Supplemental Information for "Supercurrent diode effect and finite momentum superconductivity"

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GINZBURG-LANDAU COEFFICIENT

We compute the the function $\alpha_{\boldsymbol{q}}$ microscopically within mean field theory. From Eqs. (1,4) of the main text, the mean-field free energy density is $f(\boldsymbol{q},\Delta)=|\Delta|^2/V - T\int d^2\boldsymbol{k}\mathrm{tr}\log[1+e^{-\mathcal{H}_{\boldsymbol{k}}(\boldsymbol{q},\Delta)/T}]$, where V is the attractive interaction strength and tr denotes trace in spin space.

By expansion of free energy density $f(\boldsymbol{q},\Delta)$ in terms of Δ we have

$$\alpha=\log\frac{T}{T_c}+\frac{1}{N_0}\sum_{\lambda=\pm}\oint_{\mathrm{FS}_\lambda}\frac{dk}{|\boldsymbol{v}^\lambda|}\left\{\phi\left(\frac{Q+\lambda\mathcal{E}_+}{2\pi T}\right)\cos^2\frac{\theta}{2}+\phi\left(\frac{Q+\lambda\mathcal{E}_-}{2\pi T}\right)\sin^2\frac{\theta}{2}\right\}, \tag{S1}$$

where $\lambda=\pm$ denote contributions from inner ($\lambda=+$) and outer ($\lambda=-$) Fermi surfaces respectively, $\phi(x)=\mathrm{Re}\left[\psi\left(\frac{1+ix}{2}\right)\right]-\psi\left(\frac{1}{2}\right)$, and ψ is the digamma function.

Here, $\boldsymbol{v}^\lambda=\partial_{\boldsymbol{k}}\xi^\lambda_{\boldsymbol{k}}$ is the electron velocity, $Q=\boldsymbol{v}^\lambda\cdot\boldsymbol{q}$ is the depairing energy of finite momentum pairing, $\mathcal{E}_\pm=|\boldsymbol{h}_+|\pm|\boldsymbol{h}_-|$ is the depairing energy of Zeeman splitting for inter- (+) or intra-pocket (−) Cooper pairs with $\boldsymbol{h}_\pm=\boldsymbol{B}+\boldsymbol{g}_{\frac{1}{2}\boldsymbol{q}\pm\boldsymbol{k}}$, and the angle $\theta=\langle\boldsymbol{h}_+,\boldsymbol{h}_-\rangle$ between $\boldsymbol{h}_\pm$ controls the ratio between inter- or intra-pocket Cooper pairs. Supercurrent affects $\boldsymbol{q}$ and hence depairing energy Q, while magnetic field $\boldsymbol{B}$ together with SOC affects depairing energies $\mathcal{E}_\pm$ and angle θ.

Near T_c, the temperature dependence of $\alpha_{\boldsymbol{q}}$ can be captured by the first term $\log(T/T_c)$, and we can set $T=T_c$ in the Fermi surface integrals. To evaluate the integral, notice that the field is weak $B\ll B_P$, one can expand the special function

$$\phi(x)=2.10x^2-2.01x^4+2.00x^6+O(x^8) \tag{S2}$$

and then integrate order by order to obtain Eq. (14) of the main text.

NONRECIPROCAL CRITICAL CURRENT AND POLARITY-DEPENDENT CRITICAL FIELD

For $\alpha_{\boldsymbol{q}}=\overline{\alpha}+a\delta q_\parallel^2-b\delta q_\parallel^3$, we find the supercurrent is

$$\beta J_\parallel/e=|\alpha_{\boldsymbol{q}}|\partial_\parallel\alpha_{\boldsymbol{q}}=2a\overline{\alpha}\delta q_\parallel-3\left(\overline{\alpha}b\right)\delta q_\parallel^2 \tag{S3}$$
$$+2a^2\delta q_\parallel^3-5(ab)\delta q_\parallel^4+3b^2\delta q_\parallel^5. \tag{S4}$$

Notice that to the leading order of $|\overline{\alpha}|^{\frac{1}{2}}$, critical currents $\pm J_c^\pm$ correspond to $\delta q_\parallel=\mp\delta q_c$ respectively, where $\delta q_c=\sqrt{|\overline{\alpha}|/3a}$. Then we have

$$\beta J_c^\pm/e=\frac{4|\overline{\alpha}|^{3/2}}{9a}\left(\sqrt{3a^3}\pm b\sqrt{|\overline{\alpha}|}\right)+O(|\overline{\alpha}|^{5/2}). \tag{S5}$$

Since $\overline{\alpha},a,b$ are all functions of B and T, the equation above determines two phase boundaries parametrized by J, B and T as depicted in Fig. 1b of the main text. At weak field, a can be treated as a constant, $b\propto B$ and $\overline{\alpha}=t(1-B^2/B_c^2)$ with reduced temperature $t=(T-T_c)/T_c$ and critical field $B_c\propto|t|^{1/2}$. Then Eqs. (16-22) of the main text can be obtained.

