Supplementary Materials for "Simultaneous Detection of Signal Regions Using Quadratic Scan Statistics With Applications to Whole Genome Association Studies"

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Contents

1 Supplementary Note

In this supplementary note, we give the detail proof of the theorems in the paper. In order to proof the theorem, we introduce the following three lemmas first.

Lemma 1 (Refinement of Bernstein's Inequality (Petrov 1968)) *Consider a sequence of independent random variables* $\{\boldsymbol{X}_j\}, j = 1, 2, \cdots$. *Assume* $\mathbb{E}(\boldsymbol{X}_j) = 0$, $\mathbb{E}(\boldsymbol{X}_j^2) = \sigma_j^2$ *and* $L_j(z) = \log(\mathbb{E}(\exp(z\boldsymbol{X}_j)))$. *We introduce the following notation:*

$$\boldsymbol{S}_n = \sum_{j=1}^{n} X_j, \quad B_n = \sum_{j=1}^{n} \sigma_j^2, \quad F_n(x) = \mathbb{P}(\boldsymbol{S}_n < x\sqrt{B_n}), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})dt.$$

If the following three conditions hold:

(1) *There exists positive constants* $A, c_1, c_2, \cdots$ *such that* $|L_j(z)| \leq c_j$ *for* $|z| < A$, $j = 1, 2, \cdots$,

(2) $\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} c_j^{\frac{3}{2}} < \infty$,

(3) $\liminf_{n\to\infty} \frac{B_n}{n} > 0$,

then there exists a positive constant w *such that for sufficiently large* n,

$$1 - F_n(x) = [1 - \Phi(x)] \exp\left\{\frac{x^3}{\sqrt{n}} \lambda_n(\frac{x}{\sqrt{n}})\right\}(1 + l_1 w)$$

$$F_n(-x) = \Phi(-x) \exp\{-\frac{x^3}{\sqrt{n}} \lambda_n(-\frac{x}{\sqrt{n}})(1 + l_2 w)\}$$

in the region $0 \leq x \leq w\sqrt{n}$. *Here* $|l_1| \leq l$ *and* $|l_2| \leq l$ *being some constant.*

The next two lemmas are two inequalities for the tail probability of normal distribution and chi-square distribution, respectively.

Lemma 2 (Mills' Ratio Inequality) *For arbitrary positive number* $x > 0$, *the inequalities*

$$\frac{x}{1 + x^2} \exp(-\frac{x^2}{2}) < \int_{x}^{\infty} \exp(-\frac{u^2}{2})du < \frac{1}{x} \exp(-\frac{x^2}{2}).$$

Lemma 3 (Exponential inequality for chi-square distribution(Laurent and Massart 2000)) *Let* $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_D$ *be i.i.d. standard Gaussian variables. Let* $a_1, \cdots, a_D$ *be nonnegative constant. We set*

$$||\boldsymbol{a}||_\infty = \sup_{1\leq i\leq D} |a_i| \quad ||\boldsymbol{a}||_2^2 = \sum_{i=1}^{D} a_i^2.$$

Let $\boldsymbol{Z} = \sum_{i=1}^{D} a_i(\boldsymbol{Y}_i^2 - 1)$, *then the following inequalities hold for any positive x:*

$$\mathbb{P}(\boldsymbol{Z} \geq 2||\boldsymbol{a}||_2\sqrt{x} + 2||\boldsymbol{a}||_\infty x) \leq \exp(-x),$$

$$\mathbb{P}(\boldsymbol{Z} \leq -2||\boldsymbol{a}||_2\sqrt{x}) \leq \exp(-x).$$

1.1 Bound of Empirical Threshold

Now we go back to the proof of the main theorem in the paper. First recall the notation of Q-scan procedure. For any vector $\boldsymbol{a}$, set $||\boldsymbol{a}||_1 = \sum_i |a_i|$, $||\boldsymbol{a}||_2 = \sqrt{\sum_i a_i^2}$ and $||\boldsymbol{a}||_\infty = \sup_i |a_i|$. Let $U_i = \frac{1}{\sqrt{n}}\boldsymbol{G}_i^T(\boldsymbol{Y} - \hat{\boldsymbol{\mu}})$ be the score statistic for variant i, $i = 1, 2, \cdots, p$. Under the null hypothesis, $U_i \sim N(0, \sigma_i^2)$ with $\sigma_i^2 = \frac{1}{n}\boldsymbol{G}_i^T\boldsymbol{P}\boldsymbol{G}_i$ for all $i = 1, 2, \cdots, p$, where $\boldsymbol{P}$ is the projection matrix in the null model. Assume there exists constant $c > 0$ such that $\sigma_i^2 \geq c$. For any region I, assume $\boldsymbol{U}_I = (U_i)_{i\in I}$ and $\boldsymbol{\Sigma}_I = \frac{1}{n}\boldsymbol{G}_I^T\boldsymbol{P}\boldsymbol{G}_I$, then $\boldsymbol{U}_I \sim N(0, \boldsymbol{\Sigma}_I)$. Let $\boldsymbol{\lambda}_I$ is the eigenvalues of $\boldsymbol{\Sigma}_I$, the Q-scan statistic of region is defined as

$$Q(I) = \frac{\sum_{i\in I} U_i^2 - \sum_{i=1}^{|I|}\lambda_{I,i}}{\sqrt{2\sum_{i=1}^{|I|}\lambda_{I,i}^2}} = \frac{\sum_{i\in I} U_i^2 - ||\boldsymbol{\lambda}_I||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2}.$$

We first proof the results of fixed length scan.

Lemma 4 *If the following conditions (A)-(C) hold,*

(A) $\max_{|I|=L_p} ||\boldsymbol{\lambda}_I||_\infty \leq K_0$*, where* K_0 *is a constant,*

(B) $\frac{L_p}{\log p} \to \infty$ *and* $\frac{\log(L_p)}{\log(p)} \to 0$*,*

(C) $\{U_i\}_{i=1}^p$ *is* M_p*-dependent and* $\frac{\log(M_p)}{\log(p)} \to 0$*,*

then

$$\frac{\max_{|I|=L_p} Q(I)}{\sqrt{2\log(p)}} \xrightarrow{p} 1.$$

Proof For any I satisfies that $|I| = L_p$, assume $\boldsymbol{\Omega}_I\boldsymbol{\Sigma}_I\boldsymbol{\Omega}_I^T = \mathrm{diag}(\boldsymbol{\lambda}_I)$ is the SVD of $\boldsymbol{\Sigma}_I$ where $\boldsymbol{\Omega}_I$ is an orthogonal matrix. Let $\tilde{\boldsymbol{U}}_I = \boldsymbol{\Omega}_I\boldsymbol{U}_I$, then $\tilde{\boldsymbol{U}}_I \sim N\big(0, \mathrm{diag}(\boldsymbol{\lambda}_I)\big)$ and $\tilde{\boldsymbol{U}}_I^T\tilde{\boldsymbol{U}}_I = \boldsymbol{U}_I^T\boldsymbol{U}_I$. Hence, for $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}(Q(I) > (1+\epsilon)\sqrt{2\log(p)}) &= \mathbb{P}\big(\sum_{i\in I} U_i^2 - ||\boldsymbol{\lambda}_I||_1 > 2(2+\epsilon)||\boldsymbol{\lambda}_I||_2\sqrt{\log(p)}\big) \\ &= \mathbb{P}\big(\sum_{i\in I}\lambda_{I,i}(\frac{\tilde{U}_i^2}{\lambda_{I,i}} - 1) > 2(1+\epsilon)||\boldsymbol{\lambda}_I||_2\sqrt{\log(p)}\big).\end{aligned}$$

Note that $||\boldsymbol{\lambda}_I||_2 \geq ||\boldsymbol{\lambda}_I||_1^2/\sqrt{|I|} \geq c\sqrt{|I|}$, by condition (A) and (B), we have

$$(1+\frac{\epsilon}{2})^2||\boldsymbol{\lambda}_I||_\infty\log(p) = O\big(\log(p)\big) = o(||\boldsymbol{\lambda}_I||_2\sqrt{\log p}).$$

Then by Lemma 3, for p sufficiently large,

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i\in I}\lambda_{I,i}(\frac{\tilde{U}_i^2}{\lambda_{I,i}}-1) > 2(1+\epsilon)||\boldsymbol{\lambda}_I||_2\sqrt{\log(p)}\Big) \\
\leq\ &\mathbb{P}\big(\sum_{i\in I}\lambda_{I,i}(\frac{\tilde{U}_i^2}{\lambda_{I,i}}-1) > 2(1+\frac{\epsilon}{2})^2||\boldsymbol{\lambda}_I||_2\sqrt{\log(p)} + 2(1+\frac{\epsilon}{2})||\boldsymbol{\lambda}_I||_\infty \log(p)\big) \\
\leq\ &\exp\{-(1+\frac{\epsilon}{2})^2\log(p)\}.
\end{aligned}$$

Hence, using Boole's inequality,

$$\mathbb{P}\Big(\max_{|I|=L_p} Q(I) > (1+\epsilon)\sqrt{2\log(p)}\Big) \leq \sum_{|I|=L_p}\mathbb{P}\Big(Q(I) > (1+\epsilon)\sqrt{2\log(p)}\Big) \leq p\times\exp\{-(1+\frac{\epsilon}{2})^2\log(p)\} = o(1) \tag{1}$$

On the other hand,

$$\begin{aligned}
\mathbb{P}\Big(\max_{|I|=L_p} Q(I) < (1-\epsilon)\sqrt{2\log(p)}\Big) &= \mathbb{P}\Big(\bigcap_{|I|=L_p}\{Q(I) < (1-\epsilon)\sqrt{2\log(p)}\}\Big) \\
&= 1-\mathbb{P}\big(\bigcup_{|I|=L_p}\{Q(I) \geq (1-\epsilon)\sqrt{2\log(p)}\}\big)
\end{aligned}$$

