \to \infty$. By (d), the last term, $\mathrm{E}\|\mathbf{X} - \mathbf{X}^{[J]}\|^2$, converges to 0 as $J \to P$. Therefore, $\|\mathbf{Z} - \tilde{\mathbf{Z}}\|_{\mathcal{H}}$ converges in the second mean to 0 as $L \to \infty$ and $J \to P$, and hence $\|\mathbf{Z} - \tilde{\mathbf{Z}}\|_{\mathcal{H}} = o_p(1)$.

Step-VI. We show that $\tilde{\rho}_m$ converges in probability to ρ_m. Firstly, we show that $\|\mathbf{Z}\|_{\mathcal{H}}$ is bounded in probability. For any $\epsilon > 0$, let $c = \{\frac{2}{\epsilon}(KT\|\sigma_y\|_\infty + p\|\sigma_x\|_\infty)\}^{1/2}$. Then by the Markov's inequality,

$$\begin{aligned}
P(\|\mathbf{Z}\|_{\mathcal{H}} > c) &\leq \frac{1}{c^2}\mathrm{E}(\|\mathbf{Z}\|^2_{\mathcal{H}}) \\
&= \frac{1}{c^2}\mathrm{E}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k} Y^{(k)}(t_k)^2 dt_k + \sum_{r=1}^{p} X_r^2\right\} \\
&\overset{Fubini}{=} \frac{1}{c^2}\left[\sum_{k=1}^{K}\int_{\mathcal{T}_k} \mathrm{E}\left\{Y^{(k)}(t_k)^2\right\} dt_k + \sum_{r=1}^{p}\mathrm{E}(X_r^2)\right] \\
&= \frac{1}{c^2}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k} \sigma_y^{(kk)}(t_k, t_k) dt_k + \sum_{r=1}^{p}\sigma_x(r, r)\right\} \\
&\leq \frac{1}{c^2}\left\{\sum_{k=1}^{K}\int_{\mathcal{T}_k} \|\sigma_y^{(kk)}\|_\infty \, dt_k + \sum_{r=1}^{p}\|\sigma_x\|_\infty\right\} \\
&\leq \frac{1}{c^2}\left\{KT\|\sigma_y\|_\infty + p\|\sigma_x\|_\infty\right\} \quad [\because (a), (c)] \\
&= \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Thus, $\|\mathbf{Z}\|_{\mathcal{H}} = O_p(1)$. Then, noting that $\|\boldsymbol{\xi}_m - \tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}} = o(1)$, $\|\tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}} = 1$ and $\|\mathbf{Z} - \tilde{\mathbf{Z}}\|_{\mathcal{H}} = o_p(1)$, we can show

$$\begin{aligned}
|\rho_m - \tilde{\rho}_m| &= |\langle \mathbf{Z}, \boldsymbol{\xi}_m\rangle_{\mathcal{H}} - \langle \tilde{\mathbf{Z}}, \tilde{\boldsymbol{\xi}}_m\rangle_{\mathcal{H}}| \\
&\leq |\langle \mathbf{Z}, \boldsymbol{\xi}_m - \tilde{\boldsymbol{\xi}}_m\rangle_{\mathcal{H}}| + |\langle \mathbf{Z} - \tilde{\mathbf{Z}}, \tilde{\boldsymbol{\xi}}_m\rangle_{\mathcal{H}}| \\
&\overset{CS}{\leq} \|\mathbf{Z}\|_{\mathcal{H}} \, \|\boldsymbol{\xi}_m - \tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}} + \|\mathbf{Z} - \tilde{\mathbf{Z}}\|_{\mathcal{H}} \, \|\tilde{\boldsymbol{\xi}}_m\|_{\mathcal{H}} \\
&= O_p(1)o(1) + o_p(1) = o_p(1).
\end{aligned}$$

That is, $\tilde{\rho}_m$ converges in probability to ρ_m as $L \to \infty$ and $J \to p$.

□

· Proof of Theorem 6

We first state regularity conditions that are needed for the proof.

R1. $\Delta_L^y = \sup_{h=1,\ldots,L} (\tau_h - \tau_{h+1})^{-1} < \infty$

R2. $\Delta_J^x = \sup_{j=1,\ldots,J} (\kappa_j - \kappa_{j+1})^{-1} < \infty$

R3. $\|\mathcal{C}_y - \hat{\mathcal{C}}_y\|_{op} = O_p(c_n^y)$, where $c_n^y \to 0$ as $n \to \infty$;

R4. $\|\mathcal{C}_x - \hat{\mathcal{C}}_x\|_{op} = O_p(c_n^x)$, where $c_n^x \to 0$ as $n \to \infty$;

R5. $\sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \mathrm{E}\{Y^{(k)}(t_k)^2 X_r^2\} dt_k < \infty$

R6. $\langle \phi_h, \hat{\phi}_h \rangle_2 \geq 0$ for all $h = 1, \ldots L$;

R7. $\langle \mathbf{w}_j, \hat{\mathbf{w}}_j \rangle \geq 0$ for all $j = 1, \ldots J$.

R8. $\hat{\eta}_{ih} = \langle Y_i, \hat{\phi}_h \rangle_2$ for all $h = 1, \ldots, L$ and $i = 1, \ldots, n$.

R9. $\hat{\gamma}_{ij} = \langle \mathbf{X}_i, \hat{\mathbf{w}}_j \rangle$ for all $j = 1, \ldots, J$ and $i = 1, \ldots, n$.

Before we present the proof of Theorem 6, we state and prove the following lemmas.

Lemma 1 *For all* $h = 1, \ldots, L$, $\|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} = O_p(\Delta_L^y c_n^y)$. *For all* $j = 1, \ldots, J$, $\|\mathbf{w}_j - \hat{\mathbf{w}}_j\| = O_p(\Delta_J^x c_n^x)$.

Proof: By Lemma 4.3 of Bosq (2000) and regularity conditions R1–R4, we can establish the following. For $h = 1$,

$$\|\phi_1 - \hat{\phi}_1\|_{\mathcal{F}} \leq 2\sqrt{2}(\tau_1 - \tau_2)^{-1} \|\mathcal{C}_y - \hat{\mathcal{C}}_y\|_{op} \leq 2\sqrt{2}\Delta_L^y O_p(c_n^y) = O_p(\Delta_L^y c_n^y).$$

For $h = 2, \ldots, L$,

$$\|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} \leq 2\sqrt{2} \max(\tau_{h-1} - \tau_h, \tau_h - \tau_{h+1})^{-1} \|\mathcal{C}_y - \hat{\mathcal{C}}_y\|_{op} \leq 2\sqrt{2}\Delta_L^y O_p(c_n^y) = O_p(\Delta_L^y c_n^y).$$

For $j = 1$,

$$\|\mathbf{w}_1 - \hat{\mathbf{w}}_1\| \leq 2\sqrt{2}(\kappa_1 - \kappa_2)^{-1} \|\mathcal{C}_x - \hat{\mathcal{C}}_x\|_{op} \leq 2\sqrt{2}\Delta_J^x O_p(c_n^x) = O_p(\Delta_J^x c_n^x).$$

For $j = 2, \ldots, J$,

$$\|\mathbf{w}_j - \hat{\mathbf{w}}_j\| \le 2\sqrt{2}\max(\kappa_{j-1} - \kappa_j, \kappa_j - \kappa_{j+1})^{-1}\|\mathcal{C}_x - \hat{\mathcal{C}}_x\|_{op} \le 2\sqrt{2}\Delta_J^x O_p(c_n^x) = O_p(\Delta_J^x c_n^x).$$

