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Supplementary Material for *"Depthgram: Visualizing Outliers in High-Dimensional Functional Data with Application to fMRI Data Exploration"*

Summary

This supplementary material contains the proof to Proposition 1 and additional simulation results and fMRI data analysis.

1 | PROOF OF PROPOSITION 1

Proof. To simplify the notation, let us denote by $U = (u_{ij})_{i=1,\ldots,n}^{j=1,\ldots,p}$ the matrix $\mathbf{MEI}_d(\mathbf{x})$, whose rows u_i will be considered as functional observations recorded as discrete points $j = 1, \ldots, p$. In the same way, let us denote by $V = (v_{ij})_{i=1,\ldots,n}^{j=1,\ldots,p}$ the matrix $\mathbf{MBD}_d(\mathbf{x})$. Under assumption *a)*, that is, if the original curves $\mathbf{x}$ do not cross in any of the dimensions $1, \ldots, p$, then, by (1) (in the main text), it holds that

$$v_{ij} = MBD_{\{x_1^j,\ldots,x_n^j\}}(x_i^j) = f_n\left(MEI_{\{x_1^j,\ldots,x_n^j\}}(x_i^j)\right) = f_n(u_{ij}), \quad i = 1, \ldots, n,\ j = 1, \ldots, p.$$

Moreover, if *b)* also holds, that is, if for any time point, the order of individual curves across dimensions is preserved, then both U and V *functional data sets* consist on constant functions, since the values of MBD and MEI will be constant across dimensions. In that case, again by (1) (in the main text) applied to the data set of non-crossing curves U, we have that

$$MBD_{\{u_1,\ldots,u_n\}}(u_i) = f_n\left(MEI_{\{u_1,\ldots,u_n\}}(u_i)\right), \quad i = 1, \ldots, n. \tag{4}$$

But since $v_i = f_n(u_i)$, where f_n is now applied to all the components of u_i, then, by the symmetry around $x_0 = \frac{n+1}{2n}$ of the parabola f_n and its monotonicity in $(-\infty, x_0)$ and $[x_0, \infty)$, we get $MEI_{\{v_1,\ldots,v_n\}}(v_i) = \frac{1}{n} + 2|MEI_{\{u_1,\ldots,u_n\}}(u_i) - x_0|$, $i = 1, \ldots, n$. Equivalently, we get

$$MEI_{\{u_1,\ldots,u_n\}}(u_i) = \begin{cases} x_0 + \frac{1}{2}\left(MEI_{\{v_1,\ldots,v_n\}}(v_i) - \frac{1}{n}\right), & \text{if } MEI_{\{u_1,\ldots,u_n\}}(u_i) \geq x_0 \\ x_0 - \frac{1}{2}\left(MEI_{\{v_1,\ldots,v_n\}}(v_i) - \frac{1}{n}\right), & \text{if } MEI_{\{u_1,\ldots,u_n\}}(u_i) < x_0 \end{cases}.$$

Because of the symmetry of f_n around x_0, replacing this last expression in (4) yields

$$MBD_{\{u_1,\ldots,u_n\}}(u_i) = f_n\left(x_0 - \frac{1}{2}\left(MEI_{\{v_1,\ldots,v_n\}}(v_i) - \frac{1}{n}\right)\right) = g_n\left(1 - MEI_{\{v_1,\ldots,v_n\}}(v_i)\right), \quad i = 1, \ldots, n$$

which, by switching back to the original notation, is the stated result.

Notice that if *b)* does not hold, then (4) becomes an inequality and so does the final result. □

2 | HIGH-DIMENSIONAL SIMULATION STUDY

In this section we present the graphical summaries for the high-dimensional simulation study (section 3), except for Model 1 and $p = 50000$, which is included in the main text. These are provided in Figures 1 to 7.

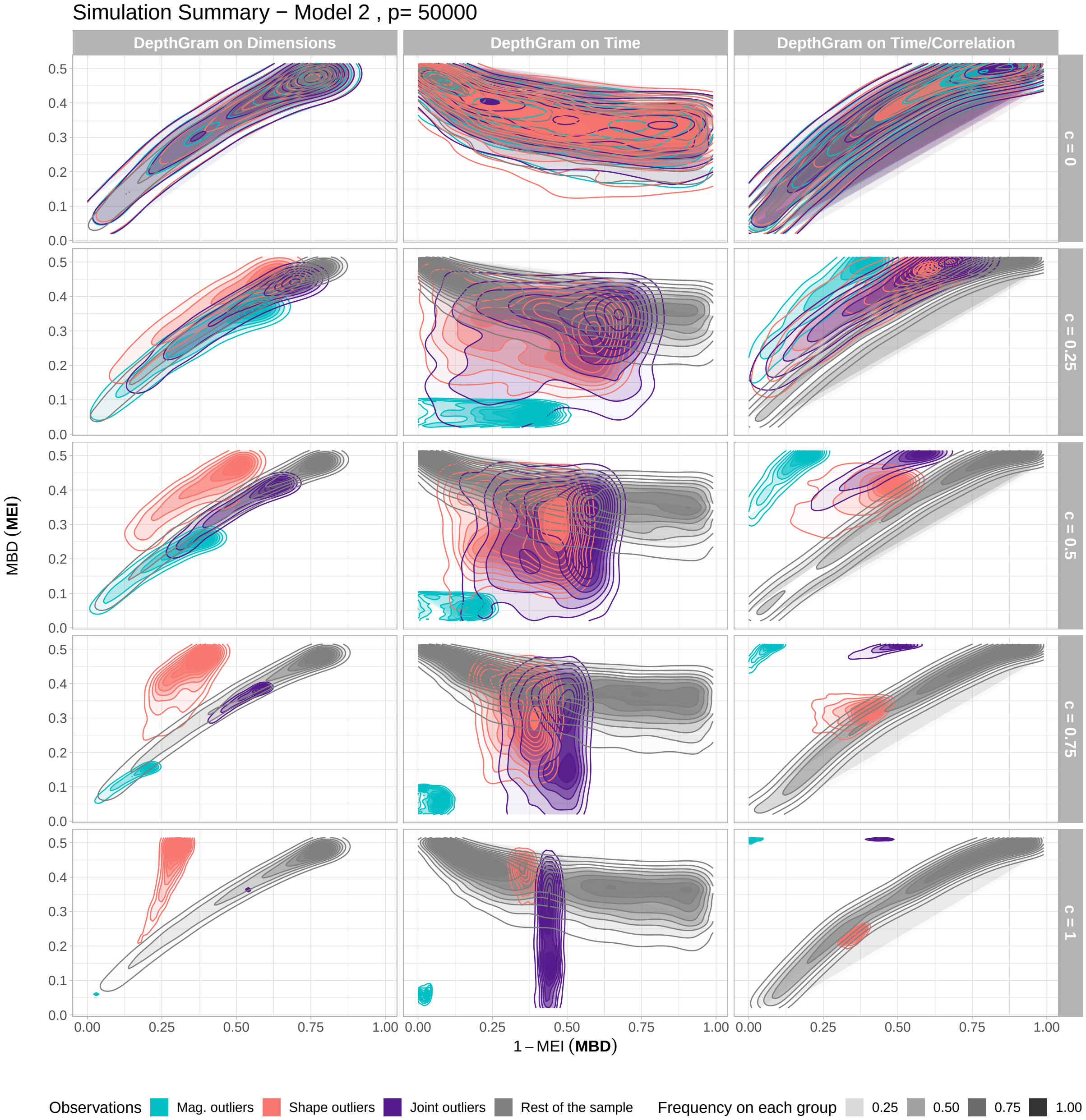

FIGURE 1 Summary of 200 simulation runs under Model 2, with $p = 50000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

Additionally to the DepthGram analysis, in each data set we have conducted a marginal outlier detection through standard methods for magnitude and shape univariate functional outlier detection. We have used the functional boxplot[?] for magnitude outliers and the outliergram[?] for shape outlier detection on each dimension of every high-dimensional functional data set. Notice that this can be easily incorporated to the DepthGram algorithm since both procedures rely on the same modified band depth and modified epigraph index quantities that are already computed dimension-wise for the DepthGram. But more interestingly, the incorporation of this step is also desirable since, by definition, magnitude and shape outliers in multivariate functional data sets are eminently marginal outliers, and informing the dimensions on which they are actually having an atypical behavior, and not only reporting an average outlyingness measure over all dimensions, is an advantage. However, because of the computational burden of the high-dimensional setting, through the simulation study we used unoptimized versions the outliergram and the