Let $I_k = \{k, k+1\cdots, k+L_p-1\}$ and $A_k = \Big\{\frac{\sum_{i\in I_k}U_i^2 - ||\boldsymbol{\lambda}_{I_k}||_1}{\sqrt{2}||\boldsymbol{\lambda}_{I_k}||_2} \geq (1-\epsilon)\sqrt{2\log(p)}\Big\}$, by Chung-Erdös inequality,

$$\mathbb{P}\big(\bigcup_{|I|=L_p}\{Q(I) \geq (1-\epsilon)\sqrt{2\log(p)}\}\big) \geq \frac{\{\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)\}^2}{\sum_{i=1}^{p-L_p+1}\sum_{j=1}^{p-L_p+1}\mathbb{P}(A_i\cap A_j)}.$$

Note that

$$\begin{aligned}
\sum_{i=1}^{p-L_p+1}\sum_{j=1}^{p-L_p+1}\mathbb{P}(A_i\cap A_j) &= \sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i) + \sum_{\substack{i\neq j\\ 1\leq i,j\leq p-L_p+1}}\mathbb{P}(A_i\cap A_j) \\
&= \sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i) + \sum_{|i-j|\leq M_p+L_p}\mathbb{P}(A_i\cap A_j) + \sum_{|i-j|>M_p+L_p}\mathbb{P}(A_i\cap A_j) \\
&= \sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i) + \sum_{|i-j|\leq M_p+L_p}\mathbb{P}(A_i\cap A_j) + \sum_{|i-j|>M_p+L_p}\mathbb{P}(A_i)\mathbb{P}(A_j) \\
&\leq (M_p+L_p+1)\sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i) + \sum_{|i-j|>M_p+L_p}\mathbb{P}(A_i)\mathbb{P}(A_j),
\end{aligned}$$

then by condition (C), we get

$$
\begin{aligned}
& \frac{\{\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)\}^2}{\sum_{i=1}^{p-L_p+1}\sum_{j=1}^{p-L_p+1}\mathbb{P}(A_i\cap A_j)} \\
\geq\ & \frac{\{\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)\}^2}{(M_p+L_p+1)\sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i)+\sum_{|i-j|>M_p+L_p}\mathbb{P}(A_i)\mathbb{P}(A_j)} \\
\geq\ & 1-\frac{(M_p+L_p+1)\sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i)}{(M_p+L_p+1)\sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i)+\sum_{|i-j|>M_p+L_p}\mathbb{P}(A_i)\mathbb{P}(A_j)} \\
\geq\ & 1-\frac{(M_p+L_p+1)\sum_{i=1}^{p-L_p+1}\mathbb{P}(A_i)}{\{\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)\}^2} \\
\geq\ & 1-\frac{M_p+L_p+1}{\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)}.
\end{aligned}
$$

By condition (A) and (B), using Lemma 1, there exists constant $c_1, c_2 > 0$ such that

$$
\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)\geq\mathbb{P}\big[1-\Phi\{(1-\epsilon)\sqrt{2\log(p)}\}\big]\exp\big\{\frac{2^{\frac{3}{2}}(1-\epsilon)^3(\log(p))^{\frac{3}{2}}}{\sqrt{L}}c_1\big\}c_2
$$

Then, using Lemma 2, we have

$$
\begin{aligned}
\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k) &\geq \frac{p(1-\epsilon)\sqrt{2\log(p)}}{1+2(1-\epsilon)^2\log(p)}\exp\big\{\frac{2^{\frac{3}{2}}(1-\epsilon)^3(\log(p))^{\frac{3}{2}}}{\sqrt{L}}c_1\big\}c_2 \\
&\geq \frac{c_2}{4(1-\epsilon)}\exp\big\{\log(p)(2\epsilon-\epsilon^2+2^{\frac{3}{2}}c_1(1-\epsilon)^3\sqrt{\frac{\log(p)}{L_p}})-\frac{1}{2}\log(\log(p))\big\}.
\end{aligned}
$$

Hence by condition (B) and (C),

$$
\frac{\sum_{k=1}^{p-L_p+1}\mathbb{P}(A_k)}{M_p+L_p+1}\geq\frac{c_2}{4(1-\epsilon)}\exp\Big[\log(p)\big\{2\epsilon-\epsilon^2+2^{\frac{3}{2}}c_1(1-\epsilon)^3\sqrt{\frac{\log(p)}{L_p}}\big\}-\frac{1}{2}\log(\log(p))-\log(M_p+L_p+1)\Big]\to\infty.
$$

This indicates that

$$
\mathbb{P}\big(\max_{|I|=L_p}Q(I)<(1-\epsilon)\sqrt{2\log(p)}\big)\to 0 \tag{2}
$$

Combine (1) and (2), we get

$$
\mathbb{P}\Big(\big|\frac{\max_{|I|=L_p}Q(I)}{\sqrt{2\log(p)}}-1\big|>\epsilon\Big)\to 0.
$$

For the arbitrary of ϵ, we complete the proof. □

We then prove the result of multi-length scan, which is the Theorem 1 in the paper.

Theorem 1 *If the following conditions (A)-(C) hold,* *(A)* $\max_{|I|=L_{\max}} ||\boldsymbol{\lambda}_I||_\infty \le K_0$*, where* K_0 *is a constant,*

(B) $\frac{L_{\min}}{\log(p)} \to \infty$ *and* $\frac{\log(L_{\max})}{\log(p)} \to 0$,

(C) $\{U_i\}_{i=1}^p$ *is* M_p*-dependent and* $\frac{\log(M_p)}{\log(p)} \to 0$,

then

$$\frac{\max_{L_{\min}\le|I|\le L_{\max}} Q(I)}{\sqrt{2\log(p)}} \xrightarrow{p} 1.$$

Proof For any $\epsilon > 0$,

$$\begin{aligned} & \mathbb{P}\Big(\Big|\frac{\max_{L_{\min}\le|I|\le L_{\max}} Q(I)}{\sqrt{2\log(p)}} - 1\Big| \ge \epsilon\Big) \\ = \; & \mathbb{P}\Big(\max_{L_{\min}\le|I|\le L_{\max}} Q(I) \ge (1+\epsilon)\sqrt{2\log(p)}\Big) + \mathbb{P}\Big(\max_{L_{\min}\le|I|\le L_{\max}} Q(I) \le (1-\epsilon)\sqrt{2\log(p)}\Big) \\ \triangleq \; & A_1(\epsilon) + A_2(\epsilon). \end{aligned}$$

Note that $\max_{L_{\min}\le|I|\le L_{\max}} Q(I) \ge \max_{|I|=L_{\max}} Q(I)$, by Lemma 4,

$$A_2(\epsilon) \le \mathbb{P}\Big(\max_{|I|=L_{\max}} Q(I) \le (1-\epsilon)\sqrt{2\log(p)}\Big) = o(1). \tag{3}$$

For $A_1(\epsilon)$, by Boole's inequality,

$$A_1(\epsilon) \le \sum_{L_{\min}\le|I|\le L_{\max}} \mathbb{P}\Big(Q(I) \ge (1+\epsilon)\sqrt{2\log(p)}\Big).$$

For any I satisfies that $L_{\min} \le |I| \le L_{\max}$, by condition (B),

$$\frac{2(1+\frac{\epsilon}{2})^2\log(p)||\boldsymbol{\lambda}_I||_\infty}{||\boldsymbol{\lambda}_I||_2} \le \frac{2K_0(1+\frac{\epsilon}{2})^2\log(p)}{\sqrt{L_{\min}}} = o(\sqrt{\log(p)}).$$

Hence, by Lemma 3,

$$A_1(\epsilon) \le \sum_{L_{\min}\le|I|\le L_{\max}} \exp\{-(1+\frac{\epsilon}{2})^2\log(p)\} \le \exp\{-(1+\frac{\epsilon}{2})^2\log(p) + \log(p) + \log(L_{\max})\} = o(1). \tag{4}$$

Using (3) and (4), we have

$$\mathbb{P}\Big(\Big|\frac{\max_{L_{\min}\le|I|\le L_{\max}} Q(I)}{\sqrt{2\log(p)}} - 1\Big| \ge \epsilon\Big) = o(1).$$

For the arbitrary of ϵ, we complete the proof. □

Assume $h(p, L_{\min}, L_{\max}, \alpha)$ is the $(1-\alpha)$th quantile of $Q_{\max}$, that is,

$$\mathbb{P}\big(Q_{\max} > h(p, L_{\min}, L_{\max}, \alpha)\big) = \alpha.$$

Hence $h(p, L_{\min}, L_{\max}, \alpha)$ is the threshold that control the family-wise error rate at α level. By Theorem 1, for p sufficiently large, we have $0.99\sqrt{2\log(p)} \le h(p, L_{\min}, L_{\max}, \alpha) \le 1.01\sqrt{2\log(p)}$. Next we give a more accurate upper bound of $h(p, L_{\min}, L_{\max}, \alpha)$.

Proposition 2 *If condition (A) and (B) hold, for p sufficiently large,*

$$h(p, L_{\min}, L_{\max}, \alpha) \le \sqrt{2\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]} + \frac{\sqrt{2}\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]}{\{L_{\min}\log(p)\}^{\frac{1}{4}}}.$$

Proof Let $\tilde{h}(p,\alpha) = \sqrt{2\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]} + \frac{\sqrt{2}\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]}{\{L_{\min}\log(p)\}^{\frac{1}{4}}}$. For any I satisfies that $L_{\min} \le |I| \le L_{\max}$,

$$\frac{2||\lambda_I||_\infty\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} = O\Big(\frac{\sqrt{2}\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]}{\sqrt{|I|}}\Big).$$

Note that $\frac{\log(p)}{|I|} \le \frac{\log(p)}{L_{\min}} = o(1)$, then for sufficiently large p,

$$\frac{2||\boldsymbol{\lambda}_I||_\infty\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} \le \frac{\sqrt{2}\big[\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big]}{\big(L_{\min}\log(p)\big)^{\frac{1}{4}}}.$$

Hence, by Lemma 3,

$$\mathbb{P}\big(Q_{\max} > \tilde{h}(p,\alpha)\big) \le \sum_{L_{\min}\le|I|\le L_{\max}} \exp\big(-\log\{p(L_{\max}-L_{\min})\} - \log(\alpha)\big) \le \alpha.$$

Thus $h(p, L_{\min}, L_{\max}, \alpha) \le \tilde{h}(p,\alpha)$ and we complete the proof. □

By Theorem 1 and Proposition 2, for p sufficiently large,

$$(1-\epsilon)\sqrt{2\log(p)} \le h(p, L_{\min}, L_{\max}, \alpha) \le \sqrt{2\gamma_p} + \frac{\sqrt{2}\gamma_p}{\{L_{\min}\log(p)\}^{\frac{1}{4}}},$$

where ϵ is a small constant and $\gamma_p = \log\{p(L_{\max}-L_{\min})\} - \log(\alpha)$.

