Supplement to accompany **On selection bias in comparison measures of smartphone-generated population mobility: an illustration of no-bias conditions with a commercial data source**

In **Web Appendix 1**, we justify the conditions when the estimated measure and the bias factor are viewed as fixed. In **Web Appendix 2**, we consider the estimates as varying between imagined study replications.

To reduce repetition in the appendices that follow, we rearrange some of the definitions from **Table 2** of the main text.

The estimated difference, $\widehat{D}_j$, is alternatively expressed as $\widehat{D}_j = \left(Y_{j,1} + \alpha_{j,1}\right) - \left(Y_{j,0} + \alpha_{j,0}\right) = Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0}$.

The estimated difference-in-differences, $\widehat{DiD}$, is alternatively expressed as, $\widehat{DiD} = \left(\left(Y_{1,1} + \alpha_{1,1}\right) - \left(Y_{1,0} + \alpha_{1,0}\right)\right) - \left(\left(Y_{0,1} + \alpha_{0,1}\right) - \left(Y_{0,0} + \alpha_{0,0}\right)\right) = \left(Y_{1,1} + \alpha_{1,1} - Y_{1,0} - \alpha_{1,0}\right) - \left(Y_{0,1} + \alpha_{0,1} - Y_{0,0} - \alpha_{0,0}\right) = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Its estimand, DiD, is similarly alternatively defined as $DiD = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0}$.

Web Appendix 1

The goal of this appendix is to show, under each stated condition from the main text (**Table 2**), that each estimated summary measure is equal to its estimand. The estimated measures and their estimands are defined in **Table 1** of the main text. As we note in the main text, estimated measures describing a contrast on the absolute scale are defined with $\alpha_{j,t}$, for group *j* at time *t*, and estimated measures describing a contrast on the relative scale are defined with $\beta_{j,t}$.

Condition 1 (C1): No bias in any group at any time point.

On the additive scale, the condition states that $\alpha_{0,0} = \alpha_{0,1} = \alpha_{1,0} = \alpha_{1,1} = 0$.

On the multiplicative scale, it states that $\beta_{0,0} = \beta_{0,1} = \beta_{1,0} = \beta_{1,1} = 1$.

Under this condition, all summary measures are unbiased: $\widehat{D}_j$, $\widehat{DiD}$, $\widehat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$.

Linear difference

By definition, $\widehat{D}_j = Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0}$. Substituting per the condition, $Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0} = Y_{j,1} - Y_{j,0} + 0 - 0 = Y_{j,1} - Y_{j,0} = D_j$.

Ratio

By definition, $\widehat{R}_j = \frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}}$. Substituting per the condition, $\frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} = \frac{Y_{j,1} * 1}{Y_{j,0} * 1} = \frac{Y_{j,1}}{Y_{j,0}} = R_j$.

Percent difference

By definition, $\widehat{PD}_j = \frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} - 1$. Substituting per the condition, $\frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} - 1 = \frac{Y_{j,1} * 1}{Y_{j,0} * 1} - 1 = \frac{Y_{j,1}}{Y_{j,0}} - 1 = PD_j$.

Difference in differences

By definition, $\widehat{DiD} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + 0 - 0 - 0 + 0 = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

By definition, $\widehat{RoR} = \frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}}}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}}}$. Substituting per the condition, $\frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}}}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}}} = \frac{\frac{Y_{1,1} * 1}{Y_{1,0} * 1}}{\frac{Y_{0,1} * 1}{Y_{0,0} * 1}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Ratio of percent differences

By definition, $\widehat{RPD} = \frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}} - 1}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}} - 1}$. Substituting per the condition, $\frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}} - 1}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}} - 1} = \frac{\frac{Y_{1,1} * 1}{Y_{1,0} * 1} - 1}{\frac{Y_{0,1} * 1}{Y_{0,0} * 1} - 1} = \frac{\frac{Y_{1,1}}{Y_{1,0}} - 1}{\frac{Y_{0,1}}{Y_{0,0}} - 1} = RPD$.

Condition 2 (C2): Bias is the same in all groups and time periods.

On the additive scale, the condition states that $\alpha_{0,0} = \alpha_{0,1} = \alpha_{1,0} = \alpha_{1,1} = \alpha \neq 0$.

On the multiplicative scale, it states that $0 \neq \beta_{0,0} = \beta_{0,1} = \beta_{1,0} = \beta_{1,1} = \beta \neq 1$. To avoid unusual results, we stipulate that $\beta_{j,t} \neq 0$.

Under this condition, all summary measures are unbiased: $\widehat{D}_j$, $\widehat{DiD}$, $\hat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$.

Linear difference

By definition, $\widehat{D}_j = Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0}$. Substituting per the condition, $Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0} = Y_{j,1} - Y_{j,0} + \alpha - \alpha$. Canceling terms, $Y_{j,1} - Y_{j,0} + \alpha - \alpha = Y_{j,1} - Y_{j,0} = \widehat{D}_j = D_j$.

Ratio

By definition, $\hat{R}_j = \frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}}$. Substituting per the condition, $\frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} = \frac{Y_{j,1} * \beta}{Y_{j,0} * \beta}$. Canceling terms, $\frac{Y_{j,1} * \beta}{Y_{j,0} * \beta} = \frac{Y_{j,1}}{Y_{j,0}} = R_j$.

Percent difference

By definition, $\widehat{PD}_j = \frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} - 1$. Substituting per the condition, $\frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} - 1 = \frac{Y_{j,1} * \beta}{Y_{j,0} * \beta} - 1$. Canceling terms, $\frac{Y_{j,1} * \beta}{Y_{j,0} * \beta} - 1 = \frac{Y_{j,1}}{Y_{j,0}} - 1 = PD_j$.

Difference in differences

By definition, $\widehat{DiD} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha - \alpha - \alpha + \alpha$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha - \alpha - \alpha + \alpha = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

By definition, $\widehat{RoR} = \frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}}}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}}}$. Substituting per the condition, $\frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}}}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}}} = \frac{\frac{Y_{1,1} * \beta}{Y_{1,0} * \beta}}{\frac{Y_{0,1} * \beta}{Y_{0,0} * \beta}}$. Canceling terms, $\frac{\frac{Y_{1,1} * \beta}{Y_{1,0} * \beta}}{\frac{Y_{0,1} * \beta}{Y_{0,0} * \beta}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Ratio of percent differences

By definition, $\widehat{RPD} = \frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}} - 1}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}} - 1}$. Substituting per the condition, $\frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}} - 1}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}} - 1} = \frac{\frac{Y_{1,1} * \beta}{Y_{1,0} * \beta} - 1}{\frac{Y_{0,1} * \beta}{Y_{0,0} * \beta} - 1}$. Canceling terms, $\frac{\frac{Y_{1,1} * \beta}{Y_{1,0} * \beta} - 1}{\frac{Y_{0,1} * \beta}{Y_{0,0} * \beta} - 1} = \frac{\frac{Y_{1,1}}{Y_{1,0}} - 1}{\frac{Y_{0,1}}{Y_{0,0}} - 1} = RPD$.

Condition 3.1 (C3.1): Bias is the same between time periods within group.

On the additive scale, the condition states that $\alpha_{0,0} = \alpha_{0,1} = \alpha_{0,t}$ and $\alpha_{1,0} = \alpha_{1,1} = \alpha_{1,t}$.

On the multiplicative scale, it states that $0 \neq \beta_{0,0} = \beta_{0,1} = \beta_{0,t}$ and $0 \neq \beta_{1,0} = \beta_{1,1} = \beta_{1,t}$. We stipulate that $\beta_{0,t} \neq 0$ so the estimate does not drop out of the expression.

Under this condition, all summary measures are unbiased: $\widehat{D}_j$, $\widehat{DiD}$, $\widehat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$.

Linear difference

By definition, $\widehat{D}_j = Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0}$. Under the condition, bias does not differ over time within group j, so $\alpha_{j,t} = \alpha_j$. Substituting per the condition, $Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0} = Y_{j,1} - Y_{j,0} + \alpha_j - \alpha_j$. Canceling terms, $Y_{j,1} - Y_{j,0} + \alpha_j - \alpha_j = Y_{j,1} - Y_{j,0} = D_j$.