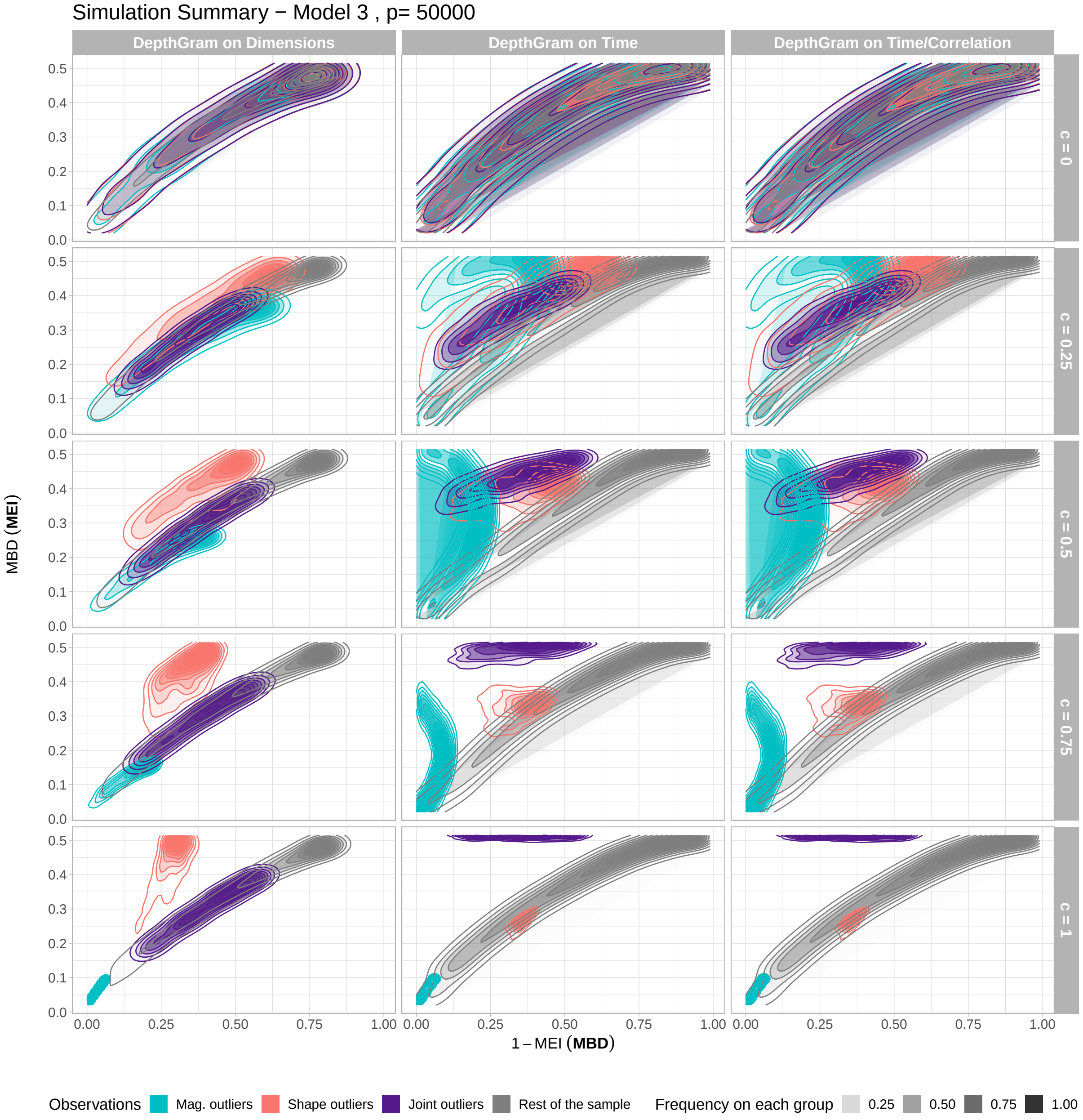

FIGURE 2 Summary of 200 simulation runs under Model 3, with $p = 50000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

functional boxplot. The outlier detection rule in both cases mimics that of the univariate boxplot in which a factor value F (typically $F = 1.5$) needs to be specified so that the outlying observations are those lying below (resp. above) the first (resp. third) quartile minus (resp. plus) the interquartile range times F. Both methods are recommended to be used with data-driven choice of F (see[?] for the adjusted version of the functional boxplot), which significantly increases their performances. However, because of time constraints when performing an extensive high-dimensional simulation study, the sub-efficient rule given by $F = 1.5$ is used. Nonetheless, results are satisfactory as shown in table 1, although the use of the optimized detection rule is feasible (and encouraged) when analyzing a single high-dimensional data set.

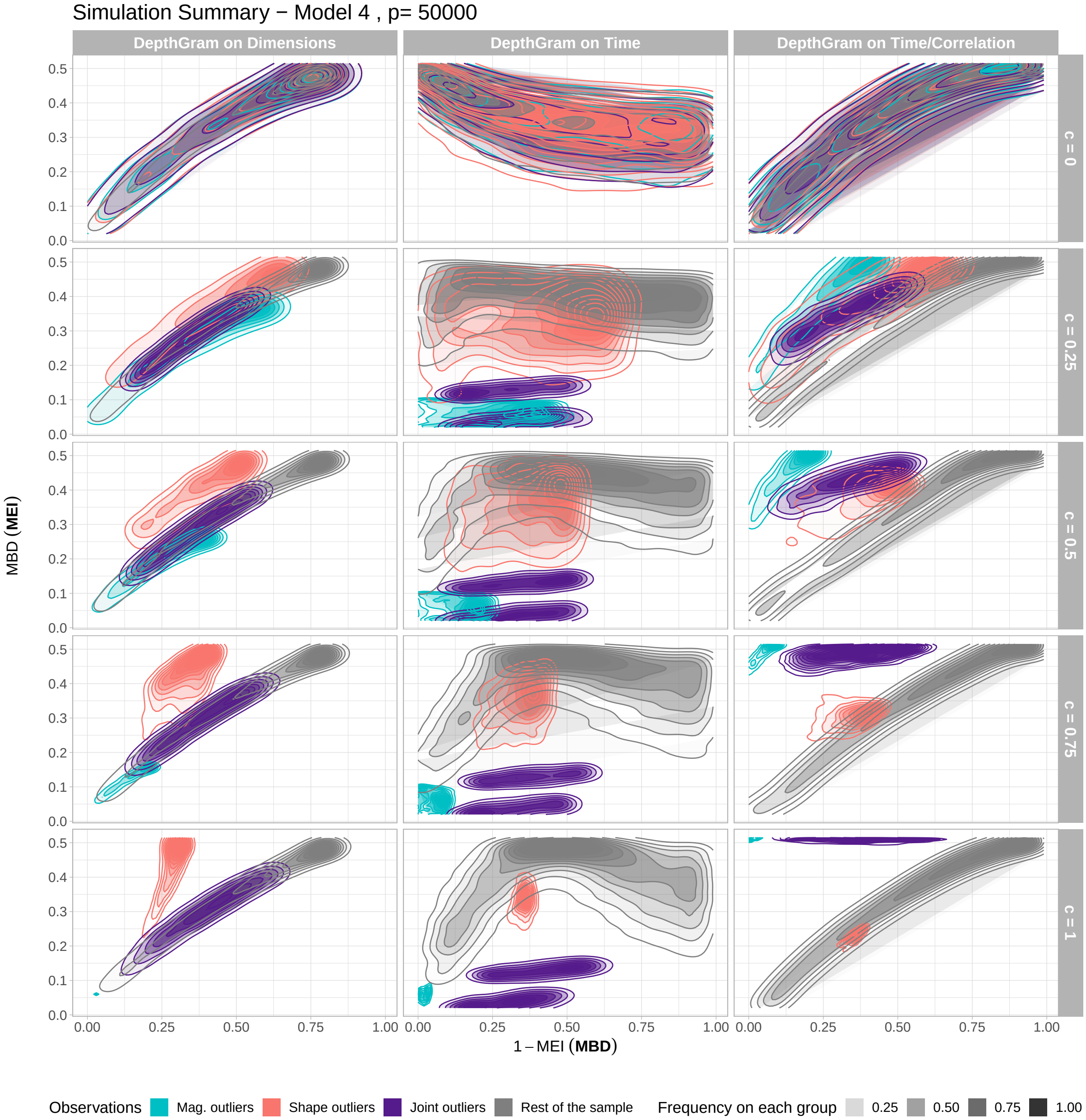

FIGURE 3 Summary of 200 simulation runs under Model 4, with $p = 50000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

3 | LOW-DIMENSIONAL SIMULATION STUDY

We present the full results of the low-dimensional simulation study (section 3.1). These are shown in Figures 8 and 9 and Tables 2 and 3.