Especially for Rashba superconductors at weak field, from Eq. (13) of the main text and (S5) we obtain

$$\frac{\beta\boldsymbol{J}}{eN_0}=\frac{4}{3\sqrt{3}}|\alpha_{\boldsymbol{q}_0}|^2\xi\hat{\boldsymbol{n}}+\frac{4}{9}\frac{b_1|\alpha_{\boldsymbol{q}_0}|}{\xi^2}\boldsymbol{B}\times\hat{\boldsymbol{z}}, \tag{S6}$$

where $\hat{\boldsymbol{n}}=(\boldsymbol{q}-\boldsymbol{q}_0)/|\boldsymbol{q}-\boldsymbol{q}_0|$, and under magnetic field

$$\alpha_{\boldsymbol{q}_0}=t\left(1-\frac{B^2}{B_c^2}\right),\ \xi=C\frac{v_{\mathrm{F}}}{\pi T_c}\left|t\left(1-\frac{B^2}{B_c^2}\right)\right|^{-\frac{1}{2}}. \tag{S7}$$

Then we obtain the skewed phase boundary

$$\left(\frac{B}{B_c}\right)^2+\left|\frac{\boldsymbol{J}}{J_c}-\gamma\frac{\boldsymbol{B}\times\hat{\boldsymbol{z}}}{B_c}\left(1-\frac{B^2}{B_c^2}\right)^2\right|^{2/3}=1, \tag{S8}$$

with zero-field critical current

$$J_c=\frac{4eN_0^2C}{3\sqrt{3}\beta}\frac{v_{\mathrm{F}}}{\pi T_c}|t|^{3/2} \tag{S9}$$

and the skewness parameter

$$\gamma=\frac{b_1B_c|t|^{1/2}}{\sqrt{3}N_0}\left(\frac{\pi T_c}{v_{\mathrm{F}}}\right)^3=0.64\frac{\alpha_{\mathrm{R}}}{v_{\mathrm{F}}}\frac{B_c}{B_P}\sqrt{1-\frac{T}{T_c}}. \tag{S10}$$

To include higher order contributions, the supercurrent diode coefficient is

$$\delta = D\left(x\right), \quad x = \sqrt{\frac{|\alpha_{\boldsymbol{q}_0}|}{a^3}} b. \tag{S11}$$

The special function is

$$D(x) = \frac{J(x) - J(-x)}{J(x) + J(-x)}, \tag{S12}$$

where $J(x) = [Q(x) - \frac{3}{2}xQ(x)][-1 + Q(x)^2 - xQ(x)^3]$,

$$Q(x) = \frac{1}{3x} - \frac{\sqrt{n}}{2x} - \frac{1}{2}\sqrt{\frac{4}{5\sqrt{n}}\left(1 - \frac{2}{27x^2}\right) - \frac{n}{x^2} + \frac{8}{15x^2}}$$

and $n = \frac{2}{15}\left(z + 1/z + \frac{4}{3}\right)$,

$$z = \sqrt[3]{t + \sqrt{t^2 - 1}}, \quad t = \frac{135}{4}x^4 - 5x^2 + 1.$$

When $|x| < \frac{2}{3\sqrt{3}}$, $\frac{22}{27} < t < 1$ and z is complex. Denote $t = \cos\theta$, then $z = e^{i\theta/3}$ and $n = \frac{4}{15}(\cos\frac{\theta}{3} + \frac{2}{3})$ is real. Since $D(x) \approx x/\sqrt{3}$ we get the leading order contribution $\delta = \sqrt{\frac{|\alpha_{\boldsymbol{q}_0}|}{3a^3}} b$, namely

$$\delta = \delta_{\rm m}\sqrt{1 - \frac{T}{T_c(B)}}, \text{ with } \delta_{\rm m} = b\sqrt{\frac{a_0 T_c(B)}{3a^3}}. \tag{S13}$$