Ratio

By definition, $\widehat{R}_j = \frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}}$. Under the condition, bias does not differ over time within group j, so $\beta_{j,t} = \beta_j$. Substituting per the condition, $\frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} = \frac{Y_{j,1} * \beta_j}{Y_{j,0} * \beta_j}$. Canceling terms, $\frac{Y_{j,1} * \beta_j}{Y_{j,0} * \beta_j} = \frac{Y_{j,1}}{Y_{j,0}} = R_j$.

Percent difference

By definition, $\widehat{PD}_j = \frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} - 1$. Under the condition, bias does not differ over time within group j, so $\beta_{j,t} = \beta_j$. Substituting per the condition, $\frac{Y_{j,1} * \beta_{j,1}}{Y_{j,0} * \beta_{j,0}} = \frac{Y_{j,1} * \beta_j}{Y_{j,0} * \beta_j}$. Canceling terms, $\frac{Y_{j,1} * \beta_j}{Y_{j,0} * \beta_j} = \frac{Y_{j,1}}{Y_{j,0}} = PD_j$.

Difference in differences

By definition, $\widehat{DiD} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,t} - \alpha_{1,t} - \alpha_{0,t} + \alpha_{0,t}$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,t} - \alpha_{1,t} - \alpha_{0,t} + \alpha_{0,t} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

By definition, $\widehat{RoR} = \frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}}}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}}}$. Substituting per the condition, $\frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}}}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}}} = \frac{\frac{Y_{1,1} * \beta_{1,t}}{Y_{1,0} * \beta_{1,t}}}{\frac{Y_{0,1} * \beta_{0,t}}{Y_{0,0} * \beta_{0,t}}}$. Canceling terms, $\frac{\frac{Y_{1,1} * \beta_{1,t}}{Y_{1,0} * \beta_{1,t}}}{\frac{Y_{0,1} * \beta_{0,t}}{Y_{0,0} * \beta_{0,t}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Ratio of percent differences

By definition, $\widehat{RPD} = \frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}} - 1}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}} - 1}$. Substituting per the condition, $\frac{\frac{Y_{1,1} * \beta_{1,1}}{Y_{1,0} * \beta_{1,0}} - 1}{\frac{Y_{0,1} * \beta_{0,1}}{Y_{0,0} * \beta_{0,0}} - 1} = \frac{\frac{Y_{1,1} * \beta_{1,t}}{Y_{1,0} * \beta_{1,t}} - 1}{\frac{Y_{0,1} * \beta_{0,t}}{Y_{0,0} * \beta_{0,t}} - 1}$. Canceling terms, $\frac{\frac{Y_{1,1} * \beta_{1,t}}{Y_{1,0} * \beta_{1,t}} - 1}{\frac{Y_{0,1} * \beta_{0,t}}{Y_{0,0} * \beta_{0,t}} - 1} = \frac{\frac{Y_{1,1}}{Y_{1,0}} - 1}{\frac{Y_{0,1}}{Y_{0,0}} - 1} = RPD$.

Condition 3.2 (C3.2): Bias is the same between groups within time period.

On the additive scale, the condition states that $\alpha_{0,0} = \alpha_{1,0} = \alpha_{j,0}$ and $\alpha_{0,1} = \alpha_{1,1} = \alpha_{j,1}$
On the multiplicative scale, it states $0 \neq \beta_{0,0} = \beta_{1,0} = \beta_{j,0}$ and $0 \neq \beta_{0,1} = \beta_{1,1} = \beta_{j,1}$.

Difference in differences

By definition, $\widehat{DiD} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{j,1} - \alpha_{j,0} - \alpha_{j,1} + \alpha_{j,0}$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{j,1} - \alpha_{j,0} - \alpha_{j,1} + \alpha_{j,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

By definition, $\widehat{RoR} = \frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}}{\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}}$. Substituting per the condition, $\frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}}{\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}} = \frac{\frac{Y_{1,1}*\beta_{j,1}}{Y_{1,0}*\beta_{j,0}}}{\frac{Y_{0,1}*\beta_{j,1}}{Y_{0,0}*\beta_{j,0}}}$. Canceling terms, $\frac{\frac{Y_{1,1}*\beta_{j,1}}{Y_{1,0}*\beta_{j,0}}}{\frac{Y_{0,1}*\beta_{j,1}}{Y_{0,0}*\beta_{j,0}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Condition 4.1 (C4.1): The between-time-period trend in bias is the same between groups.

On the additive scale, the condition states $\alpha_{1,1} - \alpha_{1,0} = \alpha_{0,1} - \alpha_{0,0} = \alpha_{j,1} - \alpha_{j,0}$. This condition illustrates the classic parallel-trends assumption.[1,2]

On the multiplicative scale, it states that $\frac{\beta_{1,1}}{\beta_{1,0}} = \frac{\beta_{0,1}}{\beta_{0,0}} = \frac{\beta_{j,1}}{\beta_{j,0}}$. To facilitate algebra below, let $\frac{\beta_{j,1}}{\beta_{j,0}} = k$, so that $\beta_{1,1} = k * \beta_{1,0}$, and $\beta_{0,1} = k * \beta_{0,0}$.

Under this condition, $\widehat{DiD}$, $\widehat{RoR}$, and $\widehat{RPD}$ are unbiased.

Difference in differences

By definition, $\widehat{DiD} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Factoring the bias-factor terms to facilitate substitution, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{1,1} - \alpha_{1,0}) - (\alpha_{0,1} - \alpha_{0,0})$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{1,1} - \alpha_{1,0}) - (\alpha_{0,1} - \alpha_{0,0}) = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{j,1} - \alpha_{j,0}) - (\alpha_{j,1} - \alpha_{j,0})$. Canceling like terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{j,1} - \alpha_{j,0}) - (\alpha_{j,1} - \alpha_{j,0}) = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

By definition, $\widehat{RoR} = \frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}}{\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}}$. Equivalently, $\frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}}{\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}*\frac{\beta_{1,1}}{\beta_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}*\frac{\beta_{0,1}}{\beta_{0,0}}}$. Substituting per the condition, $\frac{\frac{Y_{1,1}}{Y_{1,0}}*\frac{\beta_{1,1}}{\beta_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}*\frac{\beta_{0,1}}{\beta_{0,0}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}*\frac{\beta_{j,1}}{\beta_{j,0}}}{\frac{Y_{0,1}}{Y_{0,0}}*\frac{\beta_{j,1}}{\beta_{j,0}}}$. Canceling terms, $\frac{\frac{Y_{1,1}}{Y_{1,0}}*\frac{\beta_{j,1}}{\beta_{j,0}}}{\frac{Y_{0,1}}{Y_{0,0}}*\frac{\beta_{j,1}}{\beta_{j,0}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Condition 4.2 (C4.2): The between-group trend in bias is the same between time periods.

On the additive scale, the condition states $\alpha_{1,1} - \alpha_{0,1} = \alpha_{1,0} - \alpha_{0,0} = \alpha_{1,t} - \alpha_{0,t}$. This condition also illustrates the parallel-trends assumption.[1,2]

On the multiplicative scale, it states that $\frac{\beta_{1,1}}{\beta_{0,1}} = \frac{\beta_{1,0}}{\beta_{0,0}} = \frac{\beta_{1,t}}{\beta_{0,t}}$. To facilitate algebra below, let $\frac{\beta_{1,t}}{\beta_{0,t}} = k$, so that $\beta_{1,1} = k * \beta_{0,1}$, and $\beta_{1,0} = k * \beta_{0,0}$.

Under this condition, $\widehat{DiD}$ and $\widehat{RoR}$ are unbiased.