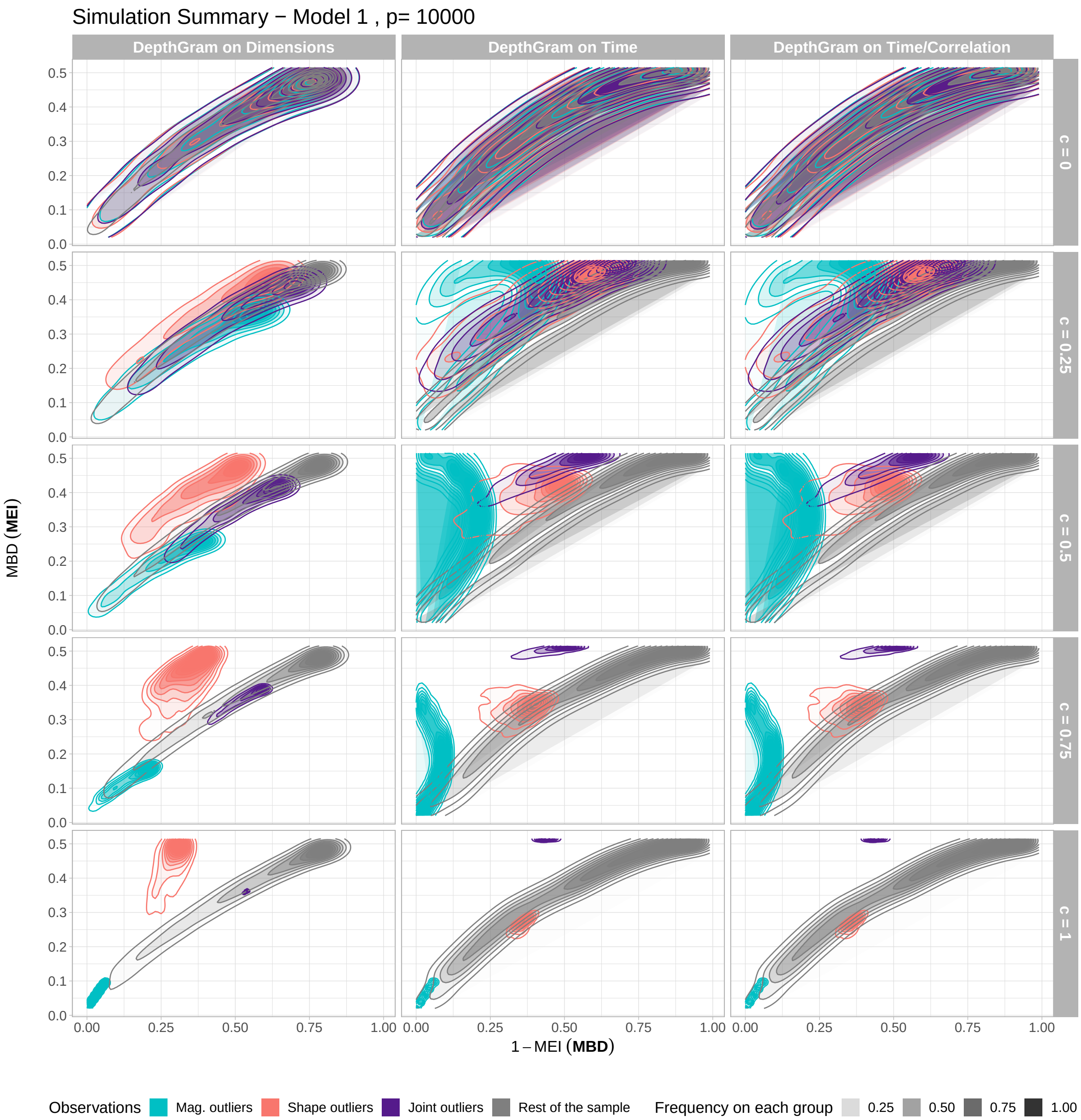

FIGURE 4 Summary of 200 simulation runs under Model 1, with $p = 10000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

4 | TASK FMRI DATA EXPLORATION

We present here the analysis of two additional datasets included in the fMRI study described in Section 4 of the main document. They consist on two task fMRI experiments on the same 100 individuals. The first one was an emotion task, in which participants were presented with blocks of trials in which they had to decide which one of two faces (with an angry or fearful expression) matched another given face. The second one was a gambling task, in which participants were asked to repeatedly guess numbers (over 5 or less than 5) on mystery cards by pressing one of two buttons. Task and resting periods were alternated during a total duration of $T = 176$ and 253 seconds for the emotion and gambling experiments, respectively.

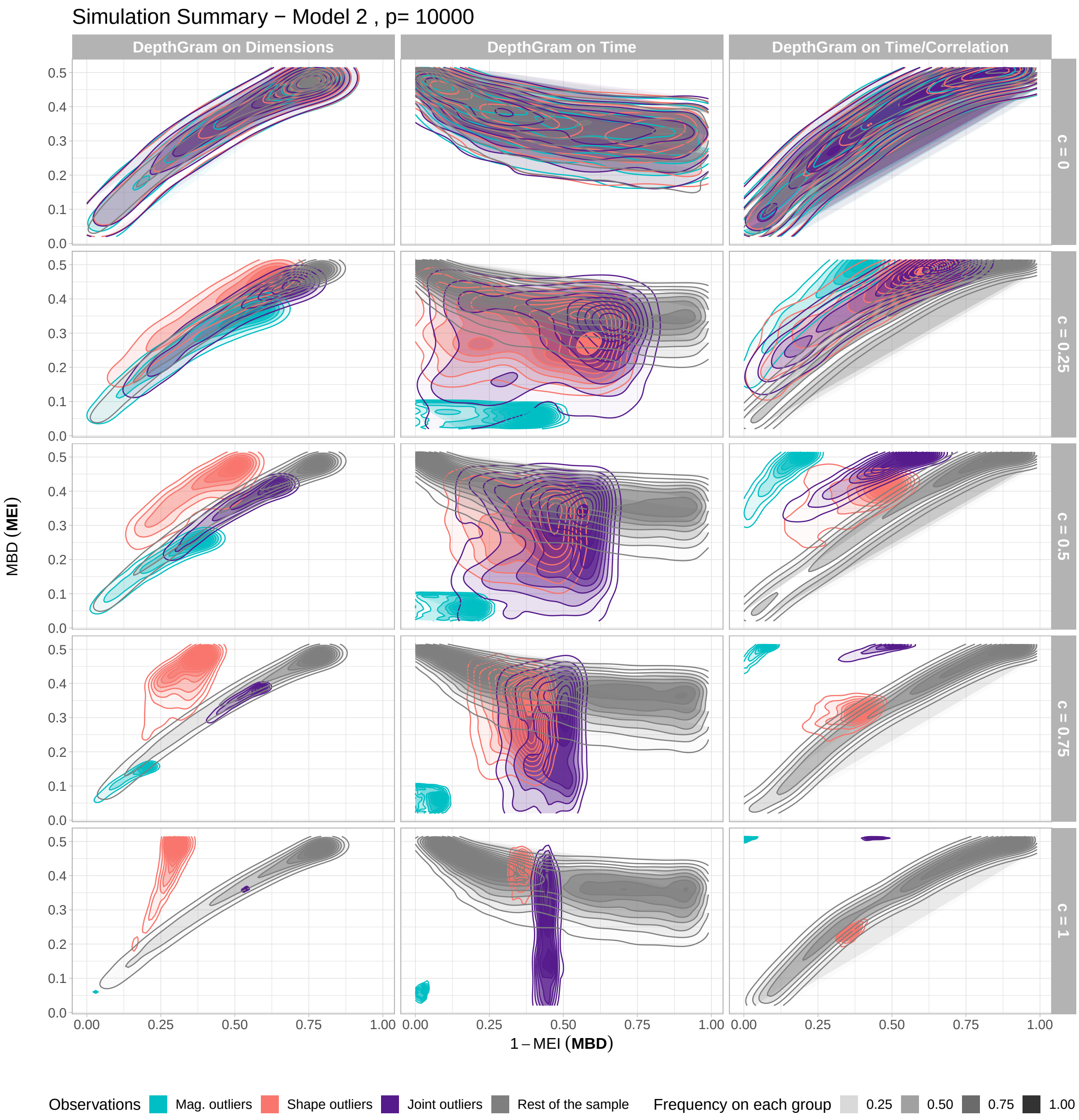

FIGURE 5 Summary of 200 simulation runs under Model 2, with $p = 10000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

Depthgrams for the emotion and gambling experiments are presented in figures 10 and 11. We can observe some differences with respect to the results for the language and motor experiment. For both the emotion and gambling tasks, the dimensions depthgram presents a very concentrated structure similar to those observed in simulations. This is representative of homogeneity of brain signals across dimensions. With respect to the time and time/correlation depthgrams we can also appreciate a more structured data cloud than those obtained for the language and motor tasks. This allows the identification of one or two potential joint outliers that stand out of the remainder of the points on the top-left area of the plots: individuals 59 (emotion task) and 55 (emotion and gambling task). We can also notice that for the two experiments, there are no significant differences between time depthgram and time/correlation depthgram (for the emotion task they are exactly the same). This indicates that in these two datasets there is none or little negative association across voxels.