1.2 Consistency of Signal Region Detection

In this section, we show the results of power analysis. We first proof the proposed Q-SCAN procedure could consistently select a signal region that overlaps with the true signal region. Let $\boldsymbol{\mu}_I = \{\mu_i\}_{i\in I}$ for any region I. Assume I^* is the true signal region with $\boldsymbol{\mu}_{I^*} \ne 0$ and $L_{\min} \le |I^*| \le L_{\max}$.

Theorem 3 *Assume condition (A)-(C) and the following condition (D) hold,*

(D) $\frac{||\boldsymbol{\mu}_{I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2} \geq 2(1+\epsilon_0)\sqrt{\log(p)}$ *for some constant* $\epsilon_0 > 0$*,*

then there exists constant $C > 0$*, such that*

$$\mathbb{P}\big(Q(I^*) > h(p, L_{\min}, L_{\max}, \alpha)\big) \geq 1 - 2p^{-C\epsilon_0^2} \to 1.$$

Proof Assume $\boldsymbol{\Omega}_{I^*}\boldsymbol{\Sigma}_{I^*}\boldsymbol{\Omega}_{I^*}^T = \text{diag}(\boldsymbol{\lambda}_{I^*})$ is the SVD of $\boldsymbol{\Sigma}_{I^*}$ where $\boldsymbol{\Omega}_{I^*}$ is an orthogonal matrix. Let $\tilde{\boldsymbol{\mu}}_{I^*} = \boldsymbol{\Omega}_{I^*}\boldsymbol{\mu}_{I^*}$ and $\tilde{\boldsymbol{U}}_{I^*} = \boldsymbol{\Omega}_{I^*}\boldsymbol{U}_{I^*} - \tilde{\boldsymbol{\mu}}_{I^*}$, then $\tilde{\boldsymbol{U}}_{I^*} \sim N\big(0, \text{diag}(\boldsymbol{\lambda}_{I^*})\big)$. Note that $\sum_{i\in I^*} U_i^2 = \sum_{i\in I^*}(\tilde{U}_i + \tilde{\mu}_i)^2$,

$$\mathbb{P}\big(Q(I^*) \leq h(p, L_{\min}, L_{\max}, \alpha)\big) = \mathbb{P}\Big(\sum_{i\in I^*}(\tilde{U}_i + \tilde{\mu}_i)^2 - ||\boldsymbol{\lambda}_{I^*}||_1 \leq \sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2 h(p, L_{\min}, L_{\max}, \alpha)\Big).$$

It is obvious that $\sum_{i\in I^*}\mu_i^2 = \sum_{i\in I^*}\tilde{\mu}_i^2 = ||\boldsymbol{\mu}_{I^*}||_2^2$. Assume $C_0 = ||\boldsymbol{\mu}_{I^*} i||_2^2 - \sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2 h(p, L_{\min}, L_{\max}, \alpha)$, then

$$\mathbb{P}\big(Q(I^*) \leq h(p, L_{\min}, L_{\max}, \alpha)\big) \leq \mathbb{P}\Big(\sum_{i\in I^*}\tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I^*}||_1 \leq -\frac{C_0}{2}\Big) + \mathbb{P}\Big(\sum_{i\in I^*}\tilde{\mu}_i\tilde{U}_i \leq -\frac{C_0}{4}\Big) \triangleq A_1(I^*) + A_2(I^*).$$

By condition (D) and Theorem 1, for p sufficiently large,

$$C_0 = ||\boldsymbol{\mu}_{I^*}||_2^2\Big\{1 - \frac{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2 h(p, L_{\min}, L_{\max}, \alpha)}{||\boldsymbol{\mu}_{I^*}||_2^2}\Big\} \geq \frac{\epsilon_0}{4}||\boldsymbol{\mu}_{I^*}||_2^2 \geq \frac{\epsilon_0}{2}||\boldsymbol{\lambda}_{I^*}||_2\sqrt{\log p}.$$

Hence, by Lemma 3, for $A_1(I^*)$, we have

$$\begin{aligned} A_1(I^*) &= \mathbb{P}\Big(\sum_{i\in I^*}\lambda_i\Big(\frac{\tilde{U}_i^2}{\lambda_i} - 1\Big) \leq -\frac{C_0}{2}\Big) \\ &\leq \mathbb{P}\Big(\sum_{i\in I^*}\lambda_i\Big(\frac{\tilde{U}_i^2}{\lambda_i} - 1\Big) \leq -\frac{\epsilon_0}{4}||\boldsymbol{\lambda}_{I^*}||_2\sqrt{\log p}\Big) \leq \exp\Big(-\frac{\epsilon_0^2}{16}\log p\Big). \end{aligned} \tag{5}$$

For $A_2(I^*)$, note that $\text{Var}\left(\sum_{i\in I^*}\tilde{\mu}_i\tilde{U}_i\right) = \sum_{i\in I^*}\lambda_i\tilde{\mu}_i^2 \leq ||\boldsymbol{\lambda}_{I^*}||_\infty \sum_{i\in I^*}\tilde{\mu}_i^2 = ||\boldsymbol{\lambda}_{I^*}||_\infty ||\boldsymbol{\mu}_{I^*}||_2^2$ and

$$\frac{C_0}{\sqrt{\text{Var}\left(\sum_{i\in I^*}\tilde{\mu}_i\tilde{U}_i\right)}} \geq \frac{\epsilon_0||\boldsymbol{\mu}_{I^*}||_2^2}{4\sqrt{||\boldsymbol{\lambda}_{I^*}||_\infty}||\boldsymbol{\mu}_{I^*}||_2} \geq \frac{\epsilon_0}{4\sqrt{K_0}}||\boldsymbol{\mu}_{I^*}||_2 \geq \frac{\epsilon_0\sqrt{\log p}}{\sqrt{8K_0}},$$

by Lemma 2, for p sufficient large,

$$A_2(I^*) = \mathbb{P}\Big(\frac{-\sum_{i\in I^*}\tilde{\mu}_i\tilde{U}_i}{\sqrt{\text{Var}\left(\sum_{i\in I^*}\tilde{\mu}_i\tilde{U}_i\right)}} \geq \frac{C_0}{4\sqrt{\text{Var}\left(\sum_{i\in I^*}\tilde{\mu}_i\tilde{U}_i\right)}}\Big) \leq \exp\Big(-\frac{\epsilon_0^2\log p}{128K_0}\Big). \tag{6}$$

Hence, by (5) and (6), let $C = \min\{1/16, 1/128K_0\}$, we complete the proof. □

For any region I_1 and I_2, define the Jaccard index between I_1 and I_2 as $J(I_1, I_2) = \frac{|I_1 \cap I_2|}{|I_1 \cup I_2|}$. It is obvious that $0 \leq J(I_1, I_2) \leq 1$. We define region I^* is consistently detected if for some $\eta_p = o(1)$, there exists $\hat{I} \in \hat{\mathcal{I}}$ such that

$$\mathbb{P}\big(J(\hat{I}, I^*) \geq 1 - \eta_p\big) \to 1.$$

Next we show that the proposed Q-SCAN could consistently detect the true signal region.

Theorem 4 *Assume condition (A)-(D) and the following condition (E) hold,*

(E) $\inf_{I \subsetneq I^*} \frac{\log(||\boldsymbol{\mu}_{I^*}||_2^2) - \log(||\boldsymbol{\mu}_I||_2^2)}{\log(||\boldsymbol{\lambda}_{I^*}||_2) - \log(||\boldsymbol{\lambda}_I||_2)} > 1,$

then

$$\mathbb{P}\big(J(\hat{I}, I^*) \geq 1 - \eta_p\big) \to 1,$$

where $\big\{\frac{\log(L_{\max})}{\log(p)}\big\}^{\frac{1}{4}} \ll \eta_p \ll 1.$

Proof Let $\mathcal{B}_1 = \{I | L_{\min} \leq |I| \leq L_{\max},\ I \cap I^* = \emptyset\}$ and $\mathcal{B}_2 = \{I | L_{\min} \leq |I| \leq L_{\max},\ I \cap I^* \neq \emptyset\}$,

$$\begin{aligned}
\mathbb{P}\big(J(\hat{I}, I^*) < 1 - \eta_p\big) &= \mathbb{P}\big(J(\hat{I}, I^*) < 1 - \eta_p,\ \hat{I} \cap I^* = \emptyset\big) + \mathbb{P}\big(J(\hat{I}, I^*) < 1 - \eta_p,\ \hat{I} \cap I^* \neq \emptyset\big) \\
&\leq \mathbb{P}\big(\sup_{I \in \mathcal{B}_1} Q(I) > Q(I^*)\big) + \mathbb{P}\big(J(\hat{I}, I^*) < 1 - \eta_p,\ \hat{I} \in \mathcal{B}_2\big) \\
&\leq \mathbb{P}\big(\sup_{I \in \mathcal{B}_1} Q(I) > (1 + \frac{\epsilon_0}{4})\sqrt{2\log(p)}\big) + \mathbb{P}\big(Q(I^*) \leq (1 + \frac{\epsilon_0}{4})\sqrt{2\log(p)}\big) \\
&\quad + \mathbb{P}\big(J(\hat{I}, I^*) < 1 - \eta_p,\ \hat{I} \in \mathcal{B}_2\big).
\end{aligned}$$