Difference in differences

By definition, $\widehat{DiD} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}$. Factoring the bias-factor terms to facilitate substitution, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0} = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{1,1} - \alpha_{0,1}) - (\alpha_{1,0} - \alpha_{0,0})$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{1,1} - \alpha_{0,1}) - (\alpha_{1,0} - \alpha_{0,0}) = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{1,t} - \alpha_{0,t}) - (\alpha_{1,t} - \alpha_{0,t})$. Canceling like terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (\alpha_{1,t} - \alpha_{0,t}) - (\alpha_{1,t} - \alpha_{0,t}) = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

By definition, $\widehat{RoR} = \frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}}{\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}}$. Rearranging terms, $\frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}}{\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}} = \frac{\frac{Y_{1,1}*\beta_{1,1}}{Y_{0,1}*\beta_{0,1}}}{\frac{Y_{1,0}*\beta_{1,0}}{Y_{0,0}*\beta_{0,0}}} = \frac{\frac{Y_{1,1}}{Y_{0,1}}*\frac{\beta_{1,1}}{\beta_{0,1}}}{\frac{Y_{1,0}}{Y_{0,0}}*\frac{\beta_{1,0}}{\beta_{0,0}}}$. Substituting per the condition, $\frac{\frac{Y_{1,1}}{Y_{0,1}}*\frac{\beta_{1,1}}{\beta_{0,1}}}{\frac{Y_{1,0}}{Y_{0,0}}*\frac{\beta_{1,0}}{\beta_{0,0}}} = \frac{\frac{Y_{1,1}}{Y_{0,1}}*\frac{\beta_{1,t}}{\beta_{0,t}}}{\frac{Y_{1,0}}{Y_{0,0}}*\frac{\beta_{1,t}}{\beta_{0,t}}}$. Canceling terms, $\frac{\frac{Y_{1,1}}{Y_{0,1}}*\frac{\beta_{1,t}}{\beta_{0,t}}}{\frac{Y_{1,0}}{Y_{0,0}}*\frac{\beta_{1,t}}{\beta_{0,t}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Web Appendix 2

In this section, we view the bias factors, $\alpha_{j,t}$ and $\beta_{j,t}$, and the estimated measures (**Table 1**, main text) as random variables rather than as fixed parameters. The conditions **(Web Table 1)** are thus defined under slightly different definitions, expressed in terms of expected values. Under these conditions, we show that the estimated absolute measures ($\widehat{D}_j$ and $\widehat{DiD}$) are statistically unbiased, $E\left[\widehat{D}_j\right] = D_j$ and $E\left[\widehat{DiD}\right] = DiD$. To do so, we follow the same general algebra as outlined in **Web Appendix 1** while also invoking linearity of expectation and the fact that the value of the outcome in each group and time in the total population, $Y_{j,t}$, is considered constant, so $E\left[Y_{j,t}\right] = Y_{j,t}$. The estimated relative measures ($\widehat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$), however, are generally not statistically unbiased, based on our definitions. In our formulations, demonstrating no bias in the estimated relative measures would involve taking expectations of quotients of random variables. The expectation of a quotient of random variables is not generally the quotient of their expectations. Instead, we show that the estimators of relative measures are statistically consistent, $\hat{\theta} \xrightarrow{p} \theta$, as is commonly applied in epidemiology.[3]

One additional note: our focus in this manuscript is the patterns of bias between groups and time periods. We have thus defined the estimated measures as a function of their constituent measures with bias factors applied in each group and time period (**Table 1**). An alternative, perhaps more straightforward, way to define the estimated ratio measures could be with an overall bias factor for the summary measure. Under this alternative conceptualization, it can be shown that the relative measures are unbiased if the multiplicative bias factor is expected to equal one. Suppose the estimated ratio, $\widehat{R}_j$, were alternatively defined as $\widehat{R}_j = R_j * \beta_j$, where β_j is a multiplicative bias factor. By the definition of bias, $\widehat{R}_j$ is unbiased if $E\left[\widehat{R}_j\right] = R_j$. Substituting the definition just noted for $\widehat{R}_j$, $E\left[\widehat{R}_j\right] = E\left[R_j * \beta_j\right]$. R_j is a constant population parameter and is independent of the random variable β_j: $E\left[R_j * \beta_j\right] = E\left[R_j\right] * E\left[\beta_j\right]$. Again, because R_j is a constant, $E\left[R_j\right] * E\left[\beta_j\right] = R_j * E\left[\beta_j\right]$. If $E\left[\beta_j\right] = 1$, then $E\left[\widehat{R}_j\right] = R_j * E\left[\beta_j\right] = R_j * 1 = R_j$.

We can use the analogous approach to show that the estimated percent difference, $\widehat{PD}_j$, ratio of ratios, $\widehat{RoR}$, and ratio of percent differences, $\widehat{RPD}$, are unbiased. For completeness, we do so here. Define $\widehat{PD}_j$ as $\widehat{PD}_j = PD_j * \beta_j$, where β_j is a multiplicative bias factor. $E\left[\widehat{PD}_j\right] = E\left[PD_j * \beta_j\right]$. PD_j is constant, so $E\left[PD_j * \beta_j\right] = E\left[PD_j\right] * E\left[\beta_j\right] = PD_j * E\left[\beta_j\right]$. If $E\left[\beta_j\right] = 1$, then $E\left[\widehat{PD}_j\right] = PD_j * E\left[\beta_j\right] = PD_j * 1 = PD_j$. Similarly, define $\widehat{RoR}$ as $\widehat{RoR} = RoR * \beta$, where β is a multiplicative bias factor. RoR is constant, so $E\left[\widehat{RoR}\right] = E[RoR * \beta] = RoR * E[\beta]$. If $E[\beta] = 1$, then $E\left[\widehat{RoR}\right] = RoR * E[\beta] = RoR * 1 = RoR$. Finally, define $\widehat{RPD}$ as $\widehat{RPD} = RPD * \beta$, where β is a multiplicative bias factor. RPD is constant, so $E\left[\widehat{RPD}\right] = E[RPD * \beta] = RPD * E[\beta]$. If $E[\beta] = 1$, then $E\left[\widehat{RPD}\right] = RPD * E[\beta] = RPD * 1 = RPD$.

Web Table 2. Bias conditions considering the estimates and bias factors as random variables.

No.	Description	Scale	Notation[b]	Sufficient for these summary measures to be statistically unbiased or consistent.
1E	In expectation, there is no bias in any group in any time period.	Absolute	$E[\alpha_{0,0}] = E[\alpha_{0,1}] = E[\alpha_{1,0}] = E[\alpha_{1,1}] = 0.$	$\hat{D}_j$, $\widehat{D\iota D}$
		Relative	$E[\beta_{0,0}] = E[\beta_{0,1}] = E[\beta_{1,0}] = E[\beta_{1,1}] = 1.$	$\hat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, $\widehat{RPD}$
2E	In expectation, bias is the same across all groups and time periods.	Absolute	$E[\alpha_{0,0}] = E[\alpha_{0,1}] = E[\alpha_{1,0}] = E[\alpha_{1,1}] = E[\alpha].$	$\hat{D}_j$, $\widehat{D\iota D}$
		Relative	$0 \neq E[\beta_{0,0}] = E[\beta_{0,1}] = E[\beta_{1,0}] = E[\beta_{1,1}] = E[\beta].$	$\hat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, $\widehat{RPD}$
3.1.E	In expectation, bias is the same between time periods within group.	Absolute	$E[\alpha_{0,0}] = E[\alpha_{0,1}] = E[\alpha_{0,t}]$, and $E[\alpha_{1,0}] = E[\alpha_{1,1}] = E[\alpha_{1,t}].$	$\hat{D}_j$, $\widehat{D\iota D}$.
		Relative	$0 \neq E[\beta_{0,0}] = E[\beta_{0,1}] = E[\beta_{0,t}]$, and $0 \neq E[\beta_{1,0}] = E[\beta_{1,1}] = E[\beta_{1,t}].$	$\hat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, $\widehat{RPD}$
3.2.E	In expectation, bias is the same between groups within time period.	Absolute	$E[\alpha_{0,0}] = E[\alpha_{1,0}] = E[\alpha_{j,0}]$, and $E[\alpha_{0,1}] = E[\alpha_{1,1}] = E[\alpha_{j,1}].$	$\widehat{D\iota D}$
		Relative	$0 \neq E[\beta_{0,0}] = E[\beta_{1,0}] = E[\beta_{j,0}]$, and $0 \neq E[\beta_{0,1}] = E[\beta_{1,1}] = E[\beta_{j,1}].$	$\widehat{RoR}$
4.1.E	The expected between-time-period trend in bias is the same between groups.	Absolute	$E[\alpha_{1,1} - \alpha_{1,0}] = E[\alpha_{0,1} - \alpha_{0,0}] = E[\alpha_{j,1} - \alpha_{j,0}].$	$\widehat{D\iota D}$
		Relative	$E\left[\frac{\beta_{1,1}}{\beta_{1,0}}\right] = E\left[\frac{\beta_{0,1}}{\beta_{0,0}}\right] = E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right].$	$\widehat{RoR}$
4.2.E	The expected between-group trend in bias is the same between time periods.	Absolute	$E[\alpha_{1,1} - \alpha_{0,1}] = E[\alpha_{1,0} - \alpha_{0,0}] = E[\alpha_{1,t} - \alpha_{0,t}].$	$\widehat{D\iota D}$
		Relative	$E\left[\frac{\beta_{1,1}}{\beta_{0,1}}\right] = E\left[\frac{\beta_{1,0}}{\beta_{0,0}}\right] = E\left[\frac{\beta_{1,t}}{\beta_{0,t}}\right].$	$\widehat{RoR}$
5E	Neither the expected between-time-period in bias is the same between groups, nor is the expected between-group trend in bias the same between time periods.	Absolute	$E[\alpha_{1,1} - \alpha_{1,0}] \neq E[\alpha_{0,1} - \alpha_{0,0}]$, and $E[\alpha_{1,1} - \alpha_{0,1}] \neq E[\alpha_{1,0} - \alpha_{0,0}].$	None
		Relative	$E\left[\frac{\beta_{1,1}}{\beta_{1,0}}\right] \neq E\left[\frac{\beta_{0,1}}{\beta_{0,0}}\right]$, and $E\left[\frac{\beta_{1,1}}{\beta_{0,1}}\right] \neq E\left[\frac{\beta_{1,0}}{\beta_{0,0}}\right].$	None