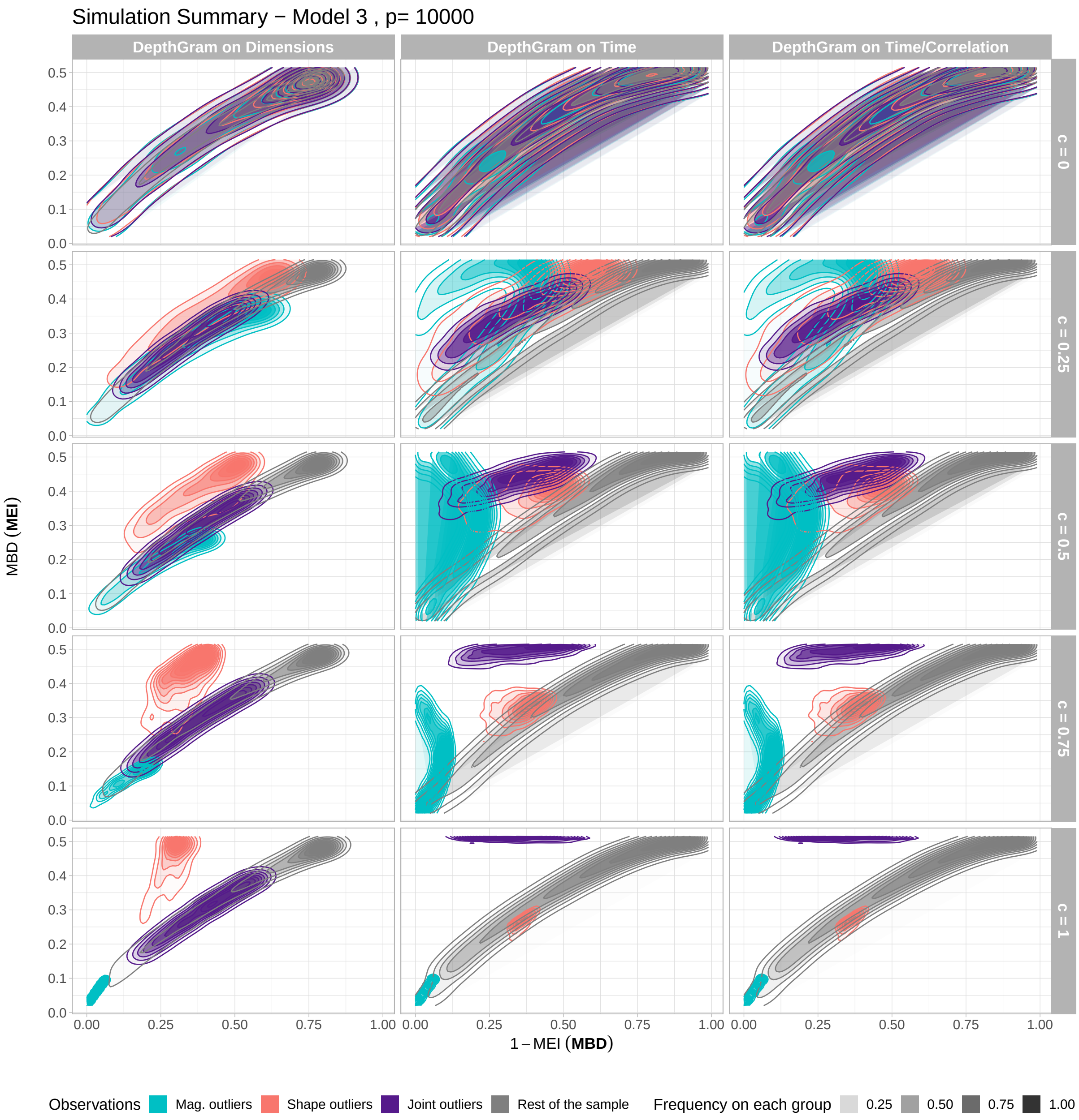

FIGURE 6 Summary of 200 simulation runs under Model 3, with $p = 10000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

https://www.overleaf.com/project/615eb1cc8ab45fce62f0c773

5 | COMPUTATIONAL COMPLEXITY

Regarding the alternative methods considered for the low-dimensional simulation study (section 3.1), we present here a comparative analysis of the computation times of these procedures. Experiments have been carried out in R (version 3.4.4) in an Intel(R) Xeon(R) CPU E5-1650 v3 (x64) @ 3.50GHz with 128GiB of RAM under Windows 10.

For the DepthGram implementation, the code is provided (`DepthGram.R`), whereas for the FOM the R package `mrfDepth` has been used, and for the MS-plot the code is the one provided by the authors in the supplementary materials of their paper

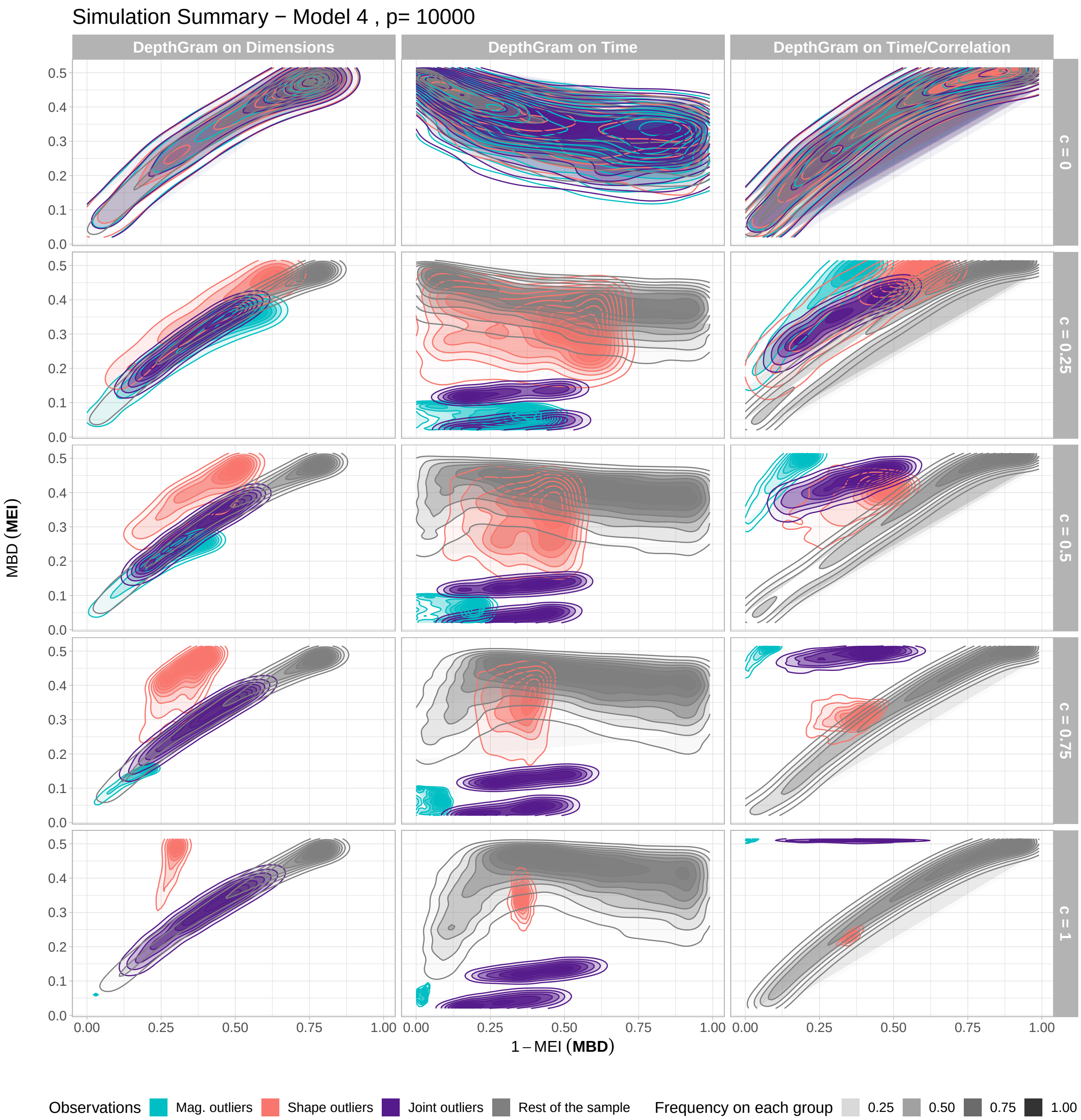

FIGURE 7 Summary of 200 simulation runs under Model 4, with $p = 10000$, and different outlyingness values c. Summary Depthgrams are obtained as the density contours of mbd(epi) and 1-epi(mbd) points over the 200 simulated datasets. The colors represent outlier classification (including non-outlying observations).