For the first part,

$$\mathbb{P}\big(\sup_{I \in \mathcal{B}_1} Q(I) > (1 + \frac{\epsilon_0}{4})\sqrt{2\log(p)}\big) \leq \sum_{I \in \mathcal{B}_1} P\big(Q(I) > (1 + \frac{\epsilon_0}{4})\sqrt{2\log(p)}\big).$$

Note that, for p sufficiently large, $2||\boldsymbol{\lambda}_I||_2\sqrt{(1 + \frac{\epsilon_0}{8})\log p} + 2||\boldsymbol{\lambda}_I||_\infty(1 + \frac{\epsilon_0}{8})\log p \leq 2(1 + \frac{\epsilon_0}{4})||\boldsymbol{\lambda}_I||_2\sqrt{\log p}$, then by Lemma 3, for any $I \in \mathcal{B}_1$, $\mathbb{P}(I) \leq \exp(-(1 + \frac{\epsilon_0}{8}))$. Hence, by condition (B),

$$\mathbb{P}\big(\sup_{I \cap I^* \neq \emptyset} Q(I) > (1 + \frac{\epsilon}{4})\sqrt{2\log(p)}\big) \leq \exp\{-(1 + \frac{\epsilon_0}{8})\log p + \log p + \log L_{\max}\} \leq p^{-C_1\epsilon_0^2}. \tag{7}$$

where $C_1 = \frac{1}{16\epsilon_0}$. For the second part, by the same approach discussed in Theorem 3, there exists $C_2 > 0$ such that

$$\mathbb{P}\big(Q(I^*) \leq (1 + \frac{\epsilon_0}{4})\sqrt{2\log(p)}\big) \leq 2p^{-C_2\epsilon_0^2}. \tag{8}$$

We next consider the third part. Let $\tilde{U}_i = U_i - \mu_i$, for any $I \in \mathcal{B}_2$, we have

$$
\begin{aligned}
Q(I) - Q(I^*) &= \frac{\sum_{i\in I} U_i^2 - ||\boldsymbol{\lambda}_I||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} - \frac{\sum_{i\in I^*} U_i^2 - ||\boldsymbol{\lambda}_{I^*}||_1}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} \\
&= \frac{\sum_{i\in I} (\tilde{U}_i + \mu_i)^2 - ||\boldsymbol{\lambda}_I||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} - \frac{\sum_{i\in I^*} (\tilde{U}_i + \mu_i)^2 - ||\boldsymbol{\lambda}_{I^*}||_1}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} \\
&= \frac{\sum_{i\in I\setminus I^*} \tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I\setminus I^*}||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} + (1 - \frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2})\frac{\sum_{i\in I\cap I^*} \tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} \\
&\quad + \frac{2\sum_{i\in I\cap I^*} \mu_i \tilde{U}_i}{\sqrt{2}||\boldsymbol{\lambda}_I||_2}(1 - \frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2}) - \frac{2\sum_{i\in I^*\setminus I} \mu_i \tilde{U}_i}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} - \frac{\sum_{i\in I^*\setminus I} \tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I^*\setminus I}||_1}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} \\
&\quad + \frac{||\boldsymbol{\mu}_{I\cap I^*}||_2^2}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} - \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} \qquad (9)
\end{aligned}
$$

Assume $\delta = \inf_{I \subsetneq I^*} \frac{\log(||\boldsymbol{\mu}_{I^*}||_2^2) - \log(||\boldsymbol{\mu}_I||_2^2)}{\log(||\boldsymbol{\lambda}_{I^*}||_2) - \log(||\boldsymbol{\lambda}_I||_2)} - 1$, then by condition (E), for any $I \in \mathcal{B}_2$,

$$
\frac{||\boldsymbol{\mu}_{I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2} - \frac{||\boldsymbol{\mu}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2} \geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2}(1 - \frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_I||_2 ||\boldsymbol{\lambda}_{I^*}||_2^{\delta}}).
$$

Then by (9), for any $I \in \mathcal{B}_2$,

$$
\mathbb{P}\big(Q(I) - Q(I^*) > 0\big) \leq \sum_{i=1}^{5} \mathbb{P}\big(A_i(I)\big), \qquad (10)
$$

where

$$
\begin{aligned}
A_1(I) &= \Big\{ \frac{\sum_{i\in I\setminus I^*} \tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I\setminus I^*}||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} > \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{5\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2}(1 - \frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_I||_2 ||\boldsymbol{\lambda}_{I^*}||_2^{\delta}}) \Big\}, \\
A_2(I) &= \Big\{ (1 - \frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2})\frac{\sum_{i\in I\cap I^*} \tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_1}{\sqrt{2}||\boldsymbol{\lambda}_I||_2} > \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{5\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2}(1 - \frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_I||_2 ||\boldsymbol{\lambda}_{I^*}||_2^{\delta}}) \Big\}, \\
A_3(I) &= \Big\{ -\frac{\sum_{i\in I^*\setminus I} \tilde{U}_i^2 - ||\boldsymbol{\lambda}_{I^*\setminus I}||_1}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} > \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{5\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2}(1 - \frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_I||_2 ||\boldsymbol{\lambda}_{I^*}||_2^{\delta}}) \Big\}, \\
A_4(I) &= \Big\{ \frac{2\sum_{i\in I\cap I^*} \mu_i \tilde{U}_i}{\sqrt{2}||\boldsymbol{\lambda}_I||_2}(1 - \frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2}) > \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{5\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2}(1 - \frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_I||_2 ||\boldsymbol{\lambda}_{I^*}||_2^{\delta}}) \Big\}, \\
A_5(I) &= \Big\{ -\frac{2\sum_{i\in I^*\setminus I} \mu_i \tilde{U}_i}{\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2} > \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{5\sqrt{2}||\boldsymbol{\lambda}_{I^*}||_2}(1 - \frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_I||_2 ||\boldsymbol{\lambda}_{I^*}||_2^{\delta}}) \Big\}.
\end{aligned}
$$

Note that $J(\hat{I}, I^*) = \frac{|\hat{I}\cap I^*|}{|\hat{I}\cup I^*|} = \frac{|I^*| - |I^*\setminus\hat{I}|}{|I^*| + |\hat{I}\setminus I^*|}$, then $J(\hat{I}, I^*) < 1 - \eta_p$ implies that $|\hat{I}\setminus I^*| > \frac{\eta_p}{2}|I^*|$ or $|I^*\setminus\hat{I}| > \frac{\eta_p}{2}|I^*|$. Let $\mathcal{B}_{21} = \{I | I\cap I^* \neq \emptyset,\ |I\setminus I^*| > \frac{\eta_p}{2}|I^*|,\ L_{\min} \leq |I| \leq L_{\max}\}$ and $\mathcal{B}_{22} = \{I | I\cap I^* \neq \emptyset,\ |I^*\setminus I| > \frac{\eta_p}{2}|I^*|,\ L_{\min} \leq |I| \leq L_{\max}\}$, we get $\big\{I | J(I, I^*) < 1 - \eta_p,\ I \in \mathcal{B}_2\big\} \subset \mathcal{B}_{21} \cup \mathcal{B}_{22}$, and hence by (10),

$$
\mathbb{P}\big(J(\hat{I}, I^*) < 1 - \eta_p,\ \hat{I}\cap I^* \neq \emptyset\big) \leq \sum_{I\in\mathcal{B}_{21}\cup\mathcal{B}_{22}} \mathbb{P}\big(A_1(I)\big) + \mathbb{P}\big(A_3(I)\big) + \mathbb{P}\big(A_5(I)\big) + \sum_{I\in\mathcal{B}_2} \mathbb{P}\big(A_2(I)\big) + \mathbb{P}\big(A_4(I)\big) \qquad (11)
$$

We first consider $A_1(I)$. When $I \in \mathcal{B}_{21}$, we have $|I\backslash I^*| \geq \frac{\eta_p|I^*|}{2}$, and hence

$$\frac{|I\backslash I^*|}{|I|} = \frac{|I\backslash I^*|}{|I\backslash I^*| + |I \cap I^*|} \geq \frac{\frac{\eta_p}{2}|I^*|}{\frac{\eta_p}{2}|I^*| + |I^*|} = \frac{\eta_p}{2+\eta_p} \geq \frac{\eta_p}{3}.$$

Recall that $c > 0$ is the lower bound of σ_i^2, $\frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2^2} \geq \frac{||\boldsymbol{\lambda}_{I\backslash I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2^2} \geq \frac{c^2|I\backslash I^*|}{K_0^2|I|} \geq \frac{c^2\eta_p}{3K_0^2}$ and $\frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2} \geq \frac{||\boldsymbol{\lambda}_{I\backslash I^*}||_2^2}{||\boldsymbol{\lambda}_I^*||_2^2} \geq \frac{c^2|I\backslash I^*|}{K_0^2|I^*|} \geq \frac{c^2\eta_p}{2K_0^2}$. It follows that

$$\frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_{I^*}||_2 + ||\boldsymbol{\lambda}_I||_2)} \geq \frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{\frac{1}{2}||\boldsymbol{\lambda}_I||_2^2 + \frac{3}{2}||\boldsymbol{\lambda}_{I^*}||_2^2} \geq \frac{2c^2\eta_p}{9K_0^2},$$

and hence

$$\begin{aligned}
\frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}} &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2)}{||\boldsymbol{\lambda}_{I^*}||_2} \\
&= \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2)}{(||\boldsymbol{\lambda}_{I^*}||_2 + ||\boldsymbol{\lambda}_{I\cap I^*}||_2)||\boldsymbol{\lambda}_{I^*}||_2} \\
&\geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2)}{(||\boldsymbol{\lambda}_I||_2 + ||\boldsymbol{\lambda}_{I^*}||_2)||\boldsymbol{\lambda}_{I^*}||_2} \\
&\geq \frac{2c^2\eta_p||\boldsymbol{\mu}_{I^*}||_2^2}{9K_0^2}.
\end{aligned}$$