$\square$

Lemma 2 *Let* $\sigma_{yx}^{(k)}(t_k, r) = \mathrm{Cov}\{Y^{(k)}(t_k), X_r\}$ *and* $\hat{\sigma}_{yx}^{(k)}(t_k, r) = n^{-1}\sum_{i=1}^n Y_i^{(k)}(t_k)X_{ir}$. *Define new bounded operators* $\mathcal{C}_{yx} : \mathcal{F} \to \mathbb{R}^p$ *and* $\hat{\mathcal{C}}_{yx} : \mathcal{F} \to \mathbb{R}^p$ *that are respectively expressed as follows*

$$(\mathcal{C}_{yx} f)_r = \sum_{k=1}^K \int_{\mathcal{T}_k} \sigma_{yx}^{(k)}(t_k, r) f^{(k)}(t_k) dt_k \quad \text{and} \quad (\hat{\mathcal{C}}_{yx} f)_r = \sum_{k=1}^K \int_{\mathcal{T}_k} \hat{\sigma}_{yx}^{(k)}(t_k, r) f^{(k)}(t_k) dt_k$$

for $f \in \mathcal{F}$. *Then,* $\|\mathcal{C}_{yx} - \hat{\mathcal{C}}_{yx}\|_{op} = O_p(n^{-1/2})$, *where* $\|\cdot\|_{op}$ *is an operator norm.*

Proof: Let Y_i and $\mathbf{X}_i$ $(i = 1, \ldots n)$ be independent and identically distributed (i.i.d.) samples of Y and $\mathbf{X}$, respectively. Then noting that $E\{Y_i^{(k)}(t_k)X_{ir}\} = \sigma_{yx}^{(k)}(t_k, r)$ for all i, we can establish that

$$\begin{aligned}
&\mathrm{E}\left[\sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \left\{\sigma_{yx}^{(k)}(t_k, r) - \hat{\sigma}_{yx}^{(k)}(t_k, r)\right\}^2 dt_k\right] = \mathrm{E}\left[\sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \left\{\sigma_{yx}^{(k)}(t_k, r) - \frac{1}{n}\sum_{i=1}^n Y_i(t_k)X_{ir}\right\}^2 dt_k\right] \\
&\overset{Fubini}{=} \sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \left(\sigma_{yx}^{(k)}(t_k, r)^2 + \mathrm{E}\left[\left\{\frac{1}{n}\sum_{i=1}^n Y_i^{(k)}(t_k)X_r\right\}^2\right] - \frac{2}{n}\sum_{i=1}^n \mathrm{E}\left\{Y_i^{(k)}(t_k)X_{ir}\right\}\sigma_{yx}^{(k)}(t_k, r)\right) dt_k \\
&= \sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \left[\frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \mathrm{E}\left\{Y_i^{(k)}(t_k)X_{ir}Y_j^{(k)}(t_k)X_{jr}\right\} - \sigma_{yx}^{(k)}(t_k, r)^2\right] dt_k \\
&= \sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \left[\frac{1}{n^2}\sum_{i=1}^n \mathrm{E}\left\{Y_i^{(k)}(t_k)^2 X_{ir}^2\right\} + \frac{2}{n^2}\sum_{1 \le i < j \le n} \mathrm{E}\left\{Y_i^{(k)}(t_k)X_{ir}\right\}\mathrm{E}\left\{Y_j^{(k)}(t_k)X_{jr}\right\} - \sigma_{yx}^{(k)}(t_k, r)^2\right] dt_k \\
&= \frac{1}{n}\sum_{r=1}^p \sum_{k=1}^K \int_{\mathcal{T}_k} \left[\mathrm{E}\left\{Y^{(k)}(t_k)^2 X_r^2\right\} - \sigma_{yx}^{(k)}(t_k, r)^2\right] dt_k = O_p(n^{-1}),
\end{aligned}$$

where the last equality follows from regularity condition R5 and the boundedness assumptions of Theorem 1 on a covariance kernel σ_{yx}. This implies

$$
\begin{aligned}
\|\mathcal{C}_{yx} - \hat{\mathcal{C}}_{yx}\|_{op} &= \sup_{\|f\|_{\mathcal{F}}=1} \|(\mathcal{C}_{yx} - \hat{\mathcal{C}}_{yx})f\| \\
&= \sup_{\|f\|_{\mathcal{F}}=1} \left(\sum_{r=1}^{p} \left[\sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{ \sigma_{yx}^{(k)}(t_k, r) - \hat{\sigma}_{yx}^{(k)}(t_k, r) \right\} f^{(k)}(t_k) dt_k \right]^2 \right)^{1/2} \\
&= \sup_{\|f\|_{\mathcal{F}}=1} \left(\sum_{r=1}^{p} \langle \sigma_{yx}(\cdot, r) - \hat{\sigma}_{yx}(\cdot, r), f \rangle_{\mathcal{F}}^2 \right)^{1/2} \\
&\overset{CS}{\leq} \sup_{\|f\|_{\mathcal{F}}=1} \left(\sum_{r=1}^{p} \|\sigma_{yx}(\cdot, r) - \hat{\sigma}_{yx}(\cdot, r)\|_{\mathcal{F}}^2 \, \|f\|_{\mathcal{F}}^2 \right)^{1/2} \\
&= \left[\sum_{r=1}^{p} \sum_{k=1}^{K} \int_{\mathcal{T}_k} \left\{ \sigma_{yx}^{(k)}(t_k, r) - \hat{\sigma}_{yx}^{(k)}(t_k, r) \right\}^2 dt_k \right]^{1/2} \overset{Markov}{=} O_p(n^{-1/2})
\end{aligned}
$$

□

Lemma 3 *Let V be the matrix stated in Theorem 4 that characterizes covariances within and between functional and vector PC scores. Then, $\lambda_{\max}(V - \hat{V}) = O_p(L \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x))$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its argument matrix.*

Proof: Recall blocked elements of V, V_y, V_{yx} V_{xy} and V_x, and consider their respective elements: $V_y^{(hl)} = \text{Cov}(\eta_h, \eta_l)$, $V_{yx}^{(hd)} = \text{Cov}(\eta_h, \gamma_d)$, $V_{xy}^{(jl)} = \text{Cov}(\gamma_j, \eta_l)$ and $V_x^{(jd)} = \text{Cov}(\gamma_j, \gamma_d)$ for $h, l = 1, \ldots L$ and $j, d = 1, \ldots J$. Then by Corollary A4 in Garren (1968), it holds that

$$
\begin{aligned}
\lambda_{\max}(V - \hat{V}) \leq \max \Bigg[& \max_{h=1,\ldots,L} \left(\sum_{l=1}^{L} |V_y^{(hl)} - \hat{V}_y^{(hl)}| + \sum_{d=1}^{J} |V_{yx}^{(hd)} - \hat{V}_{yx}^{(hd)}| \right), \\
& \max_{j=1,\ldots,J} \left(\sum_{l=1}^{L} |V_{xy}^{(jl)} - \hat{V}_{xy}^{(jl)}| + \sum_{d=1}^{J} |V_x^{(jd)} - \hat{V}_x^{(jd)}| \right) \Bigg].
\end{aligned} \tag{2}
$$

First, note that functional and vector PC scores are expressed as $\eta_h = \langle Y, \phi_h \rangle_{\mathcal{F}}$ and $\gamma_j = \langle \mathbf{X}, \mathbf{w}_j \rangle$, respectively (Rencher & Christensen, 2012; Yao et al., 2005). Their estimated versions are $\hat{\eta}_h = \langle Y, \hat{\phi}_h \rangle_{\mathcal{F}}$ and $\hat{\gamma}_j = \langle \mathbf{X}, \hat{\mathbf{w}}_j \rangle$, following from regularity conditions R8 and