https://www.tandfonline.com/doi/ref/10.1080/10618600.2018.1473781. All the methods have been used with the options to obtain the limits of the non-outlying regions disabled. That is, only the times required to compute the two variables used in each of the two-dimensional representations are compared. FOM is used with the *functional directional outlyingness* measure (*fDO*) and MS-plot with the one based on the random projection depth for multivariate data (for the p-dimensional version). For the exact settings used, check the file `Computation_times.R` that allows to reproduce the analysis whose results are summarized here. In a first analysis, all five methods are compared in a low-dimensional setting, restricted to $p < n$ so that the FOM representation can be obtained. In a second analysis, a comparison of the two versions of the MS-plot and the DepthGram is established in higher dimensions. Results are represented in Figures 12 and 13 where we can see how the p-dimensional version of FOM is computationally very heavy compared to the rest of the methods and how the DepthGram exhibits the best performance with a

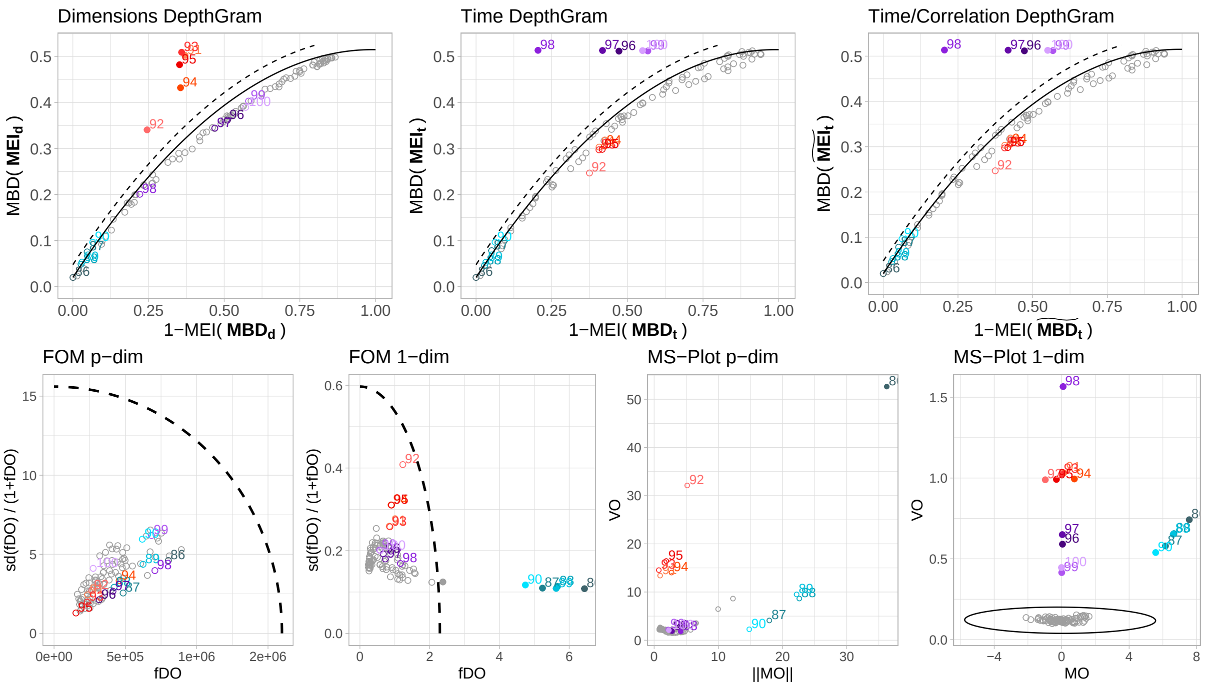

FIGURE 8 Results for a single simulation run under Model 3, with $p = 50$, and $c = 1$. In the top row we present the three DepthGram representations. In the bottom row, we present the FOM and the MS-plot in their p-dimensional and 1-dimensional versions. Except for the p-dimensional MS-plot, the boundary dividing the outlying and non-outlying observations is drawn (a dashed line for the DepthGram and FOM and a solid ellipse for the MS-plot). In all the plots, detected outliers are marked with a bullet while the rest of the observations are represented with a circle. True outliers are represented in color (blue for magnitude outliers, pink/red for shape outliers and purple for joint outliers) while non-outlying observations are drawn in gray.

computation time around 3.5 times faster than the p-dimensional version of the MS-plot and 6 times faster than its 1-dimensional version.

□

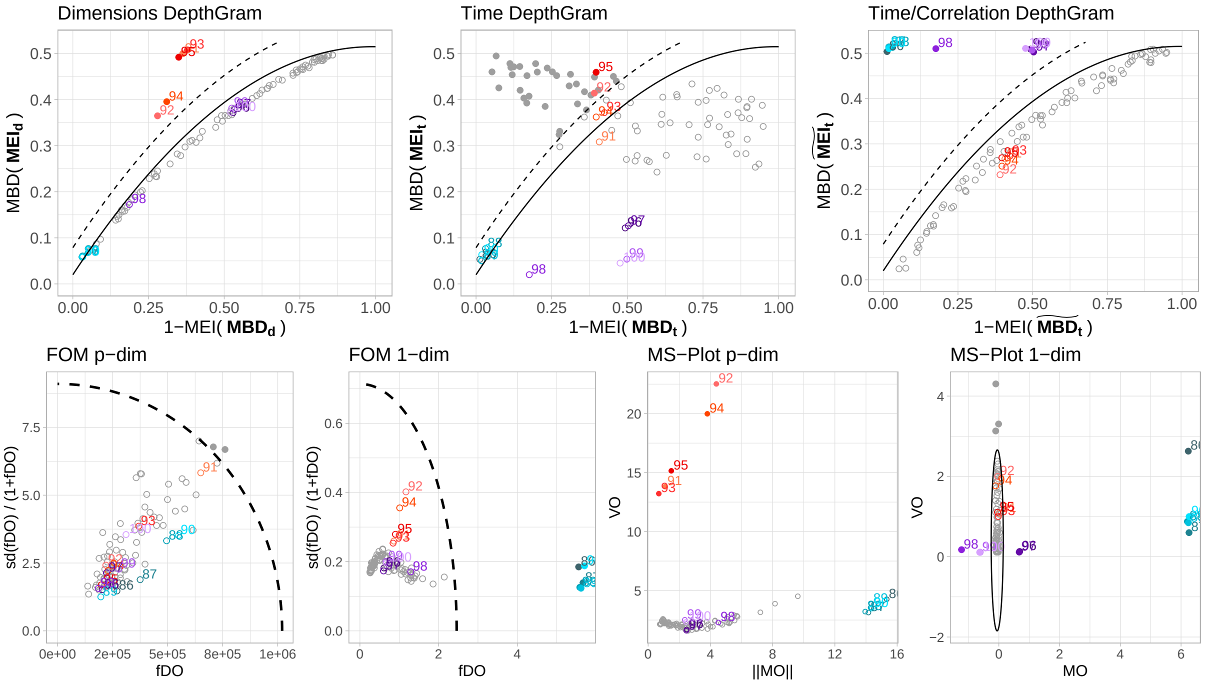

FIGURE 9 Results for a single simulation run under Model 4, with $p = 50$, and $c = 1$. In the top row we present the three DepthGram representations. In the bottom row, we present the FOM and the MS-plot in their p-dimensional and 1-dimensional versions. Except for the p-dimensional MS-plot, the boundary dividing the outlying and non-outlying observations is drawn (a dashed line for the DepthGram and FOM and a solid ellipse for the MS-plot). In all the plots, detected outliers are marked with a bullet while the rest of the observations are represented with a circle. True outliers are represented in color (blue for magnitude outliers, pink/red for shape outliers and purple for joint outliers) while non-outlying observations are drawn in gray.

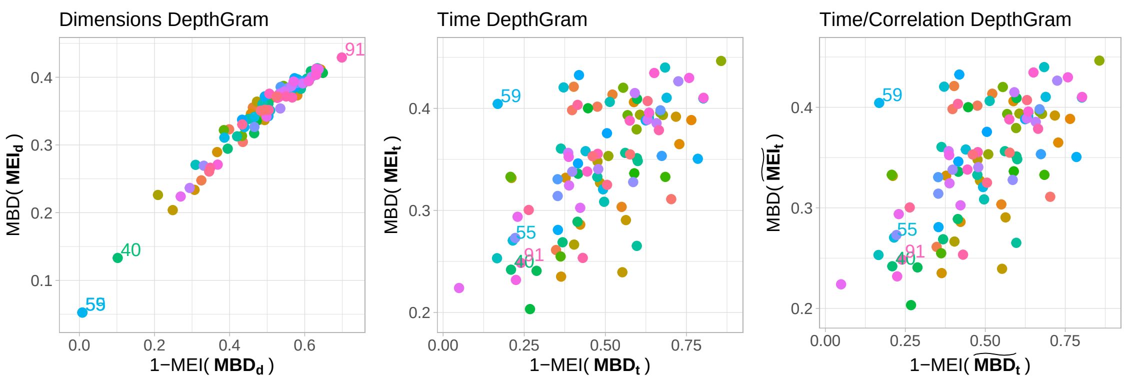

FIGURE 10 DepthGrams for the emotion experiment.