[a]The meaning of the word *trend* in Conditions 4.1. and 4.2 depends on the scale. Please refer to the notation for further precision.
[b]For expressions involving the relative bias factor, we stipulate that $\beta_{j,t} \neq 0$ so that the adjacent $Y_{j,t}$ does not drop out of the expression, creating unusual results.

Condition 1E (C1E): In expectation, there is no bias in any group in any time period.

On the additive scale, this condition states that $E[\alpha_{0,0}] = E[\alpha_{0,1}] = E[\alpha_{1,0}] = E[\alpha_{1,1}] = 0$.

On the multiplicative scale, it states that $E[\beta_{0,0}] = E[\beta_{0,1}] = E[\beta_{1,0}] = E[\beta_{1,1}] = 1$.

Under this condition, $\widehat{D}_j$ and $\widehat{DiD}$ are unbiased estimators of D_j and DiD, respectively, and $\widehat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$ are consistent estimators of R_j, PD_j, RoR, and RPD, respectively.

Linear difference

We show that under the noted condition, the expected value of the estimator is equal to its estimand: $E[\widehat{D}_j] = D_j$. By definition, $E[\widehat{D}_j] = E[Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0}]$. By linearity of expectation, $E[Y_{j,1} - Y_{j,0} + \alpha_{j,1} - \alpha_{j,0}] = E[Y_{j,1}] - E[Y_{j,0}] + E[\alpha_{j,1}] - E[\alpha_{j,0}]$. $E[Y_{j,t}] = Y_{j,t}$, so $E[Y_{j,1}] - E[Y_{j,0}] + E[\alpha_{j,1}] - E[\alpha_{j,0}] = Y_{j,1} - Y_{j,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}]$. For reference in subsequent conditions, we refer to the above sequence of statements as preliminary propositions for $\widehat{D}_j$.

Substituting per the condition, $Y_{j,1} - Y_{j,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}] = Y_{j,1} - Y_{j,0} + 0 - 0 = D_j$.

Ratio

We show that under the noted condition, the estimator converges in probability to its estimand: $\widehat{R}_j \xrightarrow{p} R_j$. By definition, $\widehat{R}_j = \frac{\hat{Y}_{j,1}}{\hat{Y}_{j,0}}$. By the Weak Law of Large Numbers (WLLN), $\hat{Y}_{j,1}$ and $\hat{Y}_{j,0}$ each converge in probability to their expected values, $E[\hat{Y}_{j,1}]$ and $E[\hat{Y}_{j,0}]$. Let $f(a,b) = \frac{a}{b}$. Given $\hat{Y}_{j,1} \xrightarrow{p} E[\hat{Y}_{j,1}]$ and $\hat{Y}_{j,0} \xrightarrow{p} E[\hat{Y}_{j,0}]$, by the Continuous Mapping Theorem, $f(\hat{Y}_{j,1}, \hat{Y}_{j,0}) = \frac{\hat{Y}_{j,1}}{\hat{Y}_{j,0}} \xrightarrow{p} \frac{E[\hat{Y}_{j,1}]}{E[\hat{Y}_{j,0}]}$, assuming $E[\hat{Y}_{j,0}] \neq 0$. By definition, $\frac{E[\hat{Y}_{j,1}]}{E[\hat{Y}_{j,0}]} = \frac{E[Y_{j,1} * \beta_{j,1}]}{E[Y_{j,0} * \beta_{j,0}]}$. All values of $Y_{j,t}$ are constant, so the expectation of each product in the quotient is thus the product of their expectations: $\frac{E[Y_{j,1} * \beta_{j,1}]}{E[Y_{j,0} * \beta_{j,0}]} = \frac{E[Y_{j,1}] * E[\beta_{j,1}]}{E[Y_{j,0}] * E[\beta_{j,0}]}$. Because $E[Y_{j,t}] = Y_{j,t}$, $\frac{E[Y_{j,1}] * E[\beta_{j,1}]}{E[Y_{j,0}] * E[\beta_{j,0}]} = \frac{Y_{j,1} * E[\beta_{j,1}]}{Y_{j,0} * E[\beta_{j,0}]}$. For reference in subsequent conditions, we refer to the above sequence of statements as preliminary propositions for $\widehat{R}_j$.

Substituting per the condition, $\frac{Y_{j,1} * E[\beta_{j,1}]}{Y_{j,0} * E[\beta_{j,0}]} = \frac{Y_{j,1} * 1}{Y_{j,0} * 1} = \frac{Y_{j,1}}{Y_{j,0}} = R_j$.

Percent difference

We show that under the noted condition, $\widehat{PD}_j \xrightarrow{p} PD_j$. By definition, $\widehat{PD}_j = \frac{\hat{Y}_{j,1}}{\hat{Y}_{j,0}} - 1$. By the WLLN, $\hat{Y}_{j,1} \xrightarrow{p} E[\hat{Y}_{j,1}]$, $\hat{Y}_{j,0} \xrightarrow{p} E[\hat{Y}_{j,0}]$, and the constant one converges to itself. Let $f(a,b) = \frac{a}{b} - 1$ where $b \neq 0$. Given $\hat{Y}_{j,1} \xrightarrow{p} E[\hat{Y}_{j,1}]$ and $\hat{Y}_{j,0} \xrightarrow{p} E[\hat{Y}_{j,0}]$, by the Continuous Mapping Theorem, $f(\hat{Y}_{j,1}, \hat{Y}_{j,0}) = \frac{\hat{Y}_{j,1}}{\hat{Y}_{j,0}} - 1 \xrightarrow{p} \frac{E[\hat{Y}_{j,1}]}{E[\hat{Y}_{j,0}]} - 1$. By definition, $\frac{E[\hat{Y}_{j,1}]}{E[\hat{Y}_{j,0}]} - 1 = \frac{E[Y_{j,1} * \beta_{j,1}]}{E[Y_{j,0} * \beta_{j,0}]} - 1$. $Y_{j,t}$ is constant, so $\frac{E[Y_{j,1} * \beta_{j,1}]}{E[Y_{j,0} * \beta_{j,0}]} - 1 = \frac{E[Y_{j,1}] * E[\beta_{j,1}]}{E[Y_{j,0}] * E[\beta_{j,0}]} - 1 = \frac{Y_{j,1} * E[\beta_{j,1}]}{Y_{j,0} * E[\beta_{j,0}]} - 1$. For reference in subsequent conditions, we refer to the above sequence of statements as preliminary propositions for $\widehat{PD}_j$.