R9, respectively. We separately consider each term of (2). Firstly,

$$
\begin{aligned}
&\left|V_y^{(hl)} - \hat{V}_y^{(hl)}\right| = \left|\mathrm{Cov}(\eta_h, \eta_l) - \frac{1}{n-1}\sum_{i=1}^n \hat{\eta}_{ih}\hat{\eta}_{il}\right| \\
&= \left|\mathrm{Cov}(\eta_h, \eta_l) - \frac{1}{n}\sum_{i=1}^n \hat{\eta}_{ih}\hat{\eta}_{il} - \frac{1}{n(n-1)}\sum_{i=1}^n \hat{\eta}_{ih}\hat{\eta}_{il}\right| \\
&\le \left|\mathrm{Cov}(\langle Y, \phi_h\rangle_{\mathcal{F}}\langle Y, \phi_h\rangle_{\mathcal{F}}) - \frac{1}{n}\sum_{i=1}^n \langle Y_i, \hat{\phi}_h\rangle_{\mathcal{F}}\langle Y_i, \hat{\phi}_h\rangle_{\mathcal{F}}\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{i=1}^n \left|\langle Y_i, \hat{\phi}_h\rangle_{\mathcal{F}}\langle Y_i, \hat{\phi}_h\rangle_{\mathcal{F}}\right| \\
&\overset{\substack{\text{CS}\\\text{Fubini}}}{\le} \left|\sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \mathrm{E}\{Y^{(u)}(s_u)Y^{(k)}(t_k)\}\phi_h^{(u)}(s_u)\phi_l^{(k)}(t_k)ds_u dt_k\right. \\
&\quad \left.-\sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \frac{1}{n}\sum_{i=1}^n Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)\hat{\phi}_h^{(u)}(s_u)\hat{\phi}_l^{(k)}(t_k)ds_u dt_k\right| + \frac{1}{n(n-1)}\sum_{i=1}^n \|Y_i\|_{\mathcal{F}}\,\|\hat{\phi}_h\|_{\mathcal{F}}\,\|Y_i\|_{\mathcal{F}}\,\|\hat{\phi}_l\|_{\mathcal{F}} \\
&= \left|\sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \sigma_y^{(uk)}(s_u, t_k)\phi_h^{(u)}(s_u)\phi_l^{(k)}(t_k)ds_u dt_k - \sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \hat{\sigma}_y^{(uk)}(s_u, t_k)\hat{\phi}_h^{(u)}(s_u)\hat{\phi}_l^{(k)}(t_k)ds_u dt_k\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{i=1}^n O_p(1)\cdot O_p(1) \\
&\le \left|\sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \sigma_y^{(uk)}(s_u, t_u)\left\{\phi_h^{(u)}(s_u)\phi_l^{(k)}(t_k) - \hat{\phi}_h^{(u)}(s_u)\hat{\phi}_l^{(k)}(t_k)\right\}ds_u dt_k\right| \\
&\quad + \left|\sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \left\{\sigma_y^{(uk)}(s_u, t_k) - \hat{\sigma}_y^{(uk)}(s_u, t_k)\right\}\hat{\phi}_h^{(u)}(s_u)\hat{\phi}_l^{(k)}(t_k)ds_u dt_k\right| + O_p(n^{-1}) \\
&\overset{CS}{\le} \sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \sigma_y^{(uu)}(s_u, s_u)^{1/2}\sigma_y^{(kk)}(t_k, t_k)^{1/2}\left|\phi_h^{(u)}(s_u)\phi_l^{(k)}(t_k) - \hat{\phi}_h^{(u)}(s_u)\hat{\phi}_l^{(k)}(t_k)\right|ds_u dt_k \\
&\quad + \left|\sum_{k=1}^K \int_{\mathcal{T}_k} \left\{(\mathcal{C}_y - \hat{\mathcal{C}}_y)\hat{\phi}_h\right\}^{(k)}(t_k)\,\hat{\phi}_l^{(k)}(t_k)dt_k\right| + O_p(n^{-1}) \\
&\overset{CS}{\le} \|\sigma_y\|_\infty\left\{\sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \left|\phi_h^{(u)}(s_u)\right|\left|\phi_l^{(k)}(t_k) - \hat{\phi}_l^{(k)}(t_k)\right|ds_u dt_k\right. \\
&\quad \left.+ \sum_{u=1}^K\sum_{k=1}^K \int_{\mathcal{T}_u}\int_{\mathcal{T}_k} \left|\phi_h^{(u)}(s_u) - \hat{\phi}_h^{(u)}(s_u)\right|\left|\hat{\phi}_l^{(k)}(t_k)\right|ds_u dt_k\right\} + \left\|(\mathcal{C}_y - \hat{\mathcal{C}}_y)\hat{\phi}_h\right\|_{\mathcal{F}}\left\|\hat{\phi}_l\right\|_{\mathcal{F}} + O_p(n^{-1}) \\
&\overset{CS}{\le} \|\sigma_y\|_\infty KT\left\{\|\phi_h\|_{\mathcal{F}}\,\|\phi_l - \hat{\phi}_l\|_{\mathcal{F}} + \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}}\,\|\hat{\phi}_l\|_{\mathcal{F}}\right\} + \|\mathcal{C}_y - \hat{\mathcal{C}}_y\|_{op}\,\|\hat{\phi}_h\|_{\mathcal{F}}\,\|\hat{\phi}_l\|_{\mathcal{F}} + O_p(n^{-1}) \\
&= \|\sigma_y\|_\infty KT\left(\left\|\phi_l - \hat{\phi}_l\right\|_{\mathcal{F}} + \left\|\phi_h - \hat{\phi}_h\right\|_{\mathcal{F}}\right) + O_p(c_n^y) + O_p(n^{-1}) \quad \cdots\cdots \quad \text{Ⓐ}
\end{aligned}
$$