TABLE 1 Mean and standard deviation (in parentheses) of the proportion of correctly, p_c, and, p_f, identified magnitude and shape outliers in the four simulation models over 200 simulation runs. Proportions are calculated considering, for each data set, the number of outlying and non-outlying curves as the sum over dimensions of the corresponding numbers on each dimension.

$p = 10000$								
Magnitude outliers	Model 1		Model 2		Model 3		Model 4	
Cont. level	p_c	p_f	p_c	p_f	p_c	p_f	p_c	p_f
$c = 0$	-	0.005(0.004)	-	0.005(0.005)	-	0.005(0.005)	-	0.005(0.005)
$c = 0.25$	1(0)	0.006(0.004)	1(0)	0.005(0.004)	1(0)	0.005(0.004)	1(0)	0.006(0.004)
$c = 0.5$	1(0)	0.006(0.005)	1(0)	0.007(0.005)	1(0)	0.006(0.005)	1(0)	0.006(0.005)
$c = 0.75$	1(0.01)	0.007(0.005)	1(0.01)	0.007(0.005)	1(0)	0.007(0.005)	1(0.01)	0.006(0.005)
$c = 1$	1(0.01)	0.007(0.007)	1(0.01)	0.007(0.006)	1(0.01)	0.007(0.006)	1(0.01)	0.007(0.006)
$p = 10000$								
Shape outliers	Model 1		Model 2		Model 3		Model 4	
Cont. level	p_c	p_f	p_c	p_f	p_c	p_f	p_c	p_f
$c = 0$	-	0.039(0.003)	-	0.039(0.003)	-	0.039(0.003)	-	0.04(0.003)
$c = 0.25$	0.955(0)	0.036(0.002)	0.956(0)	0.036(0.003)	0.957(0)	0.036(0.002)	0.961(0)	0.036(0.002)
$c = 0.5$	0.956(0)	0.033(0.002)	0.95(0)	0.032(0.002)	0.954(0)	0.032(0.002)	0.947(0)	0.033(0.002)
$c = 0.75$	0.949(0)	0.029(0.002)	0.952(0)	0.029(0.002)	0.95(0)	0.029(0.002)	0.954(0)	0.029(0.002)
$c = 1$	0.94(0)	0.026(0.002)	0.943(0)	0.026(0.002)	0.951(0)	0.026(0.002)	0.941(0)	0.026(0.002)
$p = 50000$								
Magnitude outliers	Model 1		Model 2		Model 3		Model 4	
Cont. level	p_c	p_f	p_c	p_f	p_c	p_f	p_c	p_f
$c = 0$	-	0.005(0.004)	-	0.005(0.004)	-	0.004(0.004)	-	0.004(0.004)
$c = 0.25$	1(0)	0.005(0.004)	1(0)	0.006(0.005)	1(0)	0.006(0.004)	1(0)	0.005(0.004)
$c = 0.5$	1(0)	0.006(0.004)	1(0)	0.006(0.004)	1(0)	0.007(0.005)	1(0)	0.006(0.005)
$c = 0.75$	1(0)	0.007(0.005)	1(0)	0.006(0.005)	1(0.01)	0.007(0.005)	1(0.01)	0.007(0.005)
$c = 1$	1(0.01)	0.007(0.005)	1(0.01)	0.007(0.005)	1(0.01)	0.007(0.006)	1(0)	0.006(0.005)
$p = 50000$								
Shape outliers	Model 1		Model 2		Model 3		Model 4	
Cont. level	p_c	p_f	p_c	p_f	p_c	p_f	p_c	p_f
$c = 0$	-	0.04(0.002)	-	0.04(0.002)	-	0.04(0.002)	-	0.04(0.002)
$c = 0.25$	0.949(0)	0.036(0.002)	0.949(0)	0.036(0.002)	0.953(0)	0.036(0.002)	0.954(0)	0.036(0.003)
$c = 0.5$	0.951(0)	0.032(0.002)	0.948(0)	0.032(0.002)	0.95(0)	0.032(0.002)	0.952(0)	0.033(0.002)
$c = 0.75$	0.952(0)	0.029(0.002)	0.951(0)	0.029(0.002)	0.945(0)	0.029(0.002)	0.955(0)	0.029(0.002)
$c = 1$	0.948(0)	0.026(0.002)	0.947(0)	0.026(0.002)	0.948(0)	0.025(0.002)	0.954(0)	0.026(0.002)

TABLE 2 Mean and standard deviation (in parentheses) of the proportion of correctly (by type: magnitude, p_c^m, shape, p_c^s, joint, p_c^j) and falsely, p_f, identified outliers in the four simulation models in low dimension ($p = 10$) over 200 simulation runs. The DepthGram is compared with the Functional outlier map (FOM) both in its p-dimensional and its one dimensional versions and with the Magnitude-Shape plot (MS-plot) both in its p-dimensional and its one dimensional versions.

$p = 10$ DepthGram	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.01(0.01)	-	-	-	0.01(0.01)	-	-	-	0.01(0.01)	-	-	-	0.01(0.01)
$c = 0.25$	0.65(0.21)	0.73(0.21)	0.33(0.2)	0(0)	0.84(0.18)	0.7(0.21)	0.33(0.21)	0(0)	0.63(0.21)	0.73(0.21)	0.49(0.22)	0(0)	0.84(0.17)	0.71(0.23)	0.51(0.23)	0(0)
$c = 0.5$	0.64(0.2)	0.95(0.1)	0.63(0.21)	0(0)	0.99(0.03)	0.93(0.11)	0.55(0.2)	0(0)	0.6(0.19)	0.94(0.1)	0.78(0.18)	0(0)	0.99(0.04)	0.92(0.13)	0.69(0.19)	0(0)
$c = 0.75$	0.59(0.2)	0.96(0.08)	0.87(0.17)	0(0.01)	1(0)	0.95(0.09)	0.75(0.18)	0(0)	0.57(0.2)	0.96(0.09)	0.95(0.1)	0(0)	1(0)	0.96(0.09)	0.88(0.15)	0(0)
$c = 1$	0.06(0.15)	0.98(0.06)	0.94(0.1)	0(0)	1(0)	0.98(0.06)	0.89(0.14)	0(0)	0.09(0.17)	0.98(0.06)	1(0.02)	0(0)	1(0)	0.99(0.05)	0.97(0.08)	0(0)
$p = 10$ FOM p-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.01(0.02)	-	-	-	0.01(0.01)	-	-	-	0.01(0.01)	-	-	-	0.01(0.01)
$c = 0.25$	1(0)	0.92(0.13)	0.23(0.19)	0(0.01)	1(0)	0.89(0.17)	0.27(0.21)	0(0)	1(0)	0.92(0.14)	0.35(0.21)	0(0.01)	1(0)	0.89(0.16)	0.37(0.25)	0(0.01)
$c = 0.5$	1(0)	1(0.01)	0.56(0.23)	0(0.01)	1(0)	0.99(0.04)	0.57(0.23)	0(0.01)	1(0)	1(0.03)	0.66(0.23)	0(0.01)	1(0)	1(0.03)	0.65(0.24)	0(0.01)
$c = 0.75$	1(0)	0.98(0.07)	0.68(0.22)	0(0)	1(0)	0.97(0.09)	0.72(0.23)	0(0)	1(0)	0.98(0.06)	0.74(0.24)	0(0.01)	1(0)	0.95(0.12)	0.71(0.23)	0(0.01)
$c = 1$	0.99(0.03)	0.01(0.04)	0.8(0.2)	0(0.01)	1(0)	0.01(0.05)	0.76(0.21)	0(0.01)	1(0.03)	0(0.03)	0.86(0.18)	0.01(0.01)	1(0)	0.02(0.06)	0.6(0.23)	0(0.01)
$p = 10$ FOM 1-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.03(0.02)	-	-	-	0.02(0.02)	-	-	-	0.02(0.02)	-	-	-	0.03(0.02)
$c = 0.25$	1(0.03)	0(0.02)	0(0.03)	0.01(0.01)	1(0.01)	0.01(0.04)	0.01(0.04)	0.01(0.01)	1(0)	0.01(0.05)	0(0.02)	0.01(0.01)	1(0)	0.01(0.04)	0(0)	0.01(0.01)
$c = 0.5$	1(0)	0(0.03)	0(0.01)	0.01(0.01)	1(0)	0(0.02)	0(0)	0.01(0.01)	1(0)	0.01(0.04)	0(0)	0.01(0.01)	1(0)	0(0.03)	0(0)	0.01(0.01)
$c = 0.75$	1(0)	0.01(0.05)	0(0)	0(0.01)	1(0)	0(0.03)	0(0)	0.01(0.01)	1(0)	0.01(0.05)	0(0)	0.01(0.01)	1(0)	0.01(0.04)	0(0)	0.01(0.01)
$c = 1$	1(0)	0.02(0.06)	0(0)	0.01(0.01)	1(0)	0.02(0.07)	0(0)	0.01(0.01)	1(0)	0.02(0.07)	0(0)	0.01(0.01)	1(0)	0.02(0.07)	0(0)	0.01(0.01)
$p = 10$ MS-plot p-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.05(0.02)	-	-	-	0.04(0.02)	-	-	-	0.04(0.02)	-	-	-	0.05(0.02)
$c = 0.25$	1(0)	0.97(0.08)	0.29(0.21)	0.02(0.01)	1(0)	0.96(0.1)	0.3(0.21)	0.01(0.01)	1(0)	0.97(0.09)	0.37(0.22)	0.01(0.01)	1(0)	0.96(0.09)	0.41(0.23)	0.01(0.01)
$c = 0.5$	1(0)	1(0)	0.66(0.21)	0.01(0.01)	1(0)	1(0)	0.64(0.21)	0.01(0.01)	1(0)	1(0)	0.76(0.2)	0.01(0.01)	1(0)	1(0)	0.74(0.21)	0.01(0.01)
$c = 0.75$	1(0)	1(0)	0.78(0.2)	0.01(0.01)	1(0)	1(0)	0.79(0.2)	0.01(0.01)	1(0)	1(0)	0.86(0.17)	0.01(0.01)	1(0)	1(0)	0.87(0.17)	0.01(0.01)
$c = 1$	0.89(0.12)	1(0)	0.84(0.18)	0.02(0.01)	1(0)	1(0)	0.81(0.17)	0.01(0.01)	0.88(0.12)	1(0)	0.92(0.13)	0.02(0.01)	1(0)	1(0)	0.89(0.16)	0.01(0.01)
$p = 10$ MS-plot 1-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0(0)	-	-	-	0.13(0.05)	-	-	-	0(0)	-	-	-	0.13(0.05)
$c = 0.25$	1(0)	0.92(0.13)	0.35(0.22)	0(0)	1(0)	0.11(0.15)	0.08(0.12)	0.08(0.04)	1(0)	0.9(0.15)	0.5(0.23)	0(0)	1(0)	0.11(0.16)	0.08(0.13)	0.09(0.04)
$c = 0.5$	1(0)	1(0)	0.75(0.2)	0(0)	1(0)	0.15(0.18)	0.05(0.1)	0.07(0.04)	1(0)	1(0)	0.85(0.17)	0(0)	1(0)	0.16(0.19)	0.2(0.21)	0.09(0.04)
$c = 0.75$	1(0)	1(0)	0.86(0.17)	0(0)	1(0)	0.18(0.2)	0.04(0.1)	0.07(0.04)	1(0)	1(0)	0.93(0.12)	0(0)	1(0)	0.22(0.24)	0.47(0.3)	0.09(0.05)
$c = 1$	1(0)	1(0)	0.89(0.15)	0(0)	1(0)	0.26(0.24)	0.03(0.09)	0.07(0.04)	1(0)	1(0)	0.96(0.1)	0(0)	1(0)	0.31(0.27)	0.83(0.22)	0.09(0.04)