The expansion of $\alpha_{\boldsymbol{q}}$ near its minimum $\boldsymbol{q}_0$ in general can be anisotropic

$$\alpha_{\boldsymbol{q}+\boldsymbol{q}_0} = \alpha_{\boldsymbol{q}_0} + a(1+\epsilon)q_x^2 + a(1-\epsilon)q_y^2 + 2a\eta q_x q_y - (b_1 q_x^3 + b_2 q_y^3 + b_3 q_x q_y^2 + b_4 q_x^2 q_y), \tag{S14}$$

where $a > 0$ and $\epsilon^2 + \eta^2 < 1$ for stability. The supercurrent diode coefficient for supercurrent $\boldsymbol{J} = J(\cos\theta, \sin\theta)$ can be worked out as

$$\delta = \sqrt{\frac{|\alpha_{\boldsymbol{q}_0}|}{3a^3}} \frac{\left(\frac{b_1+b_3}{2} + \frac{b_1-b_3}{2}\cos 2\theta\right)\cos\theta + \left(\frac{b_2+b_4}{2} - \frac{b_2-b_4}{2}\cos 2\theta\right)\sin\theta}{(1+\epsilon\cos 2\theta + \eta\sin 2\theta)^{3/2}}. \tag{S15}$$

SUPERCURRENT DIODE EFFECT NEAR FFLO TRANSITION

When magnetic field is high, near a phase transition where two or more local minima of $\alpha_{\boldsymbol{q}}$ compete, strong supercurrent diode effect can happen. As a concrete example, we consider the transition from BCS phase to FF phase near the upturning point (T_*, B_*) (red star in Fig. 3a of the main text), and adapt the following free energy density expanded up to quartic order in q

$$\alpha_{\boldsymbol{q}} = c_0 + c_1 q^2 + c_2 q^4, \tag{S16}$$

where $c_1 = c(B_*^2 - B^2)$, and $c, c_2 > 0$. When $B < B_*$, $c_1 > 0$ and the BCS phase with zero Cooper pair momentum is the ground state. When $B > B_*$, $c_1 < 0$ and the ground state changes to the FF phase with Cooper pair momentum $q_0 \equiv \sqrt{|c_1|/(2c_2)} \propto \sqrt{B^2 - B_*^2}$. As a result, when one lowers the temperature, the in-plane critical field $B_c(T)$ exhibits an upturn across the upturning point and hence the name.

To better understand the FFLO physics we need quartic order Ginzburg-Landau analysis. One can write the order parameter in Fourier form $\Delta(\boldsymbol{r}) = \int d^2\boldsymbol{q}\Delta_{\boldsymbol{q}}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$ and the free energy then reads

$$F \equiv \int d^2\boldsymbol{r} f(\boldsymbol{r}) = \int d^2\boldsymbol{q}\alpha_{\boldsymbol{q}}|\Delta_{\boldsymbol{q}}|^2 + \frac{1}{2}\int d^2\boldsymbol{q}d^2\boldsymbol{p}d^2\boldsymbol{p}'\beta_{\boldsymbol{q}\boldsymbol{p}\boldsymbol{p}'}\Delta_{\boldsymbol{q}+\boldsymbol{p}}\Delta_{\boldsymbol{q}-\boldsymbol{p}}\Delta^*_{\boldsymbol{q}+\boldsymbol{p}'}\Delta^*_{\boldsymbol{q}-\boldsymbol{p}'}. \tag{S17}$$

By analyzing the quartic coefficients $\beta_{\boldsymbol{q}\boldsymbol{p}\boldsymbol{p}'}$ and the quadratic coefficients $\alpha_{\boldsymbol{q}}$ one can then distinguish FF and LO phases [1, 2]. The quartic coefficient $\beta_{\boldsymbol{q}\boldsymbol{p}\boldsymbol{p}'}$ is in general a function of momenta $\boldsymbol{q}, \boldsymbol{p}$ and $\boldsymbol{p}'$, which to the leading order can be treated as momentum-independent $\beta_{\boldsymbol{q}\boldsymbol{p}\boldsymbol{p}'} = \beta N_0$ if it does not change sign in the region where we are interested. One can work out β numerically

$$\beta = -\frac{T}{2N_0}\sum_{n\in\mathbb{Z}}\int d^2\boldsymbol{k}\mathrm{Tr}[G_e(\boldsymbol{k}, i\omega_n)(i\sigma_y)G_h(\boldsymbol{k}, i\omega_n)(i\sigma_y)^\dagger]^2 \tag{S18}$$

by Mastubara Green's functions $G_e = [i\omega_n - \mathcal{H}_{\boldsymbol{k}}]^{-1}$ and $G_h = [i\omega_n + \mathcal{H}^*_{-\boldsymbol{k}}]^{-1}$ with $\omega_n = (2n+1)\pi T$.