Substituting per the condition, $\frac{Y_{j,1} * E[\beta_{j,1}]}{Y_{j,0} * E[\beta_{j,0}]} - 1 = \frac{Y_{j,1} * 1}{Y_{j,0} * 1} - 1 = \frac{Y_{j,1}}{Y_{j,0}} - 1 = PD_j$.

Difference in differences

We show that under the noted condition, $E\left[\widehat{DiD}\right] = DiD$. By definition, $E\left[\widehat{DiD}\right] = E\left[Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}\right]$. By linearity of expectation, $E\left[Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + \alpha_{1,1} - \alpha_{1,0} - \alpha_{0,1} + \alpha_{0,0}\right] = E\left[Y_{1,1}\right] - E\left[Y_{1,0}\right] - E\left[Y_{0,1}\right] + E\left[Y_{0,0}\right] - E\left[\alpha_{1,1}\right] - E\left[\alpha_{1,0}\right] - E\left[\alpha_{0,1}\right] + E\left[\alpha_{0,0}\right]$. Because $E\left[Y_{j,t}\right] = Y_{j,t}$, $E\left[Y_{1,1}\right] - E\left[Y_{1,0}\right] - E\left[Y_{0,1}\right] + E\left[Y_{0,0}\right] + E\left[\alpha_{1,1}\right] - E\left[\alpha_{1,0}\right] - E\left[\alpha_{0,1}\right] + E\left[\alpha_{0,0}\right] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E\left[\alpha_{1,1}\right] - E\left[\alpha_{1,0}\right] - E\left[\alpha_{0,1}\right] + E\left[\alpha_{0,0}\right]$. For reference in subsequent conditions, we refer to the above sequence of statements as preliminary propositions for $\widehat{DiD}$.

Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} - E\left[\alpha_{1,1}\right] - E\left[\alpha_{1,0}\right] - E\left[\alpha_{0,1}\right] + E\left[\alpha_{0,0}\right] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + 0 - 0 - 0 + 0 = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

We show that under the noted condition, $\widehat{RoR} \xrightarrow{p} RoR$. By definition, $\widehat{RoR} = \frac{\frac{\hat{Y}_{1.1}}{\hat{Y}_{1,0}}}{\frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}}}$. By the WLLN, $\hat{Y}_{1,1} \xrightarrow{p} E\left[\hat{Y}_{1,1}\right]$, $\hat{Y}_{1,0} \xrightarrow{p} E\left[\hat{Y}_{1,0}\right]$, $\hat{Y}_{0,1} \xrightarrow{p} E\left[\hat{Y}_{0,1}\right]$, and $\hat{Y}_{0,0} \xrightarrow{p} E\left[\hat{Y}_{0,0}\right]$. Let $f(a,b,c,d) = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a*d}{b*c}$. By the Continuous Mapping Theorem, $f\left(\hat{Y}_{1.1}, \hat{Y}_{1,0}, \hat{Y}_{0,1}, \hat{Y}_{0,0}\right) = \frac{\frac{\hat{Y}_{1.1}}{\hat{Y}_{1,0}}}{\frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}}} \xrightarrow{p} \frac{\frac{E[\hat{Y}_{1,1}]}{E[\hat{Y}_{1,0}]}}{\frac{E[\hat{Y}_{0,1}]}{E[\hat{Y}_{0,0}]}}$, assuming $\frac{E[\hat{Y}_{0,1}]}{E[\hat{Y}_{0,0}]} \neq 0$. By definition, $\frac{\frac{E[\hat{Y}_{1,1}]}{E[\hat{Y}_{1,0}]}}{\frac{E[\hat{Y}_{0,1}]}{E[\hat{Y}_{0,0}]}} = \frac{\frac{E[Y_{1,1}*\beta_{1,1}]}{E[Y_{1,0}*\beta_{1,0}]}}{\frac{E[Y_{0,1}*\beta_{0,1}]}{E[Y_{0,0}*\beta_{0,0}]}}$. Each value of $Y_{j,t}$ is constant, so $\frac{\frac{E[Y_{1,1}*\beta_{1,1}]}{E[Y_{1,0}*\beta_{1,0}]}}{\frac{E[Y_{0,1}*\beta_{0,1}]}{E[Y_{0,0}*\beta_{0,0}]}} = \frac{\frac{E[Y_{1,1}]*E[\beta_{1,1}]}{E[Y_{1,0}]*E[\beta_{1,0}]}}{\frac{E[Y_{0,1}]*E[\beta_{0,1}]}{E[Y_{0,0}]*E[\beta_{0,0}]}} = \frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}}$. For reference in subsequent conditions, we refer to the above sequence of statements as preliminary propositions for $\widehat{RoR}$.

Substituting per the condition, $\frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}} = \frac{\frac{Y_{1,1}*1}{Y_{1,0}*1}}{\frac{Y_{0,1}*1}{Y_{0,0}*1}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Ratio of percent differences

We show that under the noted condition, $\widehat{RPD} \xrightarrow{p} RPD$. By definition, $\widehat{RPD} = \frac{\frac{\hat{Y}_{1.1}}{\hat{Y}_{1,0}} - 1}{\frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}} - 1}$. By the WLLN, $\hat{Y}_{1,1} \xrightarrow{p} E\left[\hat{Y}_{1,1}\right]$, $\hat{Y}_{1,0} \xrightarrow{p} E\left[\hat{Y}_{1,0}\right]$, $\hat{Y}_{0,1} \xrightarrow{p} E\left[\hat{Y}_{0,1}\right]$, $\hat{Y}_{0,0} \xrightarrow{p} E\left[\hat{Y}_{0,0}\right]$, and the constant 1 converges to itself. Let $f(a,b,c,d) = \frac{\frac{a}{b} - 1}{\frac{c}{d} - 1}$. By the Continuous Mapping Theorem, $f\left(\hat{Y}_{1.1}, \hat{Y}_{1,0}, \hat{Y}_{0,1}, \hat{Y}_{0,0}\right) = \frac{\frac{\hat{Y}_{1.1}}{\hat{Y}_{1,0}} - 1}{\frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}} - 1} \xrightarrow{p} \frac{\frac{E[Y_{1,1}*\beta_{1,1}]}{E[Y_{1,0}*\beta_{1,0}]} - 1}{\frac{E[Y_{0,1}*\beta_{0,1}]}{E[Y_{0,0}*\beta_{0,0}]} - 1}$, assuming $\frac{E[Y_{0,1}*\beta_{0,1}]}{E[Y_{0,0}*\beta_{0,0}]} - 1 \neq 0$. Each value of $Y_{j,t}$ is constant, so $\frac{\frac{E[Y_{1,1}*\beta_{1,1}]}{E[Y_{1,0}*\beta_{1,0}]} - 1}{\frac{E[Y_{0,1}*\beta_{0,1}]}{E[Y_{0,0}*\beta_{0,0}]} - 1} = \frac{\frac{E[Y_{1,1}]*E[\beta_{1,1}]}{E[Y_{1,0}]*E[\beta_{1,0}]} - 1}{\frac{E[Y_{0,1}]*E[\beta_{0,1}]}{E[Y_{0,0}]*E[\beta_{0,0}]} - 1} = \frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]} - 1}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]} - 1}$. For reference in subsequent conditions, we refer to the above sequence of statements as preliminary propositions for $\widehat{RPD}$.

Substituting per the condition, $\frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]} - 1}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]} - 1} = \frac{\frac{Y_{1,1}*1}{Y_{1,0}*1} - 1}{\frac{Y_{0,1}*1}{Y_{0,0}*1} - 1} = \frac{\frac{Y_{1,1}}{Y_{1,0}} - 1}{\frac{Y_{0,1}}{Y_{0,0}} - 1} = RPD$.

Condition 2E (C2E): In expectation, bias is the same across all groups and time periods.