On the other hand,

$$\frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{(||\boldsymbol{\lambda}_I||_2 + ||\boldsymbol{\lambda}_{I^*}||_2)||\boldsymbol{\lambda}_{I^*}||_2} \geq \frac{||\boldsymbol{\lambda}_{I^*\backslash I}||_2}{||\boldsymbol{\lambda}_{I^*}||_2} \times \sqrt{\frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{(||\boldsymbol{\lambda}_I||_2 + ||\boldsymbol{\lambda}_{I^*}||_2)^2}} \geq \frac{c\eta_p||\boldsymbol{\lambda}_{I^*\backslash I}||_2}{4K_0||\boldsymbol{\lambda}_{I^*}||_2}.$$

It follows that

$$\frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}} \geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2} \times \frac{c\eta_p||\boldsymbol{\lambda}_{I^*\backslash I}||_2}{4K_0}.$$

Combined the two parts,

$$\frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{5||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}} \geq \max\{\frac{4c^2\eta_p||\boldsymbol{\lambda}_{I^*}||_2\sqrt{\log p}}{45K_0^2}, \frac{c^2\eta_p\sqrt{\log p}||\boldsymbol{\lambda}_{I^*\backslash I}||_2}{10K_0^2}\}.$$

and hence $2||\boldsymbol{\lambda}_{I\backslash I^*}||_2\sqrt{\frac{c^4}{1600K_0^4}\eta_p^2\log p} + 2||\boldsymbol{\lambda}_{I\backslash I^*}||_\infty(\frac{c^4}{1600K_0^4}\eta_p^2\log p) \leq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{5||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}}$. Then by Lemma 3, we have $\mathbb{P}\big(A_1(I)\big) \leq \exp(-\frac{c^4}{1600K_0^4}\eta_p^2\log p)$. By the definition of η_p, there exists $C_3 > 0$, such that

$$\sum_{I\in\mathcal{B}_{21}} \mathbb{P}\big(A_1(I)\big) \leq 2L_{\max}^2\exp(-\frac{c^4}{1600K_0^4}\eta_p^2\log p) \leq p^{-C_3\eta_p^2}. \tag{12}$$

When $I \in \mathcal{B}_{22}$, we have $|I^*\backslash I| \geq \frac{\eta_p}{2}|I^*|$, and hence

$$\frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2} \leq \frac{||\boldsymbol{\lambda}_{I^*}||_2^2 - ||\boldsymbol{\lambda}_{I^*\backslash I}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2} \leq 1 - \frac{c^2|I^*\backslash I|}{K_0^2|I^*|} \leq 1 - \frac{c^2\eta_p}{2K_0^2},$$

and $1-\left(\frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2}{||\boldsymbol{\lambda}_{I^*}||_2}\right)^{\delta} \geq 1-\left(1-\frac{c^2\eta_p}{2K_0^2}\right)^{\frac{\delta}{2}} \geq \frac{\delta c^2\eta_p}{4K_0^2}$. It follows that

$$\begin{aligned}
\frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}} &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{\delta})}{||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}} \\
&\geq \frac{\delta c^2\eta_p||\boldsymbol{\mu}_{I^*}||_2^2||\boldsymbol{\lambda}_I||_2}{2K_0^2||\boldsymbol{\lambda}_{I^*}||_2} \\
&\geq \frac{\delta c^2\eta_p\sqrt{\log(p)}||\boldsymbol{\lambda}_I||_2}{K_0^2}.
\end{aligned}$$

Note that $||\boldsymbol{\lambda}_I||_2\sqrt{\log p} \geq c\sqrt{|I|\log(p)} \geq c\sqrt{L_{\min}\log(p)}$, by condition (B), $\log p = o(||\boldsymbol{\lambda}_I||_2\sqrt{\log p})$, and thus for sufficiently large p,

$$2||\boldsymbol{\lambda}_{I\setminus I^*}||_2\sqrt{\frac{\delta^2c^4}{400K_0^4}\eta_p^2\log p}+2||\boldsymbol{\lambda}_{I\setminus I^*}||_\infty\frac{\delta^2c^4}{400K_0^4}\eta_p^2\log p \leq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{5||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}}.$$

Then by Lemma 3, $\mathbb{P}\big(A_1(I)\big) \leq \exp(-\frac{\delta^2c^4}{400K_0^4}\eta_p^2\log p)$, and thus there exists $C_4>0$ such that

$$\sum_{I\in\mathcal{B}_{22}}\mathbb{P}\big(A_1(I)\big) \leq 2L_{\max}^2\exp(-\frac{\delta^2c^4}{400K_0^4}\eta_p^2\log p) \leq p^{-C_4\eta_p^2}. \tag{13}$$

Next we consider $A_2(I)$, when $||\boldsymbol{\lambda}_{I^*}||_2 < ||\boldsymbol{\lambda}_I||_2$,

$$\frac{||\boldsymbol{\mu}_{I^*}||_2^2\big(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}\big)}{(||\boldsymbol{\lambda}_I||_2-||\boldsymbol{\lambda}_{I^*}||_2)} \geq ||\boldsymbol{\mu}_{I^*}||_2^2 \geq 2||\boldsymbol{\lambda}_{I^*}||_2\sqrt{\log(p)},$$

by Lemma 3, $\mathbb{P}\big(A_2(I)\big) \leq \frac{1}{p}$. When $||\boldsymbol{\lambda}_{I^*}||_2 \geq ||\boldsymbol{\lambda}_I||_2$, let $x = ||\boldsymbol{\lambda}_{I\cap I^*}||_2 \backslash ||\boldsymbol{\lambda}_{I^*}||_2$, we have

$$\frac{||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{\delta})}{(||\boldsymbol{\lambda}_{I^*}||_2-||\boldsymbol{\lambda}_{I\cap I^*}||_2)||\boldsymbol{\lambda}_{I^*}||_2^{\delta}} = \frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2}\Big(\frac{1-x^{\delta}}{1-x}\Big) \geq \frac{\delta||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2}.$$

It follows that

$$\begin{aligned}
&\frac{||\boldsymbol{\mu}_{I^*}||_2^2\big(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}\big)}{||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}} \times \frac{||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2}{||\boldsymbol{\lambda}_{I^*}||_2-||\boldsymbol{\lambda}_I||_2} \\
= &\frac{||\boldsymbol{\mu}_{I^*}||_2^2\big(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}\big)}{(||\boldsymbol{\lambda}_{I^*}||_2-||\boldsymbol{\lambda}_I||_2)||\boldsymbol{\lambda}_{I^*}||_2^{\delta}} \\
\geq &\frac{||\boldsymbol{\mu}_{I^*}||_2^2||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{\delta})}{(||\boldsymbol{\lambda}_{I^*}||_2-||\boldsymbol{\lambda}_{I\cap I^*}||_2)||\boldsymbol{\lambda}_{I^*}||_2^{\delta}} \\
\geq &\frac{\delta||\boldsymbol{\mu}_{I^*}||_2^2||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2} \geq 2\delta||\boldsymbol{\lambda}_I||_2\sqrt{\log(p)}.
\end{aligned}$$

Note that $2||\boldsymbol{\lambda}_{I\cap I^*}||_2\sqrt{\frac{\delta^2}{4}\log p}+2||\boldsymbol{\lambda}_{I\cap I^*}||_\infty(\frac{\delta^2}{4}\log p) \leq 2\delta||\boldsymbol{\lambda}_I||_2\sqrt{\log(p)}$, by Lemma 3, $\mathbb{P}\big(A_2(I)\big) \leq \exp(-\frac{\delta^2}{4}\log p)$. Hence, there exists $C_5>0$,

$$\sum_{I\in\mathcal{B}_2}\mathbb{P}\big(A_2(I)\big) = \sum_{\substack{I\in\mathcal{B}_2\\ ||\boldsymbol{\lambda}_{I^*}||_2\geq||\boldsymbol{\lambda}_I||_2}}\mathbb{P}\big(A_2(I)\big)+\sum_{\substack{I\in\mathcal{B}_2\\ ||\boldsymbol{\lambda}_{I^*}||_2<||\boldsymbol{\lambda}_I||_2}}\mathbb{P}\big(A_2(I)\big) \leq 2L_{\max}^2p^{-\frac{\delta^2}{4}}+2L_{\max}^2p^{-1} \leq p^{-C_5^2}. \tag{14}$$

Then we consider $A_3(I)$, when $I \in \mathcal{B}_{21}$, by the former discussion, $\frac{||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_{I^*}||_2 + ||\boldsymbol{\lambda}_I||_2)} \geq \frac{c^2\eta_p}{6K_0^2}$. It follows that

$$\frac{||\boldsymbol{\mu}_{I^*}||_2^2\big(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}\big)}{||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta} \geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2^2 - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^2)}{||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_I||_2 + ||\boldsymbol{\lambda}_{I^*}||_2)} \geq \frac{\eta_p||\boldsymbol{\mu}_{I^*}||_2^2}{6K_0^2},$$

and therefore $2||\boldsymbol{\lambda}_{I^*\setminus I}||_2\sqrt{\frac{1}{900K_0^2}\eta_p^2\log p} \leq \frac{||\boldsymbol{\mu}_{I^*}||_2^2\big(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}\big)}{5||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta}$. By Lemma 3, $\mathbb{P}\big(A_3(I)\big) \leq \exp(-\frac{1}{900K_0^2}\eta_p^2\log p)$. Hence, there exists $C_6 > 0$,

$$\sum_{I\in\mathcal{B}_{21}} \mathbb{P}\big(A_3(I)\big) \leq 2L_{\max}^2 \exp(-\frac{1}{3600K_0^2}\eta_p^2\log p) \leq p^{-C_6\eta_p^2}. \tag{15}$$

When $I \in \mathcal{B}_{22}$, by the former discussion, we have $\frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2} \leq 1 - \frac{c^2\eta_p}{2K_0^2}$. It follows that

$$\begin{aligned}\frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta} &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^\delta)}{||\boldsymbol{\lambda}_{I^*}||_2^\delta} \\ &\geq ||\boldsymbol{\mu}_{I^*}||_2^2\big\{1 - (1 - \frac{c^2\eta_p}{2K_0^2})^{\frac{\delta}{2}}\big\} \geq \frac{c^2\delta\eta_p||\boldsymbol{\mu}_{I^*}||_2}{4K_0^2}.\end{aligned}$$