In the absence of SOC, quartic coefficient β in Eq. (6) of the main text accidentally vanishes at the upturning point, making it a tricritical point where BCS ($q=0$), FF (single-$\boldsymbol{q}$) and LO ($\pm\boldsymbol{q}$) phases compete. With SOC considered in this work, β is finite as long as SOC is nonzero, and the supercurrent can be calculated by Eq. (11) of the main text. In the following we assume SOC is finite such that $\beta \neq 0$ and expansion Eq. (S16) also applies.

As shown in Fig. S1, along a given direction, there are five zeros of supercurrent, one at metastable BCS phase $q=0$, one at ground state FF phase q_0, one at opposite FF phase $-q_0$ and the other two at excited states where superconductivity vanishes $\alpha_{\boldsymbol{q}} = 0$. We hence expect four extremal points $q_c^\pm, -q_c^\pm$ of the supercurrent, and the resulting maximum and minimum are critical currents $\pm J_c^\pm$ respectively.

Near the metastable BCS phase, the function $\alpha_{\boldsymbol{q}} = \alpha_{-\boldsymbol{q}}$ is fully symmetric, and there seems no diode effect. However, around the true ground state FF phase $\boldsymbol{q} = \boldsymbol{q}_0$ we have the expansion up to the nonzero third order

$$\alpha_{\boldsymbol{q}+\boldsymbol{q}_0} = c_0 - \frac{c_1^2}{4c_2} + 4c_2\left\{(\boldsymbol{q}\cdot\boldsymbol{q}_0)^2 + (\boldsymbol{q}\cdot\boldsymbol{q}_0)q^2\right\}. \quad \text{(S19)}$$

Since $|\boldsymbol{q}_0| \propto \mathrm{Re}\sqrt{B^2 - B_*^2}$, near the upturning point the third order term is more important than the second order one. Consequently, $\alpha_{\boldsymbol{q}}$ is highly asymmetric near $\boldsymbol{q}_0$, and $J_c^\pm$ can be very different.

Moreover, as we approach the upturning point, $J_c^- \to 0$ if superconducting phase stays in the $q>0$ branch. In this case, the superconductor near FF transition is a perfect diode: Supercurrent cannot pass antiparallel to the Cooper pair momentum. To be precise, the diode coefficient for supercurrent along direction $\pm\hat{\boldsymbol{n}}$ is

$$\delta_{\hat{\boldsymbol{n}}} = (\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{q}}_0)F(x), \quad \text{(S20)}$$

where $x = c(B^2 - B_*^2)/\sqrt{c_0 c_2}$. The special function is

$$F(a) = \frac{J_1(a) - J_2(a)}{J_1(a) + J_2(a)}, \quad \text{(S21)}$$

where

$$J_1(a) = (ax - 2x^3)(x^4 - ax^2 - 1), \quad \text{(S22)}$$

$$J_2(a) = -(ay - 2y^3)(y^4 - ay^2 - 1) \quad \text{(S23)}$$

and ($A = \sqrt{28 + 11a^2}$, $\phi = \arccos\frac{20a^3 + 112a}{A^3}$)

$$x = \sqrt{\frac{1}{7}A\cos\left(\frac{\phi}{3}\right) + \frac{5a}{14}}, \quad \text{(S24)}$$

$$y = \sqrt{\frac{1}{7}A\cos\left(\frac{\phi - 2\pi}{3}\right) + \frac{5a}{14}}. \quad \text{(S25)}$$

Notice that $F(x) = 0$ for $x < 0$, and $F(x) = 0.90x^{\frac{3}{2}} - 1$ for $0 < x \ll 1$. As a result, when $B = B_*$ and $T < T_*$, we have $x = 0$ and the FF superconductor is a perfect diode $\delta = -1$.

It is also possible that near the upturning point, when injected supercurrent switches its direction, superconducting phase changes from the $q>0$ branch to the $q<0$ branch. In that case there is no supercurrent diode effect. Similar discussions can also be found in Ref. [3].