On the additive scale, this condition states that $E[\alpha_{0,0}] = E[\alpha_{0,1}] = E[\alpha_{1,0}] = E[\alpha_{1,1}] = E[\alpha]$.

On the multiplicative scale, it states that $0 \neq E[\beta_{0,0}] = E[\beta_{0,1}] = E[\beta_{1,0}] = E[\beta_{1,1}] = E[\beta]$.

Under this condition, $\widehat{D}_j$ and $\widehat{DiD}$ are unbiased estimators of D_j and DiD, respectively, and $\widehat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$ are consistent estimators of R_j, PD_j, RoR, and RPD, respectively.

Linear difference

We show that under the condition, $E[\widehat{D}_j] = D_j$. By the preliminary propositions noted above for $\widehat{D}_j$, $E[\widehat{D}_j] = Y_{j,1} - Y_{j,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}]$. Substituting per the condition, $Y_{j,1} - Y_{j,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}] = Y_{j,1} - Y_{j,0} + E[\alpha] - E[\alpha]$. Canceling terms, $Y_{j,1} - Y_{j,0} + E[\alpha] - E[\alpha] = Y_{j,1} - Y_{j,0} = D_j$.

Ratio

We show that under the noted condition, $\widehat{R}_j \xrightarrow{p} R_j$. By the preliminary propositions noted above for $\widehat{R}_j$, $\widehat{R}_j \xrightarrow{p} \frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]}$. Substituting per the condition, $\frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]} = \frac{Y_{j,1}*E[\beta]}{Y_{j,0}*E[\beta]}$. Canceling terms, $\frac{Y_{j,1}*E[\beta]}{Y_{j,0}*E[\beta]} = \frac{Y_{j,1}}{Y_{j,0}} = R_j$.

Percent difference

We show that under the noted condition, $\widehat{PD}_j \xrightarrow{p} PD_j$. By the preliminary propositions noted above for $\widehat{PD}_j$, $\widehat{PD}_j \xrightarrow{p} \frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]} - 1$. Substituting per the condition, $\frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]} - 1 = \frac{Y_{j,1}*E[\beta]}{Y_{j,0}*E[\beta]} - 1$. Canceling terms, $\frac{Y_{j,1}*E[\beta]}{Y_{j,0}*E[\beta]} - 1 = \frac{Y_{j,1}}{Y_{j,0}} - 1 = PD_j$.

Difference in differences

We show that under the noted condition, $E[\widehat{DiD}] = DiD$. By the preliminary propositions noted above for $\widehat{DiD}$, $E[\widehat{DiD}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}]$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha] - E[\alpha] - E[\alpha] + E[\alpha]$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha] - E[\alpha] - E[\alpha] + E[\alpha] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

We show that under the noted condition, $\widehat{RoR} \xrightarrow{p} RoR$. By the preliminary propositions noted above for $\widehat{RoR}$, $\widehat{RoR} \xrightarrow{p} \frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}}$. Substituting per the condition, $\frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}} = \frac{\frac{Y_{1,1}*E[\beta]}{Y_{1,0}*E[\beta]}}{\frac{Y_{0,1}*E[\beta]}{Y_{0,0}*E[\beta]}}$. Canceling terms, $\frac{\frac{Y_{1,1}*E[\beta]}{Y_{1,0}*E[\beta]}}{\frac{Y_{0,1}*E[\beta]}{Y_{0,0}*E[\beta]}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Ratio of percent differences

We show that under the noted condition, $\widehat{RPD} \xrightarrow{p} RPD$. By the preliminary propositions noted above for $\widehat{RPD}$, $\widehat{RPD} \xrightarrow{p} \frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}-1}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}-1}$. Substituting per the condition, $\frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}-1}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}-1} = \frac{\frac{Y_{1,1}*E[\beta]}{Y_{1,0}*E[\beta]}-1}{\frac{Y_{0,1}*E[\beta]}{Y_{0,0}*E[\beta]}-1}$. Canceling terms, $\frac{\frac{Y_{1,1}*E[\beta]}{Y_{1,0}*E[\beta]}-1}{\frac{Y_{0,1}*E[\beta]}{Y_{0,0}*E[\beta]}-1} = \frac{\frac{Y_{1,1}}{Y_{1,0}}-1}{\frac{Y_{0,1}}{Y_{0,0}}-1} = RPD$.

Condition 3.1E (C3.1E): In expectation, bias is the same between time periods within group.

On the additive scale, this condition states that $E[\alpha_{0,0}] = E[\alpha_{0,1}] = E[\alpha_{0,t}]$ and that $E[\alpha_{1,0}] = E[\alpha_{1,1}] = E[\alpha_{1,t}]$.

On the multiplicative scale, it states that $0 \neq E[\beta_{0,0}] = E[\beta_{0,1}] = E[\beta_{0,t}]$ and that $0 \neq E[\beta_{1,0}] = E[\beta_{1,1}] = E[\beta_{1,t}]$.

Under this condition, $\widehat{D}_j$ and $\widehat{DiD}$ are unbiased estimators of D_j and DiD, respectively, and $\widehat{R}_j$, $\widehat{PD}_j$, $\widehat{RoR}$, and $\widehat{RPD}$ are consistent estimators of R_j, PD_j, RoR, and RPD, respectively.

Linear difference

We show that under the condition, $E[\widehat{D}_j] = D_j$. By the preliminary propositions noted above for $\widehat{D}_j$, $E[\widehat{D}_j] = Y_{j,1} - Y_{j,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}]$. Under the condition, bias is not expected to differ over time within group j, so $E[\alpha_{j,t}] = E[\alpha_j]$. Substituting per the condition, $Y_{j,1} - Y_{j,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}] = Y_{j,1} - Y_{j,0} + E[\alpha_j] - E[\alpha_j]$. Canceling terms, $Y_{j,1} - Y_{j,0} + E[\alpha_j] - E[\alpha_j] = Y_{j,1} - Y_{j,0} = D_j$.

Ratio

We show that under the noted condition, $\widehat{R}_j \xrightarrow{p} R_j$. By the preliminary propositions noted above for $\widehat{R}_j$, $\widehat{R}_j \xrightarrow{p} \frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]}$. Under the condition, bias is not expected to differ over time within group j, so $E[\beta_{j,t}] = E[\beta_j]$. Substituting per the condition, $\frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]} = \frac{Y_{j,1}*E[\beta_j]}{Y_{j,0}*E[\beta_j]}$. Canceling terms, $\frac{Y_{j,1}*E[\beta_j]}{Y_{j,0}*E[\beta_j]} = \frac{Y_{j,1}}{Y_{j,0}} = R_j$.

Percent difference

We show that under the noted condition, $\widehat{PD}_j \xrightarrow{p} PD_j$. By the preliminary propositions noted above for $\widehat{PD}_j$, $\widehat{PD}_j \xrightarrow{p} \frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]} - 1$. Under the condition, bias is not expected to differ over time within group j, so $E[\beta_{j,t}] = E[\beta_j]$. Substituting per the condition, $\frac{Y_{j,1}*E[\beta_{j,1}]}{Y_{j,0}*E[\beta_{j,0}]} - 1 = \frac{Y_{j,1}*E[\beta_j]}{Y_{j,0}*E[\beta_j]} - 1$. Canceling terms, $\frac{Y_{j,1}*E[\beta_j]}{Y_{j,0}*E[\beta_j]} - 1 = \frac{Y_{j,1}}{Y_{j,0}} - 1 = PD_j$.