Therefore $2||\boldsymbol{\lambda}_{I^*\setminus I}||_2\sqrt{\frac{c^4\delta^2}{400K_0^4}\eta_p^2\log p} \leq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{5||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta}$. By Lemma 3, $\mathbb{P}\big(A_3(I)\big) \leq \exp(-\frac{c^4\delta^2}{400K_0^4}\eta_p^2\log p)$. Hence, there exists $C_7 > 0$,

$$\sum_{I\in\mathcal{B}_{22}} \mathbb{P}\big(A_3(I)\big) \leq \frac{2}{L_{\max}} 2L_{\max}^2 \exp(-\frac{c^4\delta^2}{400K_0^4}\eta_p^2\log p) \leq p^{-C_7\eta_p^2}. \tag{16}$$

For $A_4(I)$,

$$\mathrm{Var}\Big\{\frac{\sqrt{2}\sum_{i\in I\cap I^*}\mu_i\tilde{U}_i}{||\boldsymbol{\lambda}_I||_2}\big(1 - \frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2}\big)\Big\} = \frac{2\boldsymbol{\mu}_{I\cap I^*}\boldsymbol{\Sigma}_{I\cap I^*}\boldsymbol{\mu}_{I\cap I^*}(||\boldsymbol{\lambda}_{I^*}||_2 - ||\boldsymbol{\lambda}_I||_2)^2}{||\boldsymbol{\lambda}_I||_2^2||\boldsymbol{\lambda}_{I^*}||_2^2}$$

By condition (D) and (E),

$$\begin{aligned}\kappa &= \frac{\frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta}}}{\frac{2\sqrt{\boldsymbol{\mu}_{I\cap I^*}^T\boldsymbol{\Sigma}_{I\cap I^*}\boldsymbol{\mu}_{I\cap I^*}}\big|||\boldsymbol{\lambda}_{I^*}||_2 - ||\boldsymbol{\lambda}_I||_2\big|}{||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2}} \\ &= \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{2\sqrt{\boldsymbol{\mu}_{I\cap I^*}^T\boldsymbol{\Sigma}_{I\cap I^*}\boldsymbol{\mu}_{I\cap I^*}}||\boldsymbol{\lambda}_{I^*}||_2^\delta\big|||\boldsymbol{\lambda}_{I^*}||_2 - ||\boldsymbol{\lambda}_I||_2\big|} \\ &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2^2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{2\sqrt{K_0}||\boldsymbol{\mu}_{I\cap I^*}||_2\big|||\boldsymbol{\lambda}_{I^*}||_2^{1+\delta} - ||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta\big|} \\ &\geq \frac{\big(\log(p)\big)^{\frac{1}{4}}||\boldsymbol{\lambda}_{I^*}||_2^{1-\frac{\delta}{2}}(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^\delta - ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})}{\sqrt{K_0}\big|||\boldsymbol{\lambda}_{I^*}||_2 - ||\boldsymbol{\lambda}_I||_2\big| \times ||\boldsymbol{\lambda}_{I\cap I^*}||_2^{\frac{1+\delta}{2}}}\end{aligned}$$

When $||\boldsymbol{\lambda}_I||_2 \geq ||\boldsymbol{\lambda}_{I^*}||_2$,

$$\kappa \geq \frac{\big(\log(p)\big)^{\frac{1}{4}}||\boldsymbol{\lambda}_{I^*}||_2^{1+\frac{\delta}{2}}(||\boldsymbol{\lambda}_I||_2 - ||\boldsymbol{\lambda}_{I^*}||_2)}{\sqrt{K_0}(||\boldsymbol{\lambda}_I||_2 - ||\boldsymbol{\lambda}_{I^*}||_2)||\boldsymbol{\lambda}_{I^*}||^{\frac{1}{2}+\frac{\delta}{2}}} = \frac{\big(\log(p)\big)^{\frac{1}{4}}||\boldsymbol{\lambda}_{I^*}||_2^{\frac{1}{2}}}{\sqrt{K_0}} \geq \frac{\sqrt{c}|I^*|^{\frac{1}{4}}\big(\log(p)\big)^{\frac{1}{4}}}{\sqrt{K_0}} \geq \frac{\sqrt{\log(p)}}{\sqrt{K_0}}.$$

By Lemma 2,

$$\mathbb{P}\big(A_4(I)\big) \leq \mathbb{P}\left(\frac{\frac{\sqrt{2}\sum_{i\in I\cap I^*}\mu_i\tilde{U}_i}{||\boldsymbol{\lambda}_I||_2}(1-\frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2})}{\sqrt{\frac{4\boldsymbol{\mu}_{I\cap I^*}^T\boldsymbol{\Sigma}_{I\cap I^*}\boldsymbol{\mu}_{I\cap I^*}(||\boldsymbol{\lambda}_{I^*}||_2-||\boldsymbol{\lambda}_I||_2^2)^2}{||\boldsymbol{\lambda}_I||_2^2||\boldsymbol{\lambda}_{I^*}||_2^2}}} > \frac{\sqrt{\log(p)}}{5\sqrt{K_0}}\right) \leq \sqrt{\frac{25K_0}{\log p}}\exp(-\frac{\log p}{50K_0}).$$

When $||\boldsymbol{\lambda}_{I^*}||_2 \geq ||\boldsymbol{\lambda}_I||_2$,

$$\kappa \geq \frac{\big(\log(p)\big)^{\frac{1}{4}}||\boldsymbol{\lambda}_{I^*}||_2^{1-\frac{\delta}{2}}||\boldsymbol{\lambda}_{I^*}||_2^{1-\frac{\delta}{2}}||\boldsymbol{\lambda}_I||_2^{\frac{1}{2}-\frac{\delta}{2}}(||\boldsymbol{\lambda}_{I^*}||_2^{\delta} - ||\boldsymbol{\lambda}_I||_2^{\delta})}{\sqrt{K_0}(||\boldsymbol{\lambda}_{I^*}||_2 - ||\boldsymbol{\lambda}_I||_2)} \geq \frac{\delta\big(\log(p)\big)^{\frac{1}{4}}||\boldsymbol{\lambda}_I||_2^{\frac{1}{2}-\frac{\delta}{2}}||\boldsymbol{\lambda}_{I^*}||_2^{\frac{\delta}{2}}}{\sqrt{K_0}}.$$

Obviously $\frac{\delta\big(\log(p)\big)^{\frac{1}{4}}||\boldsymbol{\lambda}_I||_2^{\frac{1}{2}-\frac{\delta}{2}}||\boldsymbol{\lambda}_{I^*}||_2^{\frac{\delta}{2}}}{\sqrt{K_0}} \geq \frac{\delta c\big(\log(p)\big)^{\frac{1}{2}-\frac{\delta}{4}}|I^*|^{\frac{\delta}{2}}}{\sqrt{K_0}}$, it follows that

$$\mathbb{P}\big(A_4(I)\big) \leq \mathbb{P}\left(\frac{\frac{\sqrt{2}\sum_{i\in I\cap I^*}\mu_i\tilde{U}_i}{||\boldsymbol{\lambda}_I||_2}(1-\frac{||\boldsymbol{\lambda}_I||_2}{||\boldsymbol{\lambda}_{I^*}||_2})}{\sqrt{\frac{4\boldsymbol{\mu}_{I\cap I^*}^T\boldsymbol{\Sigma}_{I\cap I^*}\boldsymbol{\mu}_{I\cap I^*}(||\boldsymbol{\lambda}_{I^*}||_2-||\boldsymbol{\lambda}_I||_2^2)^2}{||\boldsymbol{\lambda}_I||_2^2||\boldsymbol{\lambda}_{I^*}||_2^2}}} > \frac{\delta c\big(\log(p)\big)^{\frac{1}{2}-\frac{\delta}{4}}|I^*|^{\frac{\delta}{4}}}{5\sqrt{K_0}}\right).$$

Hence, by Lemma 2,

$$\mathbb{P}\big(A_4(I)\big) \leq \frac{5\sqrt{K_0}}{\delta c\big(\log(p)\big)^{\frac{1}{2}-\frac{\delta}{4}}|I^*|^{\frac{\delta}{4}}}\exp(-\frac{\delta^2c^2\big(\log(p)\big)^{1-\frac{\delta}{2}}|I^*|^{\frac{\delta}{2}}}{50K_0}).$$

Therefore, note that $||\boldsymbol{\lambda}_{I^*}||_2 \geq ||\boldsymbol{\lambda}_I||_2$ implies that $|I| \leq ||\boldsymbol{\lambda}_I||_2^2/c^2 \leq ||\boldsymbol{\lambda}_{I^*}||_2^2/c^2 \leq K_0^2|I^*|/c^2$, there exists $C_8 > 0$,

$$\begin{aligned}\sum_{I\in\mathcal{B}_2}\mathbb{P}\big(A_4(I)\big) &= \sum_{\substack{I\in\mathcal{B}_2\\||\boldsymbol{\lambda}_I||_2\geq||\boldsymbol{\lambda}_{I^*}||_2}}\mathbb{P}\big(A_4(I)\big) + \sum_{\substack{I\in\mathcal{B}_2\\||\boldsymbol{\lambda}_I||_2<||\boldsymbol{\lambda}_{I^*}||_2}}\mathbb{P}\big(A_4(I)\big)\\ &\leq 2L_{\max}^2\sqrt{\frac{25K_0}{\log p}}\exp(-\frac{\log p}{50K_0}) + \exp(-\frac{\delta^2c^2\big(\log(p)\big)^{1-\frac{\delta}{2}}|I^*|^{\frac{\delta}{2}}}{50K_0})\sum_{i=1}^{K_0^2|I^*|/c^2}\{|I^*|+2(i-1)\}\\ &= p^{-C_8\epsilon_0^2}. \qquad (17)\end{aligned}$$