where the last equality follows from the regularity condition R3. Secondly,

$$
\begin{aligned}
&\left|V_{yx}^{(hd)}-\hat{V}_{yx}^{(hd)}\right| = \left|\mathrm{Cov}(\eta_h,\gamma_d)-\frac{1}{n-1}\sum_{i=1}^{n}\hat{\eta}_h\hat{\gamma}_d\right| \\
&= \left|\mathrm{Cov}(\eta_h,\gamma_d)-\frac{1}{n}\sum_{i=1}^{n}\hat{\eta}_{ih}\hat{\gamma}_{id}-\frac{1}{n(n-1)}\sum_{i=1}^{n}\hat{\eta}_{ih}\hat{\gamma}_{id}\right| \\
&\le \left|\mathrm{Cov}(\langle Y,\phi_h\rangle_{\mathcal{F}}\langle \mathbf{X},\mathbf{w}_d\rangle)-\frac{1}{n}\sum_{i=1}^{n}\langle Y_i,\hat{\phi}_h\rangle_{\mathcal{F}}\langle \mathbf{X}_i,\hat{\mathbf{w}}_d\rangle\right| + \frac{1}{n(n-1)}\sum_{i=1}^{n}\left|\langle Y_i,\hat{\phi}_h\rangle_{\mathcal{F}}\langle \mathbf{X}_i,\hat{\mathbf{w}}_d\rangle\right| \\
&\overset{\substack{\text{CS}\\ \text{Fubini}}}{\le} \left|\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\mathrm{E}\{Y^{(k)}(t_k)X_r\}\phi_h^{(k)}(t_k)w_{dr}dt_k-\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\frac{1}{n}\sum_{i=1}^{n}Y_i^{(k)}(t_k)X_{ir}\hat{\phi}_h^{(k)}(t_k)\hat{w}_{dr}dt_k\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{i=1}^{n}\|Y_i\|_{\mathcal{F}}\,\|\hat{\phi}_h\|_{\mathcal{F}}\,\|\mathbf{X}_i\|\,\|\hat{\mathbf{w}}_d\| \\
&= \left|\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_{yx}^{(k)}(t_k,r)\phi_h^{(k)}(t_k)w_{dr}dt_k-\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\hat{\sigma}_{yx}^{(k)}(t_k,r)\hat{\phi}_h^{(k)}(t_k)\hat{w}_{dr}dt_k\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{i=1}^{n}O_p(1)\cdot O_p(1) \\
&\le \left|\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_{yx}^{(k)}(t_k,r)\{\phi_h^{(k)}(t_k)w_{dr}-\hat{\phi}_h^{(k)}(t_k)\hat{w}_{dr}\}dt_k\right| \\
&\quad + \left|\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\{\sigma_{yx}^{(k)}(t_k,r)-\hat{\sigma}_{yx}^{(k)}(t_k,r)\}\hat{\phi}_h^{(k)}(t_k)\hat{w}_{dr}dt_k\right| + O_p(n^{-1}) \\
&\overset{CS}{\le} \sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\sigma_y^{(kk)}(t_k,t_k)^{1/2}\sigma_x(r,r)^{1/2}\left|\phi_h^{(k)}(t_k)w_{dr}-\hat{\phi}_h^{(k)}(t_k)\hat{w}_{dr}\right|dt_k \\
&\quad + \left|\sum_{r=1}^{p}\left\{(\mathcal{C}_{yx}-\hat{\mathcal{C}}_{yx})\hat{\phi}_h\right\}_r\hat{w}_{dr}\right| + O_p(n^{-1}) \\
&\overset{CS}{\le} \|\sigma_y\|_\infty^{1/2}\|\sigma_x\|_\infty^{1/2}\left\{\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\left|\phi_h^{(k)}(t_k)\right||w_{dr}-\hat{w}_{dr}|dt_k+\sum_{r=1}^{p}\sum_{k=1}^{K}\int_{\mathcal{T}_k}\left|\phi_h^{(k)}(t_k)-\hat{\phi}_h^{(k)}(t_k)\right||\hat{w}_{dr}|dt_k\right\} \\
&\quad + \left\|(\mathcal{C}_{yx}-\hat{\mathcal{C}}_{yx})\hat{\phi}_h\right\|\,\|\hat{\mathbf{w}}_d\| + O_p(n^{-1}) \\
&\overset{CS}{\le} \{\|\sigma_y\|_\infty\,\|\sigma_x\|_\infty\,KTp\}^{1/2}\left\{\|\phi_h\|_{\mathcal{F}}\,\|\mathbf{w}_d-\hat{\mathbf{w}}_d\|+\left\|\phi_h-\hat{\phi}_h\right\|_{\mathcal{F}}\,\|\hat{\mathbf{w}}_d\|\right\} \\
&\quad + \left\|\mathcal{C}_{yx}-\hat{\mathcal{C}}_{yx}\right\|_{op}\left\|\hat{\phi}_h\right\|_{\mathcal{F}}\,\|\hat{\mathbf{w}}_d\| + O_p(n^{-1}) \\
&= \{\|\sigma_y\|_\infty\,\|\sigma_x\|_\infty\,KTp\}^{1/2}\left(\|\mathbf{w}_d-\hat{\mathbf{w}}_d\|+\left\|\phi_h-\hat{\phi}_h\right\|_{\mathcal{F}}\right)+O_p(n^{-1/2})\ \cdots\cdots\ \text{Ⓑ}
\end{aligned}
$$

where the last equality follows from Lemma 2. Thirdly, similar derivations imply that

$$
\left|V_{xy}^{(jl)}-\hat{V}_{xy}^{(jl)}\right| = \{\|\sigma_y\|_\infty\,\|\sigma_x\|_\infty\,KTP\}^{1/2}\left(\|\mathbf{w}_j-\hat{\mathbf{w}}_j\|+\left\|\phi_l-\hat{\phi}_l\right\|_{\mathcal{F}}\right)+O_p(n^{-1/2})\ \cdots\cdots\ \text{Ⓒ}
$$

Lastly,

$$
\begin{aligned}
|V_x^{(jd)} - \hat{V}_x^{(jd)}| &= \left|\mathrm{Cov}(\gamma_j, \gamma_d) - \frac{1}{n-1}\sum_{i=1}^{n} \hat{\gamma}_{ij}\hat{\gamma}_{id}\right| \\
&= \left|\mathrm{Cov}(\gamma_j, \gamma_d) - \frac{1}{n}\sum_{i=1}^{n} \hat{\gamma}_{ij}\hat{\gamma}_{id} - \frac{1}{n(n-1)}\sum_{i=1}^{n} \hat{\gamma}_{ij}\hat{\gamma}_{id}\right| \\
&\leq \left|\mathrm{Cov}(\langle \mathbf{X}, \mathbf{w}_j\rangle\langle \mathbf{X}, \mathbf{w}_d\rangle) - \frac{1}{n}\sum_{i=1}^{n}\langle X_i, \hat{\mathbf{w}}_j\rangle\langle \mathbf{X}_i, \hat{\mathbf{w}}_d\rangle\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{i=1}^{n} |\langle \mathbf{X}_i, \hat{\mathbf{w}}_j\rangle\langle \mathbf{X}_i, \hat{\mathbf{w}}_d\rangle| \\
&\overset{CS}{\leq} \left|\sum_{r=1}^{p}\sum_{q=1}^{p} \mathrm{E}(X_r X_q) w_{jr} w_{dq} - \sum_{r=1}^{p}\sum_{q=1}^{p}\frac{1}{n}\sum_{i=1}^{n} X_{ir}X_{iq}\hat{w}_{jr}\hat{w}_{dq}\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{i=1}^{n} \|\mathbf{X}_i\|\,\|\hat{\mathbf{w}}_j\|\,\|\mathbf{X}_i\|\,\|\hat{\mathbf{w}}_d\| \\
&= \left|\sum_{r=1}^{p}\sum_{q=1}^{p} \sigma_x(r,q) w_{jr} w_{dq} - \sum_{r=1}^{p}\sum_{q=1}^{p} \hat{\sigma}_x(r,q)\hat{w}_{jr}\hat{w}_{dq}\right| + \frac{1}{n(n-1)}\sum_{i=1}^{n} O_p(1)\cdot O_p(1) \\
&\leq \left|\sum_{r=1}^{p}\sum_{q=1}^{p} \sigma_x(r,q)\{w_{jr}w_{dq} - \hat{w}_{jr}\hat{w}_{dq}\}\right| \\
&\quad + \left|\sum_{r=1}^{p}\sum_{q=1}^{p}\{\sigma_x(r,q) - \hat{\sigma}_x(r,q)\}\hat{w}_{jr}\hat{w}_{dq}\right| + O_p(n^{-1}) \\
&\overset{CS}{\leq} \sum_{r=1}^{p}\sum_{q=1}^{p} \sigma_x(r,r)^{1/2}\sigma_x(q,q)^{1/2}\,|w_{jr}w_{dq} - \hat{w}_{jr}\hat{w}_{dq}| \\
&\quad + \left|\sum_{q=1}^{p}\left\{(\mathcal{C}_x - \hat{\mathcal{C}}_x)\hat{\mathbf{w}}_j\right\}_q \hat{w}_{dq}\right| + O_p(n^{-1}) \\
&\overset{CS}{\leq} \|\sigma_x\|_\infty\left\{\sum_{r=1}^{p}\sum_{q=1}^{p}|w_{jr}||w_{dq} - \hat{w}_{dq}| + \sum_{r=1}^{p}\sum_{q=1}^{p}|w_{jr} - \hat{w}_{jr}||\hat{w}_{dq}|\right\} \\
&\quad + \|(\mathcal{C}_x - \hat{\mathcal{C}}_x)\hat{\mathbf{w}}_j\|\,\|\hat{\mathbf{w}}_d\| + O_p(n^{-1}) \\
&\overset{CS}{\leq} \|\sigma_x\|_\infty\, p\,\{\|\mathbf{w}_j\|\|\mathbf{w}_d - \hat{\mathbf{w}}_d\| + \|\mathbf{w}_j - \hat{\mathbf{w}}_j\|\|\hat{\mathbf{w}}_d\|\} \\
&\quad + \|\mathcal{C}_x - \hat{\mathcal{C}}_x\|_{op}\,\|\hat{\mathbf{w}}_j\|\,\|\hat{\mathbf{w}}_d\| + O_p(n^{-1}) \\
&= \|\sigma_x\|_\infty\, p\,(\|\mathbf{w}_d - \hat{\mathbf{w}}_d\| + \|\mathbf{w}_j - \hat{\mathbf{w}}_j\|) + O_p(c_n^x) + O_p(n^{-1}) \;\cdots\cdots\; \text{(D)}
\end{aligned}
$$