Difference in differences

We show that under the noted condition, $E[\widehat{DiD}] = DiD$. By the preliminary propositions noted above for $\widehat{DiD}$, $E[\widehat{DiD}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}]$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,t}] - E[\alpha_{1,t}] - E[\alpha_{0,t}] + E[\alpha_{0,t}]$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,t}] - E[\alpha_{1,t}] - E[\alpha_{0,t}] + E[\alpha_{0,t}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

We show that under the noted condition, $\widehat{RoR} \xrightarrow{p} RoR$. By the preliminary propositions noted above for $\widehat{RoR}$, $\widehat{RoR} \xrightarrow{p} \frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}}$. Substituting per the condition, $\frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}} = \frac{\frac{Y_{1,1}*E[\beta_{1,t}]}{Y_{1,0}*E[\beta_{1,t}]}}{\frac{Y_{0,1}*E[\beta_{0,t}]}{Y_{0,0}*E[\beta_{0,t}]}}$. Canceling terms, $\frac{\frac{Y_{1,1}*E[\beta_{1,t}]}{Y_{1,0}*E[\beta_{1,t}]}}{\frac{Y_{0,1}*E[\beta_{0,t}]}{Y_{0,0}*E[\beta_{0,t}]}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Ratio of percent differences

We show that under the noted condition, $\widehat{RPD} \xrightarrow{p} RPD$. By the preliminary propositions noted above for $\widehat{RPD}$, $\widehat{RPD} \xrightarrow{p} \frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}-1}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}-1}$. Substituting per the condition, $\frac{\frac{Y_{1,1}*E[\beta_{1,1}]}{Y_{1,0}*E[\beta_{1,0}]}-1}{\frac{Y_{0,1}*E[\beta_{0,1}]}{Y_{0,0}*E[\beta_{0,0}]}-1} = \frac{\frac{Y_{1,1}*E[\beta_{1,t}]}{Y_{1,0}*E[\beta_{1,t}]}-1}{\frac{Y_{0,1}*E[\beta_{0,t}]}{Y_{0,0}*E[\beta_{0,t}]}-1}$. Canceling terms, $\frac{\frac{Y_{1,1}*E[\beta_{1,t}]}{Y_{1,0}*E[\beta_{1,t}]}-1}{\frac{Y_{0,1}*E[\beta_{0,t}]}{Y_{0,0}*E[\beta_{0,t}]}-1} = \frac{\frac{Y_{1,1}}{Y_{1,0}}-1}{\frac{Y_{0,1}}{Y_{0,0}}-1} = RPD$.

Condition 3.2E (C3.2E): In expectation, bias is the same between groups within time period.

On the additive scale, this condition states that
$E[\alpha_{0,0}] = E[\alpha_{1,0}] = E[\alpha_{j,0}]$, and $E[\alpha_{0,1}] = E[\alpha_{1,1}] = E[\alpha_{j,1}]$.
On the multiplicative scale, it states that
$0 \neq E[\beta_{0,0}] = E[\beta_{1,0}] = E[\beta_{j,0}]$, and $0 \neq E[\beta_{0,1}] = E[\beta_{1,1}] = E[\beta_{j,1}]$.

Under this condition, $\widehat{DiD}$ is an unbiased estimator of DiD, and $\widehat{RoR}$ is a consistent estimator RoR.

Difference in differences

We show that under the noted condition, $E[\widehat{DiD}] = DiD$. By the preliminary propositions noted above for $\widehat{DiD}$, $E[\widehat{DiD}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}]$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}] - E[\alpha_{j,1}] + E[\alpha_{j,0}]$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{j,1}] - E[\alpha_{j,0}] - E[\alpha_{j,1}] + E[\alpha_{j,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

We show that under the noted condition, $\widehat{RoR} \xrightarrow{p} RoR$. By the preliminary propositions noted above for $\widehat{RoR}$, $\widehat{RoR} \xrightarrow{p} \frac{\frac{Y_{1,1} * E[\beta_{1,1}]}{Y_{1,0} * E[\beta_{1,0}]}}{\frac{Y_{0,1} * E[\beta_{0,1}]}{Y_{0,0} * E[\beta_{0,0}]}} = \frac{\frac{Y_{1,1}}{Y_{1,0}} * \frac{E[\beta_{1,1}]}{E[\beta_{1,0}]}}{\frac{Y_{0,1}}{Y_{0,0}} * \frac{E[\beta_{0,1}]}{E[\beta_{0,0}]}}$. Substituting per the condition, $\frac{\frac{Y_{1,1}}{Y_{1,0}} * \frac{E[\beta_{1,1}]}{E[\beta_{1,0}]}}{\frac{Y_{0,1}}{Y_{0,0}} * \frac{E[\beta_{0,1}]}{E[\beta_{0,0}]}} = \frac{\frac{Y_{1,1}}{Y_{1,0}} * \frac{E[\beta_{j,1}]}{E[\beta_{j,0}]}}{\frac{Y_{0,1}}{Y_{0,0}} * \frac{E[\beta_{j,1}]}{E[\beta_{j,0}]}}$.

Canceling terms, $\frac{\frac{Y_{1,1} * E[\beta_{j,1}]}{Y_{1,0} * E[\beta_{j,0}]}}{\frac{Y_{0,1} * E[\beta_{j,1}]}{Y_{0,0} * E[\beta_{j,0}]}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Condition 4.1E (C4.1E): The expected between-time-period trend in bias is the same between groups.

On the additive scale, this condition states that $E[\alpha_{1,1} - \alpha_{1,0}] = E[\alpha_{0,1} - \alpha_{0,0}] = E[\alpha_{j,1} - \alpha_{j,0}]$. This is the classic parallel-trends assumption.[1]

On the multiplicative scale, it states that $E\left[\frac{\beta_{1,1}}{\beta_{1,0}}\right] = E\left[\frac{\beta_{0,1}}{\beta_{0,0}}\right] = E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right]$. To facilitate algebra below, let $E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right] = k$.

Under this condition, $\widehat{DiD}$ is an unbiased estimator of DiD, and $\widehat{RoR}$ and $\widehat{RPD}$ are consistent estimators, respectively, of RoR and RPD.

Difference in differences

We show that under the noted condition, $E[\widehat{DiD}] = DiD$. By the preliminary propositions noted above for $\widehat{DiD}$, $E[\widehat{DiD}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}]$. Factoring the expected values of the bias-factor terms to facilitate substitution, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (E[\alpha_{1,1}] - E[\alpha_{1,0}]) - (E[\alpha_{0,1}] - E[\alpha_{0,0}])$. By linearity of expectation, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1} - \alpha_{1,0}] - E[\alpha_{0,1} - \alpha_{0,0}]$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1} - \alpha_{1,0}] - E[\alpha_{0,1} - \alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{j,1} - \alpha_{j,0}] - E[\alpha_{j,1} - \alpha_{j,0}]$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{j,1} - \alpha_{j,0}] - E[\alpha_{j,1} - \alpha_{j,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

We show that under the noted condition, $\widehat{RoR} \xrightarrow{p} RoR$. Note that $\widehat{RoR} = \frac{\frac{\hat{Y}_{1,1}}{\hat{Y}_{1,0}}}{\frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}}}$. Let $\hat{R}_{j=1} = \frac{\hat{Y}_{1.1}}{\hat{Y}_{1,0}}$ and $\hat{R}_{j=0} = \frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}}$, so that $\widehat{RoR}$ is alternatively defined as $\frac{\hat{R}_{j=1}}{\hat{R}_{j=0}}$. By the WLLN, $\hat{R}_{j=1} \xrightarrow{p} E[\hat{R}_{j=1}]$, and $\hat{R}_{j=0} \xrightarrow{p} E[\hat{R}_{j=0}]$. Let $f(a,b) = \frac{a}{b}$. By the Continuous Mapping Theorem, $f(\hat{R}_{j=1}, \hat{R}_{j=0}) = \frac{\hat{R}_{j=1}}{\hat{R}_{j=0}} = \widehat{RoR} \xrightarrow{p} \frac{E[\hat{R}_{j=1}]}{E[\hat{R}_{j=0}]}$, assuming $E[\hat{R}_{j=0}] \neq 0$. By definition, $\frac{E[\hat{R}_{j=1}]}{E[\hat{R}_{j=0}]} = \frac{E\left[\frac{\hat{Y}_{1.1}}{\hat{Y}_{1,0}}\right]}{E\left[\frac{\hat{Y}_{0,1}}{\hat{Y}_{0,0}}\right]} = \frac{E\left[\frac{Y_{1,1}*\beta_{1,1}}{Y_{1,0}*\beta_{1,0}}\right]}{E\left[\frac{Y_{0,1}*\beta_{0,1}}{Y_{0,0}*\beta_{0,0}}\right]} = \frac{E\left[\frac{Y_{1,1}}{Y_{1,0}}*\frac{\beta_{1,1}}{\beta_{1,0}}\right]}{E\left[\frac{Y_{0,1}}{Y_{0,0}}*\frac{\beta_{0,1}}{\beta_{0,0}}\right]}$. $\frac{Y_{1,1}}{Y_{1,0}}$ and $\frac{Y_{0,1}}{Y_{0,0}}$ are each constants, so $\frac{E\left[\frac{Y_{1,1}}{Y_{1,0}}*\frac{\beta_{1,1}}{\beta_{1,0}}\right]}{E\left[\frac{Y_{0,1}}{Y_{0,0}}*\frac{\beta_{0,1}}{\beta_{0,0}}\right]} = \frac{\frac{Y_{1,1}}{Y_{1,0}}*E\left[\frac{\beta_{1,1}}{\beta_{1,0}}\right]}{\frac{Y_{0,1}}{Y_{0,0}}*E\left[\frac{\beta_{0,1}}{\beta_{0,0}}\right]}$. Substituting per the condition, $\frac{\frac{Y_{1,1}}{Y_{1,0}}*E\left[\frac{\beta_{1,1}}{\beta_{1,0}}\right]}{\frac{Y_{0,1}}{Y_{0,0}}*E\left[\frac{\beta_{0,1}}{\beta_{0,0}}\right]} = \frac{\frac{Y_{1,1}}{Y_{1,0}}*E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right]}{\frac{Y_{0,1}}{Y_{0,0}}*E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right]}$. Canceling terms, $\frac{\frac{Y_{1,1}}{Y_{1,0}}*E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right]}{\frac{Y_{0,1}}{Y_{0,0}}*E\left[\frac{\beta_{j,1}}{\beta_{j,0}}\right]} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