TABLE 3 Mean and standard deviation (in parentheses) of the proportion of correctly (by type: magnitude, p_c^m, shape, p_c^s, joint, p_c^j) and falsely, p_f identified outliers in the four simulation models in low dimension ($p = 50$) over 200 simulation runs. The DepthGram is compared with the Functional outlier map (FOM) both in its p-dimensional and its one dimensional versions and with the Magnitude-Shape plot (MS-plot) both in its p-dimensional and its one dimensional versions.

$p = 50$ DepthGram	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.01(0.01)	-	-	-	0.01(0.02)	-	-	-	0.01(0.01)	-	-	-	0.01(0.02)
$c = 0.25$	0.68(0.19)	0.89(0.14)	0.51(0.25)	0(0.01)	1(0.02)	0.87(0.16)	0.47(0.23)	0(0)	0.67(0.21)	0.89(0.14)	0.62(0.26)	0(0.01)	0.99(0.06)	0.88(0.13)	0.61(0.21)	0(0)
$c = 0.5$	0.66(0.18)	0.96(0.09)	0.88(0.17)	0(0.01)	1(0)	0.94(0.1)	0.75(0.2)	0(0)	0.59(0.21)	0.96(0.1)	0.91(0.14)	0(0.01)	1(0)	0.92(0.12)	0.81(0.17)	0(0)
$c = 0.75$	0.59(0.21)	0.97(0.07)	1(0.01)	0(0.01)	1(0)	0.97(0.07)	0.98(0.07)	0(0)	0.58(0.19)	0.97(0.09)	1(0.02)	0.01(0.01)	1(0)	0.96(0.08)	0.97(0.08)	0(0)
$c = 1$	0.26(0.27)	0.99(0.04)	1(0)	0(0.01)	1(0)	0.98(0.05)	1(0)	0(0)	0.24(0.25)	0.99(0.06)	1(0)	0(0.01)	1(0)	0.98(0.06)	1(0.02)	0(0)
$p = 50$ FOM p-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.01(0.02)	-	-	-	0.01(0.03)	-	-	-	0(0.01)	-	-	-	0.01(0.03)
$c = 0.25$	0.82(0.28)	0.07(0.13)	0.03(0.08)	0(0)	0.84(0.26)	0.07(0.13)	0.03(0.08)	0(0)	0.82(0.28)	0.07(0.13)	0.03(0.08)	0(0)	0.79(0.3)	0.07(0.13)	0.04(0.09)	0(0)
$c = 0.5$	0.85(0.26)	0.09(0.14)	0.06(0.14)	0(0.01)	0.83(0.28)	0.1(0.14)	0.06(0.11)	0(0)	0.8(0.27)	0.09(0.14)	0.08(0.14)	0(0)	0.86(0.26)	0.1(0.14)	0.05(0.11)	0(0)
$c = 0.75$	0.8(0.3)	0.07(0.13)	0.1(0.16)	0(0)	0.86(0.26)	0.07(0.13)	0.1(0.18)	0(0)	0.77(0.31)	0.07(0.13)	0.06(0.13)	0(0)	0.82(0.27)	0.07(0.13)	0.06(0.12)	0(0)
$c = 1$	0.01(0.04)	0(0.01)	0.17(0.22)	0(0.01)	0.13(0.31)	0(0.02)	0.15(0.2)	0(0.01)	0.01(0.04)	0(0.02)	0.02(0.11)	0(0.01)	0.04(0.13)	0(0.02)	0(0.02)	0(0.01)
$p = 50$ FOM 1-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.02(0.02)	-	-	-	0.02(0.02)	-	-	-	0.02(0.02)	-	-	-	0.03(0.02)
$c = 0.25$	1(0.01)	0(0.03)	0(0.03)	0.01(0.01)	1(0)	0(0.03)	0(0.02)	0.01(0.01)	1(0.01)	0(0.03)	0(0)	0.01(0.01)	1(0)	0.01(0.04)	0(0)	0.01(0.01)
$c = 0.5$	1(0)	0(0.03)	0(0.01)	0(0.01)	1(0)	0.01(0.04)	0(0.01)	0(0.01)	1(0)	0(0.02)	0(0)	0.01(0.01)	1(0)	0.01(0.04)	0(0)	0.01(0.01)
$c = 0.75$	1(0)	0.01(0.04)	0(0)	0(0.01)	1(0)	0.01(0.04)	0(0)	0(0.01)	1(0)	0.02(0.05)	0(0)	0.01(0.01)	1(0)	0(0.03)	0(0)	0.01(0.01)
$c = 1$	1(0)	0.02(0.06)	0(0)	0(0.01)	1(0)	0.02(0.07)	0(0)	0(0.01)	1(0)	0.03(0.08)	0(0)	0.01(0.01)	1(0)	0.03(0.09)	0(0)	0.01(0.01)
$p = 50$ MS-plot p-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0.02(0.02)	-	-	-	0.02(0.01)	-	-	-	0.02(0.01)	-	-	-	0.02(0.02)
$c = 0.25$	1(0.01)	0.22(0.2)	0.24(0.19)	0(0)	1(0.01)	0.25(0.2)	0.22(0.19)	0(0)	1(0.01)	0.24(0.23)	0.26(0.21)	0(0)	1(0)	0.24(0.18)	0.28(0.21)	0(0)
$c = 0.5$	0.99(0.04)	0.47(0.34)	0.47(0.22)	0(0)	1(0.03)	0.47(0.33)	0.42(0.22)	0(0)	0.99(0.04)	0.49(0.36)	0.48(0.22)	0(0)	1(0.02)	0.47(0.33)	0.47(0.23)	0(0)
$c = 0.75$	0.98(0.06)	0.51(0.41)	0.6(0.23)	0(0)	0.99(0.03)	0.51(0.4)	0.58(0.23)	0(0)	0.98(0.06)	0.59(0.39)	0.48(0.28)	0(0)	0.99(0.04)	0.53(0.4)	0.48(0.28)	0(0)
$c = 1$	0.14(0.18)	0.38(0.43)	0.83(0.2)	0(0)	0.6(0.49)	0.66(0.47)	0.74(0.22)	0(0)	0.27(0.24)	0.65(0.45)	0.36(0.44)	0(0.01)	0.15(0.35)	1(0)	0.01(0.07)	0(0.01)
$p = 50$ MS-plot 1-dim	Model 1				Model 2				Model 3				Model 4			
Outlyingness	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f	p_c^m	p_c^s	p_c^j	p_f
$c = 0$	-	-	-	0(0)	-	-	-	0.13(0.05)	-	-	-	0(0)	-	-	-	0.14(0.05)
$c = 0.25$	1(0)	1(0)	0.84(0.17)	0(0)	1(0)	0.13(0.16)	0.08(0.12)	0.08(0.04)	1(0)	1(0)	0.9(0.15)	0(0)	1(0)	0.14(0.16)	0.33(0.23)	0.09(0.05)
$c = 0.5$	1(0)	1(0)	0.99(0.03)	0(0)	1(0)	0.16(0.21)	0.04(0.1)	0.07(0.04)	1(0)	1(0)	0.99(0.04)	0(0)	1(0)	0.21(0.21)	0.89(0.16)	0.09(0.05)
$c = 0.75$	1(0)	1(0)	1(0)	0(0)	1(0)	0.2(0.24)	0.04(0.1)	0.07(0.04)	1(0)	1(0)	1(0)	0(0)	1(0)	0.22(0.24)	1(0.02)	0.09(0.04)
$c = 1$	1(0)	1(0)	1(0)	0(0)	1(0)	0.21(0.21)	0.05(0.1)	0.06(0.04)	1(0)	1(0)	1(0)	0(0)	1(0)	0.37(0.3)	1(0)	0.09(0.05)