SUPERCURRENT DIODE EFFECT NEAR ZERO TEMPERATURE

Near the transition between FF superconductor and normal phase at low temperature $T \to 0$, we have

$$\phi\left(\frac{M}{2\pi T}\right) \sim -\log\frac{T}{T_c} + \log\left|\frac{M}{\Delta_0}\right| \quad \text{(S26)}$$

where $\Delta_0 = 4\pi e^{-\psi(1/2)}T_c$ is the pairing gap at zero temperature and zero field. Thus in the absence of SOC,

$$\alpha_{\boldsymbol{q}} = \frac{1}{2}\sum_{\lambda=\pm}\int_0^{2\pi}\frac{d\varphi}{2\pi}\left\{\log\left|\frac{B + \lambda v_{\mathrm{F}} q\cos\varphi}{\Delta_0}\right|\right\} \quad \text{(S27)}$$

which can be worked out as $\alpha_{\boldsymbol{q}} = \log(v_{\mathrm{F}}q/\Delta_0)$ when $v_{\mathrm{F}}q > B$, and in general can be written as the following piecewise function

$$\alpha_{\boldsymbol{q}} = \mathrm{Re}\left[\log\left(\frac{B + \sqrt{B^2 - v_{\mathrm{F}}^2 q^2}}{\Delta_0}\right)\right]. \quad \text{(S28)}$$

Minimizing $\alpha_{\boldsymbol{q}}$ over $\boldsymbol{q}$ yields a large Cooper pair momentum $q_0 = B/v_{\mathrm{F}}$ approaching $1/\xi_0$ as $B \to B_c$. In this case, due to the non-analytic dependence of $\boldsymbol{q}$, $\alpha_{\boldsymbol{q}}$ is highly skewed with respect to $\boldsymbol{q}_0$: it rises steeply as q decreases from q_0. This leads to the maximum possible diode effect with $\delta = (J_c^+ - J_c^-)/(J_c^+ + J_c^-) \sim -1$ near B_c, taking opposite sign as the one near tricritical point.

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[3] K. V. Samokhin and B. P. Truong, Phys. Rev. B **96**, 214501 (2017).

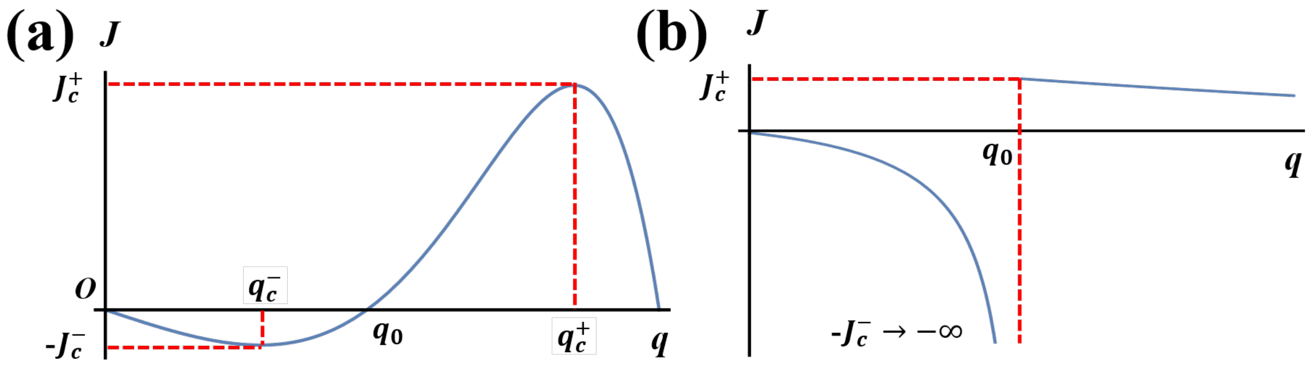

FIG. S1: (a) Near the FF transition, supercurrent J as a function of Cooper pair momentum q and $J(-q) = -J(q)$. Here q_0 is the momentum of FF phase, and $q_c^{\pm}$ are momenta for critical currents $\pm J_c^{\pm}$ respectively in the positive branch $q > 0$. (b) At $T = 0$, supercurrent J as a function of Cooper pair momentum q and $J(-q) = -J(q)$. Here q_0 is the momentum of FF phase, and $q_c^{\pm} = q_0^{\pm}$ are momenta for critical currents $\pm J_c^{\pm}$ respectively in the positive branch $q > 0$.