where the last equality follows from the regularity condition R4. By inserting and combining

Ⓐ, Ⓑ, Ⓒ and Ⓓ into equation (2), and applying Lemma 1, we can establish that

$$\lambda_{\max}(V - \hat{V}) \le O_p(L \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)).$$

□

Now we move on to the proof of Theorem 6.

I. Eigenvalues

Let $\boldsymbol{\eta} = [\eta_1, \dots, \eta_L]^T$ denote a vector of first L functional PC scores, where $\eta_h = \langle Y, \phi_h \rangle_{\mathcal{F}} = \langle Y^{[L]}, \phi_h \rangle_{\mathcal{F}}$, $h = 1, \dots, L$. Let $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_J]^T$ denote a vector of first J vector PC scores, where $\gamma_j = \langle \mathbf{X}, \mathbf{w}_j \rangle = \langle \mathbf{X}^{[J]}, \mathbf{w}_j \rangle$, $j = 1, \dots, J$. Define a $(L + J)$-dimensional vector that $\boldsymbol{\chi} = [\boldsymbol{\eta}^T, \boldsymbol{\gamma}^T]^T$ that concatenates the first L functional PC scores and then the first J PC scores. For fixed $m = 1, \dots, M$, we first establish that

$$\begin{aligned}
\langle \tilde{\mathbf{Z}}, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} &= \langle Y^{[L]}, \tilde{\psi}_m \rangle_{\mathcal{F}} + \langle \mathbf{X}^{[J]}, \tilde{\boldsymbol{\theta}}_m \rangle \\
&= \left\langle Y^{[L]}, \sum_{h=1}^{L} c_{mh} \phi_h \right\rangle_{\mathcal{F}} + \left\langle \mathbf{X}^{[J]}, \sum_{j=1}^{J} d_{mj} \mathbf{w}_j \right\rangle \quad (\because \text{Theorem 4}) \\
&= \sum_{h=1}^{L} c_{mh} \eta_h + \sum_{j=1}^{J} d_{mj} \gamma_j \\
&= \mathbf{e}_m^T \boldsymbol{\chi},
\end{aligned}$$

where $\mathbf{e}_m = [c_{m1}, \dots, c_{mL}, d_{m1}, \dots, d_{mJ}]^T$. Then, it holds that

$$\begin{aligned}
|\tilde{\lambda}_m - \hat{\lambda}_m| &= |\mathrm{Var}(\langle \tilde{\mathbf{Z}}, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}) - \mathbf{e}_m^T \hat{V} \mathbf{e}_m| \\
&= |\mathbf{e}_m^T \mathrm{Var}(\boldsymbol{\chi}) \mathbf{e}_m - \mathbf{e}_m^T \hat{V} \mathbf{e}_m| \\
&= |\mathbf{e}_m^T V \mathbf{e}_m - \mathbf{e}_m^T \hat{V} \mathbf{e}_m| \\
&= |(\mathbf{e}_m - \hat{\mathbf{e}}_m)^T V \mathbf{e}_m + \hat{\mathbf{e}}_m^T V (\mathbf{e}_m - \hat{\mathbf{e}}_m) + \hat{\mathbf{e}}_m^T (V - \hat{V}) \hat{\mathbf{e}}_m|
\end{aligned}$$

$$
\begin{aligned}
&\le \tilde{\lambda}_m \|\mathbf{e}_m - \hat{\mathbf{e}}_m\| \|\mathbf{e}_m\| + \lambda_{\max}(V)\|\mathbf{e}_m - \hat{\mathbf{e}}_m\| + \lambda_{\max}(V - \hat{V})\|\hat{\mathbf{e}}_m\| \\
&\le \|\mathbf{e}_m - \hat{\mathbf{e}}_m\|(\tilde{\lambda}_m + \tilde{\lambda}_1) + \lambda_{\max}(V - \hat{V}) \\
&\le 2\tilde{\lambda}_1 \cdot \|\mathbf{e}_m - \hat{\mathbf{e}}_m\| + \lambda_{\max}(V - \hat{V}) \quad (\because \tilde{\lambda}_m \le \tilde{\lambda}_1) \\
&\le 2\tilde{\lambda}_1 \cdot \frac{2^{3/2}\lambda_{\max}(V - \hat{V})}{\min(\tilde{\lambda}_{m-1} - \tilde{\lambda}_m, \tilde{\lambda}_m - \tilde{\lambda}_{m+1})} + \lambda_{\max}(V - \hat{V}) \\
&= \lambda_{\max}(V - \hat{V}) \cdot \left\{1 + \frac{2^{5/2}\tilde{\lambda}_1}{\min(\tilde{\lambda}_{m-1} - \tilde{\lambda}_m, \tilde{\lambda}_m - \tilde{\lambda}_{m+1})}\right\} \\
&= \lambda_{\max}(V - \hat{V}) \cdot \text{constant} \\
&= O_p(L \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)) \quad (\because \text{Lemma 3})
\end{aligned}
$$

where the fourth inequality follows from Corollary 1 in Yu et al. (2015), and the constant term emerges as λ_m is assumed to have multiplicity 1 (see the statement of Theorem 5).