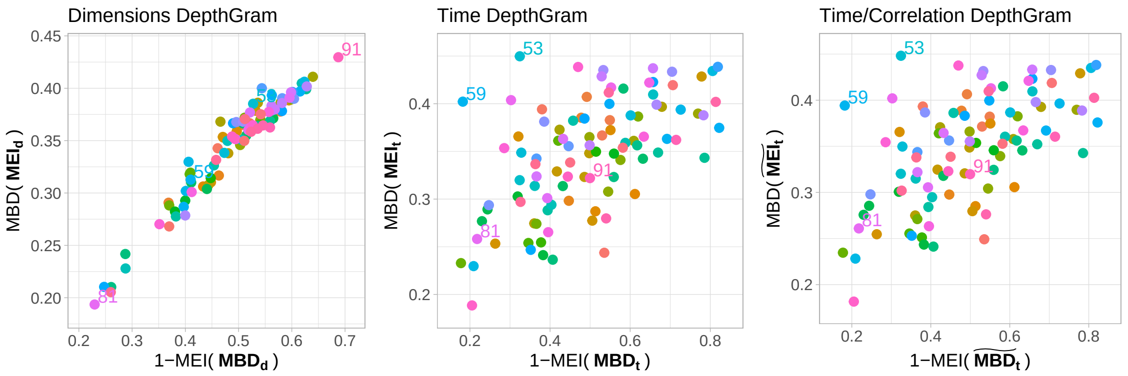

FIGURE 11 DepthGrams for the gambling experiment.

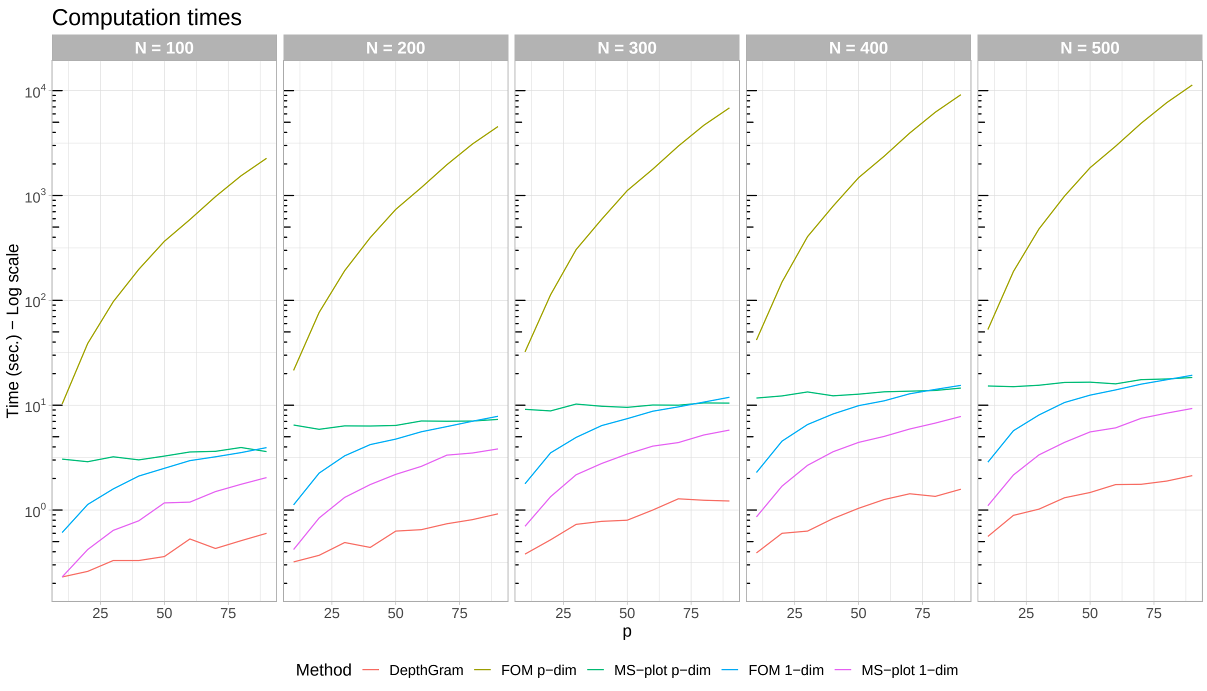

FIGURE 12 Time performance for the different algorithms on multivariate functional data sets of varying dimensions p, with $n = 100$ observations and N observation points.

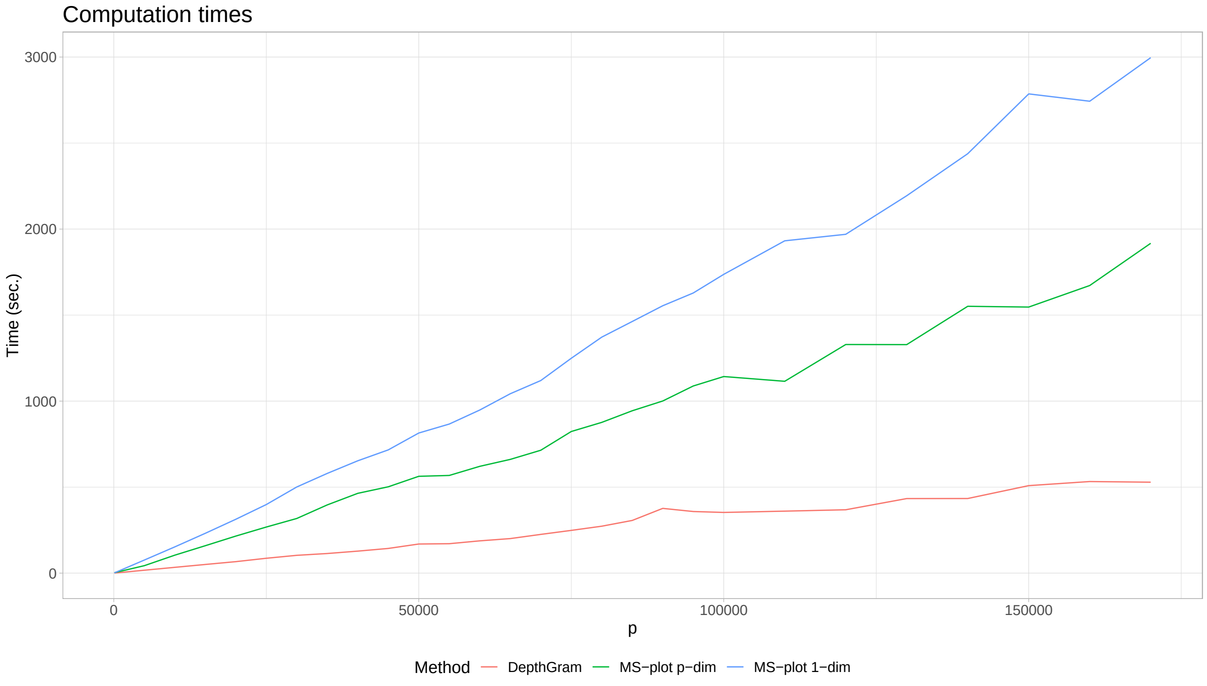

FIGURE 13 Time performance for the DepthGram and MS-plot on multivariate functional data sets of varying dimensions p, with $n = 100$ observations and $N = 100$ observation points.