Finally, we consider $A_5(I)$, by condition (E),

$$\mathrm{Var}\Big(\frac{\sum_{i\in I^*\setminus I}\mu_i\tilde{U}_i}{||\boldsymbol{\lambda}_{I^*}||_2}\Big) = \frac{\boldsymbol{\mu}_{I^*\setminus I}^T\boldsymbol{\Sigma}_{I^*\setminus I}\boldsymbol{\mu}_{I^*\setminus I}}{||\boldsymbol{\lambda}_{I^*}||_2^2} \leq \frac{K_0||\boldsymbol{\mu}_{I^*\setminus I}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2} \leq \frac{K_0||\boldsymbol{\mu}_{I^*}||_2^2||\boldsymbol{\lambda}_{I^*\setminus I}||_2^{1+\delta}}{||\boldsymbol{\lambda}_{I^*}||_2^{2+(1+\delta)}}.$$

It follows that

$$\frac{\frac{||\boldsymbol{\mu}_{I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2}\big(1-\frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta}}{||\boldsymbol{\lambda}_{I^*}||_2^{\delta}||\boldsymbol{\lambda}_I||_2}\big)}{\sqrt{\mathrm{Var}\left(\frac{\sum_{I^*\setminus I}\mu_i\tilde{U}_i}{||\boldsymbol{\lambda}_{I^*}||_2}\right)}} \geq \frac{||\boldsymbol{\mu}_{I^*}||_2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})||\boldsymbol{\lambda}_{I^*}||_2^{\frac{1-\delta}{2}}}{\sqrt{K_0}||\boldsymbol{\lambda}_{I^*\setminus I}||_2^{\frac{1+\delta}{2}}||\boldsymbol{\lambda}_I||_2}.$$

When $I\in\mathcal{B}_{21}$, by the former discussion, we have $\frac{||\boldsymbol{\lambda}_I||_2^2-||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_I||_2+||\boldsymbol{\lambda}_{I^*}||_2)}\geq\frac{c^2\eta_p}{6K_0^2}$ and thus

$$\begin{aligned}\frac{||\boldsymbol{\mu}_{I^*}||_2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})||\boldsymbol{\lambda}_{I^*}||_2^{\frac{1-\delta}{2}}}{\sqrt{K_0}||\boldsymbol{\lambda}_{I^*\setminus I}||_2^{\frac{1+\delta}{2}}||\boldsymbol{\lambda}_I||_2} &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2}{\sqrt{K_0}}\times\frac{(||\boldsymbol{\lambda}_I||_2-||\boldsymbol{\lambda}_{I\cap I^*}||_2)||\boldsymbol{\lambda}_{I^*}||_2^{\frac{1+\delta}{2}}}{||\boldsymbol{\lambda}_{I^*\setminus I}||_2^{\frac{1+\delta}{2}}||\boldsymbol{\lambda}_I||_2}\\ &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2}{\sqrt{K_0}}\times\frac{||\boldsymbol{\lambda}_I||_2^2-||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_I||_2(||\boldsymbol{\lambda}_{I^*}||_2+||\boldsymbol{\lambda}_I||_2)}\\ &\geq \frac{c^2\eta_p||\boldsymbol{\mu}_{I^*}||_2}{6K_0^{\frac{5}{2}}}\geq\frac{c^2\eta_p}{6K_0^{\frac{5}{2}}}||\boldsymbol{\lambda}_{I^*}||_2\sqrt{\log(p)}.\end{aligned}$$

Note that $\log p=o(||\boldsymbol{\lambda}_{I^*}||_2\sqrt{\log p})$, by Lemma 2, $\mathbb{P}\big(A_5(I)\big)\leq\exp(-\frac{c^4\eta_p^2}{1800K_0^5}\log p)$. Hence, there exists $C_9>0$,

$$\sum_{I\in\mathcal{B}_{21}}\mathbb{P}\big(A_5(I)\big)\leq 2L_{\max}^2\exp(-\frac{c^4\eta_p^2}{1800K_0^5}\log p)\leq p^{-C_9\epsilon_0^2}. \tag{18}$$

When $I\in\mathcal{B}_{22}$, by the former discussion, $\frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2}\leq 1-\frac{c^2\eta_p}{2K_0^2}$ and thus

$$\begin{aligned}\frac{||\boldsymbol{\mu}_{I^*}||_2(||\boldsymbol{\lambda}_I||_2||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{1+\delta})||\boldsymbol{\lambda}_{I^*}||_2^{\frac{1-\delta}{2}}}{\sqrt{K_0}||\boldsymbol{\lambda}_{I^*\setminus I}||_2^{\frac{1+\delta}{2}}||\boldsymbol{\lambda}_I||_2} &\geq \frac{||\boldsymbol{\mu}_{I^*}||_2||\boldsymbol{\lambda}_{I^*}||_2^{\frac{1-\delta}{2}}(||\boldsymbol{\lambda}_{I^*}||_2^{\delta}-||\boldsymbol{\lambda}_{I\cap I^*}||_2^{\delta})}{\sqrt{K_0}||\boldsymbol{\lambda}_{I^*\setminus I}||_2^{\frac{1+\delta}{2}}}\\ &\geq \frac{||\boldsymbol{\mu}_I^*||_2}{\sqrt{K_0}}\Big\{1-\big(\frac{||\boldsymbol{\lambda}_{I\cap I^*}||_2^2}{||\boldsymbol{\lambda}_{I^*}||_2^2}\big)^{\frac{\delta}{2}}\Big\}\\ &\geq \frac{||\boldsymbol{\mu}_I^*||_2}{\sqrt{K_0}}\Big\{1-\big(\frac{c^2\eta_p}{2K_0^2}\big)^{\frac{\delta}{2}}\Big\}\\ &\geq \frac{\eta_p\delta c^2||\boldsymbol{\mu}_{I^*}||_2}{4K_0^{\frac{5}{2}}}\geq\frac{\delta c^2\eta_p|I^*|^{\frac{1}{4}}\big(\log(p)\big)^{\frac{1}{4}}}{2\sqrt{2}K_0^{\frac{5}{2}}}.\end{aligned}$$

Note that $\log p=o(|I^*|)$, then by Lemma 2,

$$\mathbb{P}\big(A_5(I)\big)\leq\exp(-\frac{\delta^2c^4}{400K_0^5}\eta_p^2\log p).$$

and hence, there exists $C_{10}>0$,

$$\sum_{I\in\mathcal{B}_{22}}\mathbb{P}\big(A_5(I)\big)\leq 2L_{\max}^2\exp(-\frac{\delta^2c^4}{400K_0^5}\eta_p^2\log p)\leq p^{-C_{10}\epsilon_0^2}. \tag{19}$$

Therefore, assume $C=\max_{1\leq i\leq 10}C_i$, by (11) and (12)-(19),

$$\mathbb{P}\big(J(\hat{I},I^*)<1-\eta_p,\ \hat{I}\cap I^*\neq\emptyset\big)\leq p^{-C\epsilon_0^2}+p^{-C\eta_p^2}.$$

Combine with (7) and (8), we complete the proof. $\square$

2 Supplementary Figures

Figure S1: Power comparisons of Q-SCAN, M-SCAN and sliding window procedure using SKAT for multiple effects. We evaluated power via the signal region detection rate and the Jaccard index defined in the simulation section. Both criteria were calculated at the family-wise error rate at 0.05 level. The total sample size was 10,000. The number of variants in signal region p_0 was randomly select between 50 and 80. The sparsity index $\xi = 2/3$ and $s = p_0^{\xi}$ causal variants were selected randomly within each signal region. Each of causal variant has an effect size as a decreasing function of MAF, $\beta = -c\log_{10}(\mathrm{MAF})$. The value c varies from 0.11 to 0.31. From left to right, the plots consider settings in which the coefficients for the causal rare variants are 100% positive (0% negative), 80% positive (20% negative), and 50% positive (50% negative). We repeated the simulation for 1000 times. Q-SCAN and M-SCAN refer to the scan procedures using the scan statistics $\sum_{i\in I} U_i^2 - \mathbb{E}(\sum_{i\in I} U_i^2)/var(\sum_{i\in I} U_i^2)$ and $\big(\sum_{i\in I} U_i\big)^2/var\big(\sum_{i\in I} U_i\big)$, respectively. In both two scan procedures, "-1" and "-2" represents the range of the numbers of variants in searching windows $(L_{\min}, L_{\max}) = (40, 200)$ and $(L_{\min}, L_{\max}) = (50, 80)$, respectively. The sliding window length was set as 3 kb, 4 kb and 5 kb.