II. Hybrid PCs (eigenfunctions)

For fixed $m = 1, \ldots, M$, first consider the functional part of $\boldsymbol{\xi}_m = (\psi_m, \boldsymbol{\theta}_m)$.

$$
\begin{aligned}
\|\tilde{\psi}_m - \hat{\psi}_m\|_{\mathcal{F}} &= \left\| \sum_{h=1}^{L} c_{mh}\phi_h - \sum_{h=1}^{L} \hat{c}_{mh}\hat{\phi}_h \right\|_{\mathcal{F}} \quad (\because \text{Theorem 4}) \\
&= \left\| \sum_{h=1}^{L} (c_{mh} - \hat{c}_{mh})\phi_h + \sum_{h=1}^{L} \hat{c}_{mh}(\phi_h - \hat{\phi}_h) \right\|_{\mathcal{F}} \\
&\le \sum_{h=1}^{L} |c_{mh} - \hat{c}_{mh}| \|\phi_h\|_{\mathcal{F}} + \sum_{h=1}^{L} |\hat{c}_{mh}| \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} \\
&\overset{CS}{\le} L^{1/2}\|\mathbf{c}_m - \hat{\mathbf{c}}_m\| + L^{1/2}\|\mathbf{c}_m\| \, O_p(\Delta_L^y c_n^y) \quad (\because \text{Lemma 1}) \\
&\le L^{1/2}\left\{\|\mathbf{e}_m - \hat{\mathbf{e}}_m\| + O_p(\Delta_L^y c_n^y)\right\} \quad (\because \|\mathbf{c}_m\| \le \|\mathbf{e}_m\| = 1) \\
&\le L^{1/2}\left\{\frac{2^{3/2}\lambda_{\max}(V - \hat{V})}{\min(\tilde{\lambda}_{m-1} - \tilde{\lambda}_m, \tilde{\lambda}_m - \tilde{\lambda}_{m+1})} + O_p(\Delta_L^y c_n^y)\right\} \\
&= L^{1/2} \cdot O_p(L \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)) + O_p(\Delta_L^y c_n^y) \quad (\because \text{Lemma 3}) \\
&= O_p(L^{3/2} \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)) \ \cdots\cdots \ â‘
\end{aligned}
$$

where the fourth inequality follows from Corollary 1 in Yu et al. (2015). Secondly, we consider the vector part of $\boldsymbol{\xi}_m = (\psi_m, \boldsymbol{\theta}_m)$.

$$\begin{aligned}
\|\tilde{\boldsymbol{\theta}}_m - \hat{\boldsymbol{\theta}}_m\| &= \left\| \sum_{j=1}^{J} d_{mj}\mathbf{w}_j - \sum_{h=1}^{L} \hat{d}_{mj}\hat{\mathbf{w}}_j \right\| \quad (\because \text{Theorem 4}) \\
&= \left\| \sum_{j=1}^{J} (d_{mj} - \hat{d}_{mj})\mathbf{w}_j + \sum_{j=1}^{J} \hat{d}_{mj}(\mathbf{w}_j - \hat{\mathbf{w}}_j) \right\| \\
&\le \sum_{j=1}^{J} |d_{mj} - \hat{d}_{mj}| \|\mathbf{w}_j\| + \sum_{j=1}^{J} |\hat{d}_{mj}| \|\mathbf{w}_j - \hat{\mathbf{w}}_j\| \\
&\overset{CS}{\le} J^{1/2}\|\mathbf{d}_m - \hat{\mathbf{d}}_m\| + J^{1/2}\|\mathbf{d}_m\|\, O_p(\Delta_J^x c_n^x) \quad (\because \text{Lemma 1}) \\
&\le J^{1/2}\left\{\|\mathbf{e}_m - \hat{\mathbf{e}}_m\| + O_p(\Delta_J^x c_n^x)\right\} \quad (\because \|\mathbf{d}_m\| \le \|\mathbf{e}_m\| = 1) \\
&\le J^{1/2}\left\{\frac{2^{3/2}\lambda_{\max}(V - \hat{V})}{\min(\tilde{\lambda}_{m-1} - \tilde{\lambda}_m, \tilde{\lambda}_m - \tilde{\lambda}_{m+1})} + O_p(\Delta_J^x c_n^x)\right\} \\
&= J^{1/2} \cdot O_p(L \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)) + O_p(\Delta_J^x c_n^x) \quad (\because \text{Lemma 3}) \\
&= O_p(L \max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)) \ \cdots\cdots \ \text{②}
\end{aligned}$$

where the last inequality again follows from Corollary 1 in Yu et al. (2015). By combining the results ① and ②, we can establish that

$$\|\tilde{\boldsymbol{\xi}}_m - \hat{\boldsymbol{\xi}}_m\|_{\mathcal{H}} \le \|\tilde{\psi}_m - \hat{\psi}_m\|_{\mathcal{F}} + \|\tilde{\boldsymbol{\theta}}_m - \hat{\boldsymbol{\theta}}_m\| = O_p(L^{3/2}\max(n^{-1/2}, \Delta_L^y c_n^y, \Delta_J^x c_n^x)).$$

III. Hybrid objects

We first note that functional and vector PC scores are uniformly bounded in probability. To see this, for $\epsilon > 0$, set $c = (2\tau_1/\epsilon)^{1/2}$. Then,

$$P(|\eta_{ih}| > c) \overset{Markov}{\le} \frac{1}{c^2}\mathrm{E}|\eta_{ih}|^2 = \frac{1}{c^2}\mathrm{Var}(\eta_{ih}) = \frac{\tau_h}{c^2} < \epsilon,$$

which implies that $|\eta_{ih}| = O_p(1)$ for $h = 1, \ldots, L$. Similarly, it is straightforward to show that $|\gamma_{ij}| = O_p(1)$ for $j = 1, \ldots, J$.

Recall that $\tilde{\mathbf{Z}}_i = (Y_i^{[L]}, \mathbf{X}_i^{[J]}) = (\sum_{h=1}^{L} \eta_{ih}\phi_h, \sum_{j=1}^{J} \gamma_{ij}\mathbf{w}_j)$ and $\hat{\mathbf{Z}}_i = (\hat{Y}_i^{[L]}, \hat{\mathbf{X}}_i^{[J]})$ $= (\sum_{h=1}^{L} \hat{\eta}_{ih}\hat{\phi}_h, \sum_{j=1}^{J} \hat{\gamma}_{ij}\hat{\mathbf{w}}_j)$. We first consider the norm of the difference between their functional parts.

$$\begin{aligned}
\left\| Y_i^{[L]} - \hat{Y}_i^{[L]} \right\|_{\mathcal{F}} &= \left\| \sum_{h=1}^{L} \eta_{ih}\phi_h - \sum_{h=1}^{L} \hat{\eta}_{ih}\hat{\phi}_h \right\|_{\mathcal{F}} \\
&\le \sum_{h=1}^{L} |\eta_{ih}| \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} + \sum_{h=1}^{L} |\eta_{ih} - \hat{\eta}_{ih}| \|\hat{\phi}_h\|_{\mathcal{F}} \\
&= \sum_{h=1}^{L} |\eta_{ih}| \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} + \sum_{h=1}^{L} |\langle Y_i, \phi_h \rangle_{\mathcal{F}} - \langle Y_i, \hat{\phi}_h \rangle_{\mathcal{F}}| \\
&\overset{CS}{\le} \sum_{h=1}^{L} |\eta_{ih}| \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} + \sum_{h=1}^{L} \|Y_i\|_{\mathcal{F}} \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} \\
&= \sum_{h=1}^{L} (|\eta_{ih}| + \|Y_i\|_{\mathcal{F}}) \|\phi_h - \hat{\phi}_h\|_{\mathcal{F}} \\
&= \sum_{h=1}^{L} \{O_p(1) + O_p(1)\} \cdot O_p(\Delta_L^y c_n^y) \quad (\because \text{Lemma 1}) \\
&= O_p(L\, \Delta_L^y c_n^y) \;\cdots\cdots\; \text{Ⓐ}
\end{aligned}$$

Secondly, we consider the norm of the difference between their vector parts.