Condition 4.2E (C4.2E): The expected between-group trend in bias is the same between time periods.

On the additive scale, this condition states that $E[\alpha_{1,1} - \alpha_{0,1}] = E[\alpha_{1,0} - \alpha_{0,0}] = E[\alpha_{1,t} - \alpha_{0,t}]$. This is also the parallel-trends assumption.[1]

On the multiplicative scale, it states that $E\left[\frac{\beta_{1,1}}{\beta_{0,1}}\right] = E\left[\frac{\beta_{1,0}}{\beta_{0,0}}\right] = E\left[\frac{\beta_{1,t}}{\beta_{0,t}}\right]$.

Under this condition, $\widehat{DiD}$ is an unbiased estimator of DiD, and $\widehat{RoR}$ is a consistent estimator RoR.

Difference in differences

We show that under the noted condition, $E[\widehat{DiD}] = DiD$. By the preliminary propositions noted above for $\widehat{DiD}$, $E[\widehat{DiD}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}]$. Factoring the expected values of the bias-factor terms to facilitate substitution, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1}] - E[\alpha_{1,0}] - E[\alpha_{0,1}] + E[\alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + (E[\alpha_{1,1}] - E[\alpha_{0,1}]) - (E[\alpha_{1,0}] - E[\alpha_{0,0}])$. By linearity of expectation, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1} - \alpha_{0,1}] - E[\alpha_{1,0} - \alpha_{0,0}]$. Substituting per the condition, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,1} - \alpha_{0,1}] - E[\alpha_{1,0} - \alpha_{0,0}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,t} - \alpha_{0,t}] - E[\alpha_{1,t} - \alpha_{0,t}]$. Canceling terms, $Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} + E[\alpha_{1,t} - \alpha_{0,t}] - E[\alpha_{1,t} - \alpha_{0,t}] = Y_{1,1} - Y_{1,0} - Y_{0,1} + Y_{0,0} = DiD$.

Ratio of ratios

We show that under the noted condition, $\widehat{RoR} \xrightarrow{p} RoR$. Rearranging terms, $\widehat{RoR} = \frac{\frac{\hat{Y}_{1.1}}{\hat{Y}_{1.0}}}{\frac{\hat{Y}_{0.1}}{\hat{Y}_{0.0}}} = \frac{\frac{\hat{Y}_{1.1}}{\hat{Y}_{0.1}}}{\frac{\hat{Y}_{1.0}}{\hat{Y}_{0.0}}}$. Let $\hat{R}_{t=1} = \frac{\hat{Y}_{1.1}}{\hat{Y}_{0.1}}$ and $\hat{R}_{t=0} = \frac{\hat{Y}_{1.0}}{\hat{Y}_{0.0}}$, such that $\widehat{RoR}$ is alternatively defined as $\widehat{RoR} = \frac{\hat{R}_{t=1}}{\hat{R}_{t=0}}$. By the WLLN, $\hat{R}_{t=1} \xrightarrow{p} E[\hat{R}_{t=1}]$, and $\hat{R}_{t=0} \xrightarrow{p} E[\hat{R}_{t=0}]$. Let $f(a,b) = \frac{a}{b}$. By the Continuous Mapping Theorem, $f(\hat{R}_{t=1}, \hat{R}_{t=0}) = \frac{\hat{R}_{t=1}}{\hat{R}_{t=0}} = \widehat{RoR} \xrightarrow{p} \frac{E[\hat{R}_{t=1}]}{E[\hat{R}_{t=0}]}$, assuming $E[\hat{R}_{t=0}] \neq 0$. By definition, $\frac{E[\hat{R}_{t=1}]}{E[\hat{R}_{t=0}]} = \frac{E\left[\frac{\hat{Y}_{1.1}}{\hat{Y}_{0.1}}\right]}{E\left[\frac{\hat{Y}_{1.0}}{\hat{Y}_{0.0}}\right]} = \frac{E\left[\frac{Y_{1,1}*\beta_{1,1}}{Y_{0,1}*\beta_{0,1}}\right]}{E\left[\frac{Y_{1,0}*\beta_{1,0}}{Y_{0,0}*\beta_{0,0}}\right]} = \frac{E\left[\frac{Y_{1,1}}{Y_{0,1}}*\frac{\beta_{1,1}}{\beta_{0,1}}\right]}{E\left[\frac{Y_{1,0}}{Y_{0,0}}*\frac{\beta_{1,0}}{\beta_{0,0}}\right]}$. $\frac{Y_{1,1}}{Y_{0,1}}$ and $\frac{Y_{1,0}}{Y_{0,0}}$ are each constants, so $\frac{E\left[\frac{Y_{1,1}}{Y_{0,1}}*\frac{\beta_{1,1}}{\beta_{0,1}}\right]}{E\left[\frac{Y_{1,0}}{Y_{0,0}}*\frac{\beta_{1,0}}{\beta_{0,0}}\right]} = \frac{\frac{Y_{1,1}}{Y_{0,1}}*E\left[\frac{\beta_{1,1}}{\beta_{0,1}}\right]}{\frac{Y_{1,0}}{Y_{0,0}}*E\left[\frac{\beta_{1,0}}{\beta_{0,0}}\right]}$. Substituting per the condition, $\frac{\frac{Y_{1,1}}{Y_{0,1}}*E\left[\frac{\beta_{1,1}}{\beta_{0,1}}\right]}{\frac{Y_{1,0}}{Y_{0,0}}*E\left[\frac{\beta_{1,0}}{\beta_{0,0}}\right]} = \frac{\frac{Y_{1,1}}{Y_{0,1}}*E\left[\frac{\beta_{1,t}}{\beta_{0,t}}\right]}{\frac{Y_{1,0}}{Y_{0,0}}*E\left[\frac{\beta_{1,t}}{\beta_{0,t}}\right]}$. Canceling terms, $\frac{\frac{Y_{1,1}}{Y_{0,1}}*E\left[\frac{\beta_{1,t}}{\beta_{0,t}}\right]}{\frac{Y_{1,0}}{Y_{0,0}}*E\left[\frac{\beta_{1,t}}{\beta_{0,t}}\right]} = \frac{\frac{Y_{1,1}}{Y_{0,1}}}{\frac{Y_{1,0}}{Y_{0,0}}} = \frac{\frac{Y_{1,1}}{Y_{1,0}}}{\frac{Y_{0,1}}{Y_{0,0}}} = RoR$.

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