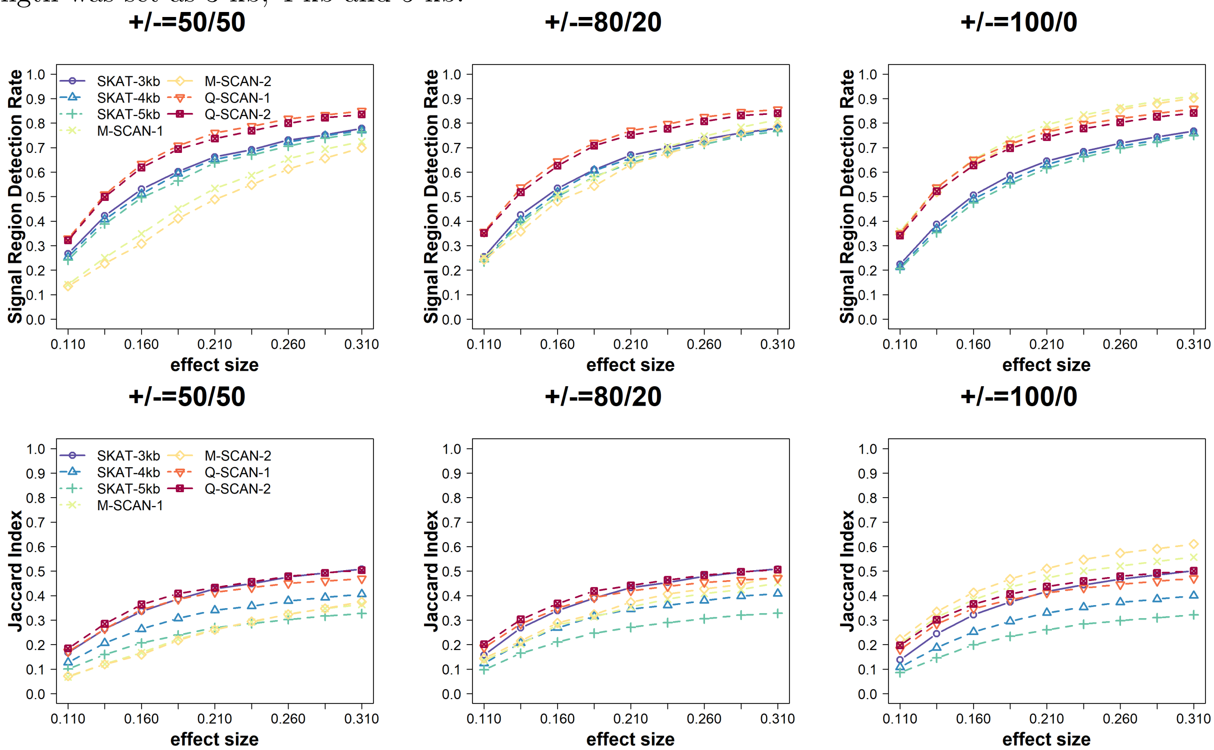


Figure S2: Power comparisons of Q-SCAN, M-SCAN and sliding window procedure using SKAT for multiple effects. We evaluated power via the signal region detection rate and the Jaccard index defined in the simulation section. Both criteria were calculated at the family-wise error rate at 0.05 level. The total sample size was 10,000. The number of variants in signal region p_0 was randomly select between 50 and 80. The sparsity index $\xi = 3/4$ and $s = p_0^{\xi}$ causal variants were selected randomly within each signal region. Each of causal variant has an effect size as a decreasing function of MAF, $\beta = -c\log_{10}(\text{MAF})$. The value c varies from 0.09 to 0.17. From left to right, the plots consider settings in which the coefficients for the causal rare variants are 100% positive (0% negative), 80% positive (20% negative), and 50% positive (50% negative). We repeated the simulation for 1000 times. Q-SCAN and M-SCAN refer to the scan procedures using the scan statistics $\sum_{i\in I} U_i^2 - \mathbb{E}(\sum_{i\in I} U_i^2)/var(\sum_{i\in I} U_i^2)$ and $\big(\sum_{i\in I} U_i\big)^2/var\big(\sum_{i\in I} U_i\big)$, respectively. In both two scan procedures, "-1" and "-2" represents the range of the numbers of variants in searching windows $(L_{\min}, L_{\max}) = (40, 200)$ and $(L_{\min}, L_{\max}) = (50, 80)$, respectively. The sliding window length was set as 3 kb, 4 kb and 5 kb.

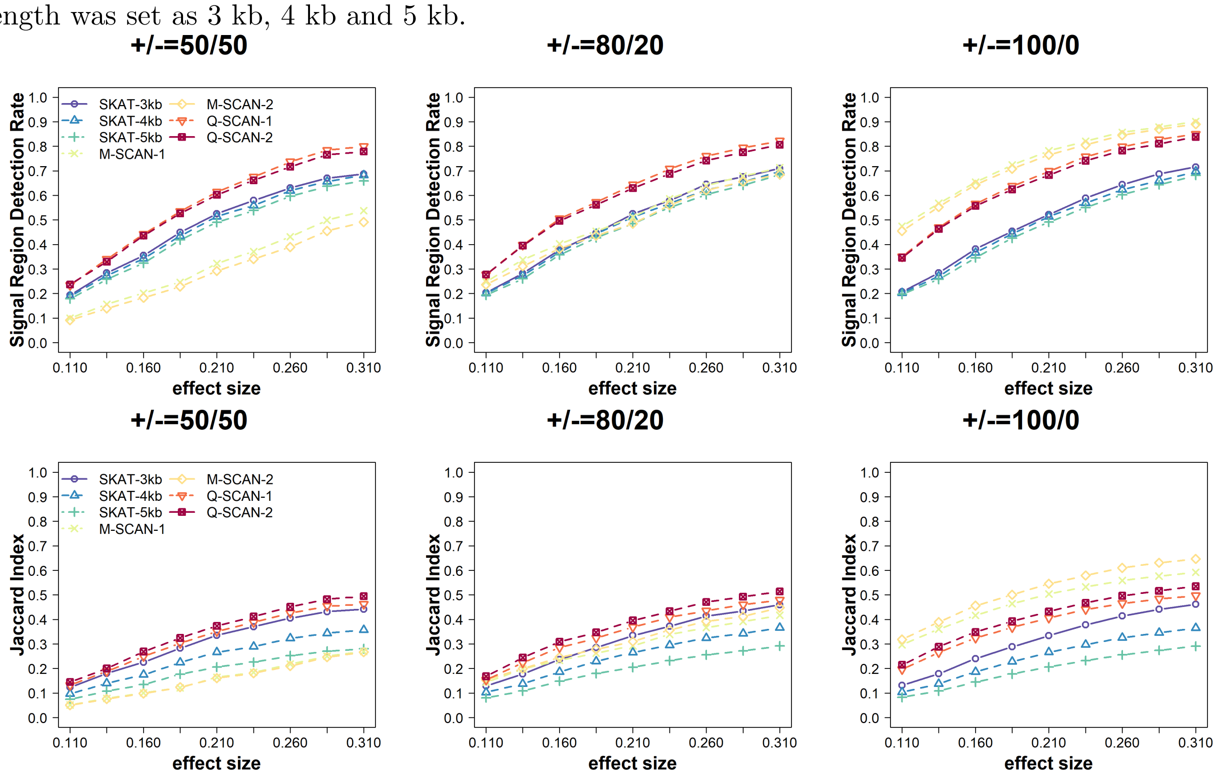
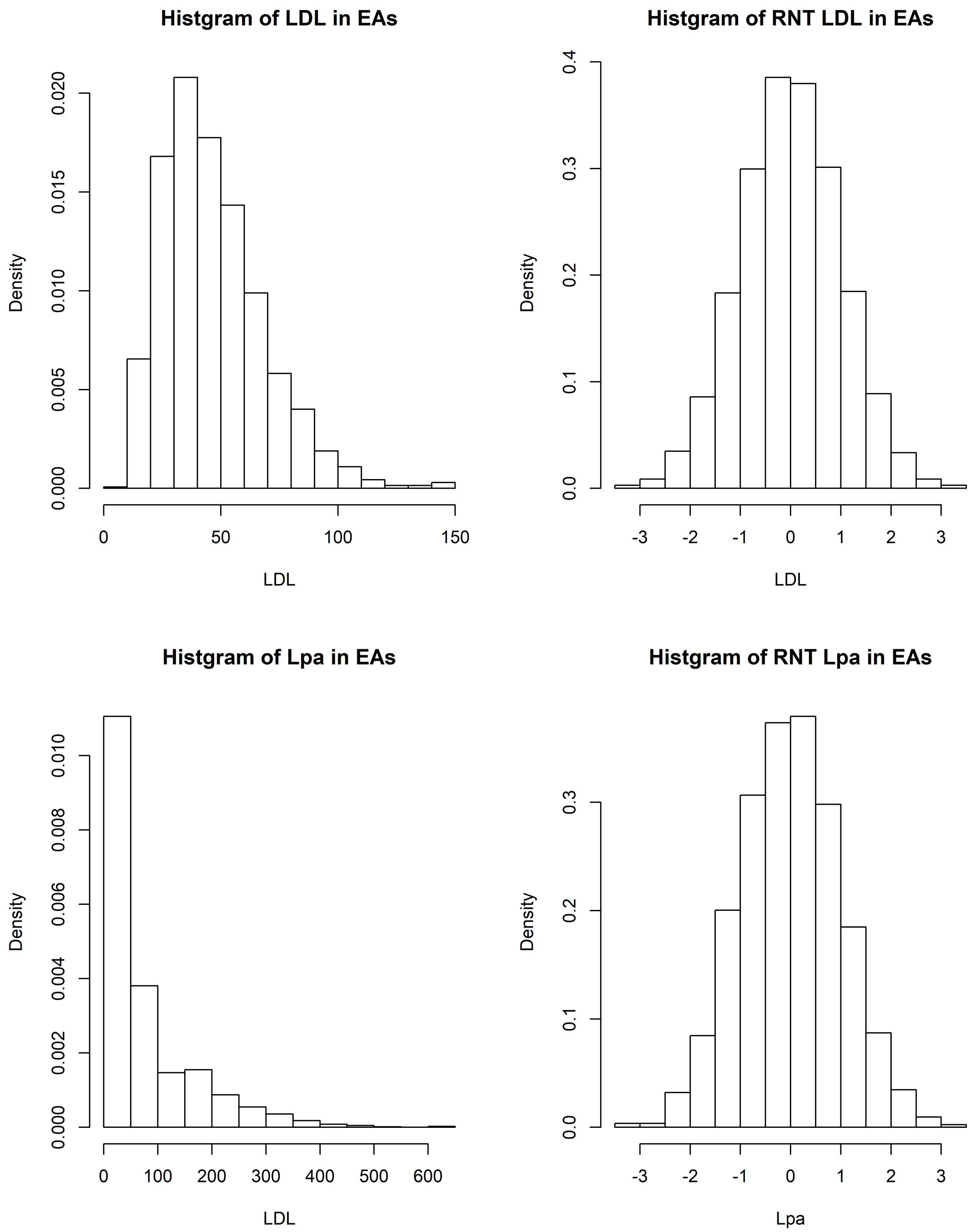


Figure S3: Distribution of small, dense, low-density lipoprotein cholesterol (LDL), lipoprotein(a) (Lp(a)) and inverse rank-normal transformed (RNT) LDL and Lpa among European Americans in ARIC data.



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