$$\begin{aligned}
\left\| \mathbf{X}_i^{[J]} - \hat{\mathbf{X}}_i^{[j]} \right\| &= \left\| \sum_{j=1}^{J} \gamma_{ij}\mathbf{w}_j - \sum_{j=1}^{J} \hat{\gamma}_{ij}\hat{\mathbf{w}}_j \right\| \\
&\le \sum_{j=1}^{J} |\gamma_{ij}| \|\mathbf{w}_j - \hat{\mathbf{w}}_j\| + \sum_{j=1}^{J} |\gamma_{ij} - \hat{\gamma}_{ij}| \|\hat{\mathbf{w}}_j\| \\
&= \sum_{j=1}^{J} |\gamma_{ij}| \|\mathbf{w}_j - \hat{\mathbf{w}}_j\| + \sum_{j=1}^{J} |\langle \mathbf{X}_i, \mathbf{w}_j \rangle - \langle \mathbf{X}_i, \hat{\mathbf{w}}_j \rangle| \\
&\overset{CS}{\le} \sum_{j=1}^{J} |\gamma_{ij}| \|\mathbf{w}_j - \hat{\mathbf{w}}_j\| + \sum_{j=1}^{L} \|\mathbf{X}_i\| \|\mathbf{w}_j - \hat{\mathbf{w}}_j\| \\
&= \sum_{j=1}^{J} (|\gamma_{ij}| + \|\mathbf{X}_i\|) \|\mathbf{w}_j - \hat{\mathbf{w}}_j\|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{J} \{O_p(1) + O_p(1)\} \cdot O_p(\Delta_J^x c_n^x) \quad (\because \text{Lemma 1}) \\
&= O_p(\Delta_J^x c_n^x) \ \cdots\cdots \ \text{(B)}
\end{aligned}
$$

By combining the results (A) and (B), we can establish that

$$
\left\|\tilde{\mathbf{Z}}_i - \hat{\mathbf{Z}}_i\right\|_{\mathcal{H}} \leq \left\|Y_i^{[L]} - \hat{Y}_i^{[L]}\right\|_{\mathcal{F}} + \left\|\mathbf{X}_i^{[J]} - \hat{\mathbf{X}}_i^{[j]}\right\| = O_p(\max(L\,\Delta_L^y c_n^y, \Delta_J^x c_n^x)).
$$

IV. Hybrid PC scores

Recall from Theorem 4 that $\tilde{\boldsymbol{\xi}}_m = (\sum_{h=1}^{L} c_{mh}\phi_h, \sum_{j=1}^{J} d_{mj} w_j)$, along with its estimated version $\hat{\boldsymbol{\xi}}_m = (\sum_{h=1}^{L} \hat{c}_{mh}\hat{\phi}_h, \sum_{j=1}^{J} \hat{d}_{mj}\hat{w}_j)$. Then, for $m = 1, \ldots M$

$$
\begin{aligned}
\langle \tilde{\mathbf{Z}}_i, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} &= \langle Y_i^{[L]}, \tilde{\psi}_m \rangle_{\mathcal{F}} + \langle \mathbf{X}_i^{[J]}, \tilde{\boldsymbol{\theta}}_m \rangle \\
&= \sum_{k=1}^{K} \int_{\mathcal{T}_k} \left(\sum_{h=1}^{L} \eta_{ih} \phi_h^{(k)}(t_k) \right) \left(\sum_{l=1}^{L} c_{ml} \phi_l^{(k)}(t_k) \right) dt_k + \sum_{r=1}^{p} \left(\sum_{j=1}^{J} \gamma_{ij} w_{jr} \right) \left(\sum_{u=1}^{J} d_{mu} w_{ur} \right) \\
&= \sum_{h=1}^{L} \sum_{l=1}^{L} \eta_{ih} c_{ml} \sum_{k=1}^{K} \int_{\mathcal{T}_k} \phi_h^{(k)}(t_k) \phi_l^{(k)}(t_k) dt_k + \sum_{j=1}^{J} \sum_{u=1}^{J} \gamma_{ij} d_{mu} \mathbf{w}_j^T \mathbf{w}_u \\
&= \sum_{h=1}^{L} \eta_{ih} c_{mh} + \sum_{j=1}^{J} \gamma_{ij} d_{mj} \\
&= \tilde{\rho}_{im},
\end{aligned}
$$

where the second and last equalities follow from Theorem 4. A similar derivation shows that $\langle \hat{\mathbf{Z}}_i, \hat{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} = \hat{\rho}_{im}$. Then,

$$
\begin{aligned}
|\tilde{\rho}_{im} - \hat{\rho}_{im}| &= |\langle \tilde{\mathbf{Z}}_i, \tilde{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} - \langle \hat{\mathbf{Z}}_i, \hat{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}| \\
&= |\langle \tilde{\mathbf{Z}}_i, \tilde{\boldsymbol{\xi}}_m - \hat{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}} + \langle \tilde{\mathbf{Z}}_i - \hat{\mathbf{Z}}_i, \hat{\boldsymbol{\xi}}_m \rangle_{\mathcal{H}}| \\
&\overset{CS}{\leq} \|\tilde{\mathbf{Z}}_i\|_{\mathcal{H}} \, \|\tilde{\boldsymbol{\xi}}_m - \hat{\boldsymbol{\xi}}_m\|_{\mathcal{H}} + \|\tilde{\mathbf{Z}}_i - \hat{\mathbf{Z}}_i\|_{\mathcal{H}} \, \|\hat{\boldsymbol{\xi}}_m\|_{\mathcal{H}}
\end{aligned}
$$

$$\leq O_p(1)\cdot O_p(L^{3/2}\max(n^{-1/2},\Delta_L^y c_n^y,\Delta_J^x c_n^x))+O_p(\max(L\,\Delta_L^y c_n^y,\Delta_J^x c_n^x))$$

$$= O_p(L^{3/2}\max(n^{-1/2},\Delta_L^y c_n^y,\Delta_J^x c_n^x)),$$

where the last inequality follows from applying the upper asymptotic bounds for the hybrid PCs and objects (see steps II and III of the proof), and the fact that $\|\tilde{\mathbf{Z}}_i\|_{\mathcal{H}}$ is bounded in probability using analogous arguments as for $\|\mathbf{Z}\|_{\mathcal{H}}$ in Step-IV of the proof of Theorem 5.

□

· **Proof of Theorem 7**

Proof: Let $\{Y_i\}_{i=1,\dots,n}$ be i.i.d. samples of Y. Then noting that $E\{Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)\} = \sigma_{yx}^{(uk)}(s_u,t_k)$ for all i, we can establish that

$$\mathrm{E}\left[\sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}\left\{\sigma_y^{(uk)}(s_u,t_k)-\hat{\sigma}_y^{(uk)}(s_u,t_k)\right\}^2 ds_u dt_k\right]$$
$$= \mathrm{E}\left[\sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}\left\{\sigma_y^{(uk)}(s_u,t_k)-\frac{1}{n}\sum_{i=1}^{n}Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)\right\}^2 ds_u dt_k\right]$$
$$\overset{Fubini}{=} \sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}\Bigg(\sigma_y^{(uk)}(s_u,t_k)^2+\mathrm{E}\left[\left\{\frac{1}{n}\sum_{i=1}^{n}Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)\right\}^2\right]$$
$$-\frac{2}{n}\sum_{i=1}^{n}\mathrm{E}\left\{Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)\right\}\sigma_y^{(uk)}(s_u,t_k)\Bigg)ds_u dt_k$$
$$= \sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left\{Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)Y_j^{(u)}(s_u)Y_j^{(k)}(t_k)\right\}-\sigma_y^{(uk)}(s_u,t_k)^2\right]ds_u dt_k$$
$$= \sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}\Bigg[\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{E}\left\{Y_i^{(u)}(s_u)^2Y_i^{(k)}(t_k)^2\right\}$$
$$+\frac{2}{n^2}\sum_{1\leq i<j\leq n}\mathrm{E}\left\{Y_i^{(u)}(s_u)Y_i^{(k)}(t_k)\right\}\mathrm{E}\left\{Y_j^{(u)}(s_u)Y_j^{(k)}(t_k)\right\}-\sigma_y^{(uk)}(s_u,t_k)^2\Bigg]ds_u dt_k$$
$$= \frac{1}{n}\sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}\left[\mathrm{E}\left\{Y^{(u)}(s_u)^2Y^{(k)}(t_k)^2\right\}-\sigma_y^{(uk)}(s_u,t_k)^2\right]ds_u dt_k = O_p(n^{-1}),$$

where the last equality follows from $\sum_{k=1}^{K}\sum_{u=1}^{K}\int_{\mathcal{T}_k}\int_{\mathcal{T}_u}E\{Y_i^{(u)}(s_u)^2Y_i^{(k)}(t_k)^2\}ds_u dt_k<\infty$ and the boundedness assumption of Theorem 1 on a covariance kernel σ_y. This implies

$$
\begin{aligned}
\|\mathcal{C}_y - \hat{\mathcal{C}}_y\|_{op} &= \sup_{\|f\|_{\mathcal{F}}=1} \|(\mathcal{C}_y - \hat{\mathcal{C}}_y)f\| \\
&= \sup_{\|f\|_{\mathcal{F}}=1} \left(\sum_{k=1}^{K} \int_{\mathcal{T}_k} \left[\sum_{u=1}^{K} \int_{\mathcal{T}_u} \left\{ \sigma_y^{(uk)}(s_u, t_k) - \hat{\sigma}_y^{(uk)}(s_u, t_k) \right\} f^{(u)}(s_u) ds_u \right]^2 dt_k \right)^{1/2} \\
&= \sup_{\|f\|_{\mathcal{F}}=1} \left(\sum_{k=1}^{K} \int_{\mathcal{T}_k} \langle \sigma_y^{(k)}(\cdot, t_k) - \hat{\sigma}_y^{(k)}(\cdot, t_k), f \rangle_{\mathcal{F}}^2 \, dt_k \right)^{1/2} \\
&\overset{CS}{\leq} \sup_{\|f\|_{\mathcal{F}}=1} \left(\sum_{k=1}^{K} \int_{\mathcal{T}_k} \|\sigma_y^{(k)}(\cdot, t_k) - \hat{\sigma}_y^{(k)}(\cdot, t_k)\|_{\mathcal{F}}^2 \, \|f\|_{\mathcal{F}}^2 \, dt_k \right)^{1/2} \\
&= \left[\sum_{k=1}^{K} \sum_{u=1}^{K} \int_{\mathcal{T}_k} \int_{\mathcal{T}_u} \left\{ \sigma_y^{(uk)}(s_u, t_k) - \hat{\sigma}_y^{(uk)}(s_u, t_k) \right\}^2 ds_u dt_k \right]^{1/2} \overset{Markov}{=} O_p(n^{-1/2})
\end{aligned}
$$

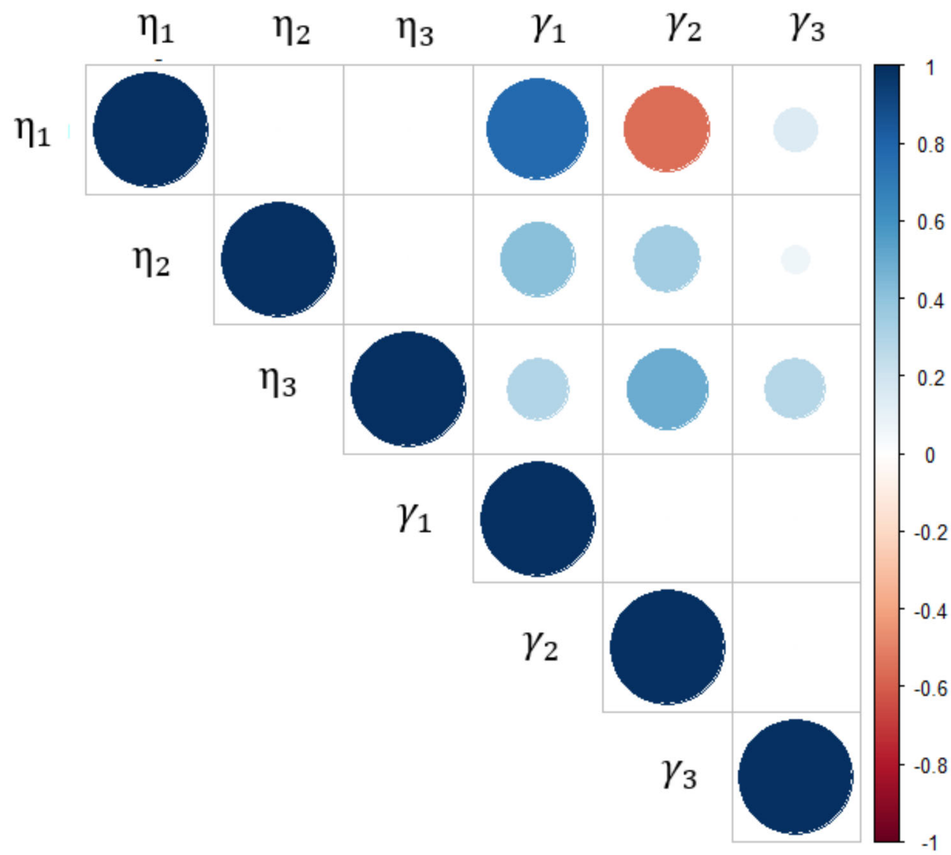
□

Appendix S2: Specification of $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{10}\}$ in Section 3

In the simulation study presented in Section 3 of the main paper, the vector part of the first 10 hybrid PC, $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{10}\}$, is set as the first 10 eigenvectors of a 10×10 compound symmetry correlation matrix with off-diagonal elements equal to 0.2. The exact values these eigenvectors are given as following:

$$
\Theta = [\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{10}] = \begin{bmatrix}
-0.224 & -0.014 & 0.000 & 0.410 & 0.000 & 0.344 & 0.000 & -0.404 & 0.000 & 0.000 \\
-0.224 & -0.086 & 0.000 & 0.393 & 0.000 & 0.014 & 0.000 & 0.537 & 0.000 & 0.000 \\
-0.224 & 0.569 & 0.193 & -0.166 & 0.000 & 0.208 & 0.010 & 0.114 & 0.071 & -0.011 \\
-0.224 & -0.376 & 0.024 & -0.367 & 0.000 & 0.400 & 0.005 & 0.105 & 0.044 & -0.002 \\
-0.224 & -0.108 & 0.320 & -0.013 & 0.000 & -0.245 & 0.513 & -0.095 & 0.061 & -0.016 \\
-0.224 & 0.040 & -0.322 & -0.054 & 0.000 & -0.131 & 0.026 & -0.044 & 0.251 & 0.509 \\
-0.224 & 0.040 & -0.368 & -0.054 & 0.000 & -0.131 & 0.026 & -0.044 & 0.224 & -0.490 \\
-0.224 & -0.108 & 0.326 & -0.013 & 0.000 & -0.245 & -0.480 & -0.095 & 0.179 & -0.020 \\
-0.224 & 0.021 & -0.086 & -0.068 & -0.500 & -0.107 & -0.050 & -0.037 & -0.415 & 0.015 \\
-0.224 & 0.021 & -0.086 & -0.068 & 0.500 & -0.107 & -0.050 & -0.037 & -0.415 & 0.015
\end{bmatrix}
$$

Figure S1: Correlations between the first three functional (η_1, η_2, η_3) and vector $(\gamma_1, \gamma_2, \gamma_3)$ PC scores of 253 kidneys from the Emory renal study data.



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