Supplementary Material.

Cumulative order-preserving image transforming network

In the original paper, the Google DeepMind demonstrated their spatial transformer network with three different popular transformations: affine, perspective and thin-plate spline transformation.[16] It is clear that affine or perspective transformation will not be suitable for modeling changes of structures in the anterior chamber. While we suspect thin plate spline transformation to be capable of modeling anterior chamber changes, we have decided to develop a new image transforming network, which is computationally simpler and easier to implement.[27] The only condition imposed on the new transformation is preservation of the original order of x and y coordinates, because anatomic structures are not likely to change their order; e.g., the corneosclera or the iris is not likely to be twisted.

Following the original definition, the output pixels are defined to lie on a regular grid $G = \{G_{i,j}\}$of pixels $G_{i,j} = (x^t_{i,j}, y^t_{i,j})$. As AS-OCT images have been aligned so that horizontal midline of an image lies approximately at the center of the $x$axis and the top of the cornea lies at almost the same position on the $y$axis, we define $G$with two $m \times n$ matrices $X^t$and Y^t corresponding to $x$and $y$coordinates as follows:

$$X^t = \begin{bmatrix} x^t_{1,1} & \cdots & x^t_{1,p} & x^t_{1,p+1} & x^t_{1,p+2} & \cdots & x^t_{1,n} \\ x^t_{2,1} & \cdots & x^t_{2,p} & x^t_{2,p+1} & x^t_{2,p+2} & \cdots & x^t_{2,n} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x^t_{m,1} & \cdots & x^t_{m,p} & x^t_{m,p+1} & x^t_{m,p+2} & \cdots & x^t_{m,n} \end{bmatrix} = \begin{bmatrix} -p & \cdots & -1 & 0 & 1 & \cdots & p \\ -p & \cdots & -1 & 0 & 1 & \cdots & p \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -p & \cdots & -1 & 0 & 1 & \cdots & p \end{bmatrix}$$

$$\text{and } Y^t = \begin{bmatrix} y^t_{1,1} & y^t_{1,2} & \cdots & y^t_{1,n} \\ y^t_{2,1} & y^t_{2,2} & \cdots & y^t_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y^t_{m,1} & y^t_{m,2} & \cdots & y^t_{m,n} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ m-1 & m-1 & \cdots & m-1 \end{bmatrix} \quad (3)$$

where $p = \frac{n-1}{2}$for odd number n. For even values of n, we modify X^t in the equation (3) to make it centered:

$$X^t = \begin{bmatrix} x^t_{1,1} & \cdots & x^t_{1,p-1} & x^t_{1,p} & x^t_{1,p+1} & x^t_{1,p+2} & \cdots & x^t_{1,n} \\ x^t_{2,1} & \cdots & x^t_{2,p-1} & x^t_{2,p} & x^t_{2,p+1} & x^t_{2,p+2} & \cdots & x^t_{2,n} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x^t_{m,1} & \cdots & x^t_{m,p-1} & x^t_{m,p} & x^t_{m,p+1} & x^t_{m,p+2} & \cdots & x^t_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} -(p-0.5) & \cdots & -1.5 & -0.5 & 0.5 & 1.5 & \cdots & p-0.5 \\ -(p-0.5) & \cdots & -1.5 & -0.5 & 0.5 & 1.5 & \cdots & p-0.5 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(p-0.5) & \cdots & -1.5 & -0.5 & 0.5 & 1.5 & \cdots & p-0.5 \end{bmatrix} \quad (4)$$

where and $p = \frac{n}{2}$.

For even number n, we define two $m \times n$ matrices R and S, which corresponds to the intervals between coordinates of the regular grid $X^t$and Y^t:

$$R = \begin{bmatrix} -1 & \cdots & -1 & -0.5 & 0.5 & 1 & \cdots & 1 \\ -1 & \cdots & -1 & -0.5 & 0.5 & 1 & \cdots & 1 \\ \vdots & ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & -0.5 & 0.5 & 1 & \cdots & 1 \end{bmatrix} \text{and } S = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad (5)$$

Then the regular grid can also be defined as the cumulative sum of intervals from the origin:

$$X^t = \left[R_l \cdot L_p \mid R_r \cdot U_p\right] \text{and } Y^t = ((S^T) \cdot U_m)^T \quad (6)$$

were $R_l$and$R_r$are the left and right half of the matrix R, $L$is a square lower triangular matrix of ones and $U$is a square upper triangular matrix of ones.

The central idea behind COPIT is to adjust the distances between the coordinates such that the order of the x and y coordinates are preserved. A COPIT parametrized by $\theta$is definedwith two components as $\left(x_{ij}^s, y_{ij}^s\right) = T_\theta\left(G_{i,j}\right) = M_\theta\left(G_{i,j}\right) + A_\theta\left(G_{i,j}\right)$,where $M_\theta$is a multiplicative component of T_θ and $A_\theta$is an additive component of T_θ.

A multiplicative component is calculated according to the formula (6) but with modified intervals:

$$M_\theta(X^t) = M_x(W_\theta) = \left[\phi(W_l) \circ R_l \cdot L_p \mid \phi(W_r) \circ R_r \cdot U_p\right] \text{and}$$

$$M_\theta(Y^t) = M_y(V_\theta) = \{(\phi(V_\theta) \circ S)^T \cdot U_m\}^T \quad (7)$$

where $\phi(z) = \begin{cases} \frac{1}{1-z} & z < 0, \\ z + 1 & z \geq 0 \end{cases}$, $\circ$is the Hadamard product, $W_l$and W_r are the left and right half of the$m \times n$ matrix, andW_θ, $V_\theta$is a matrix of size $m \times n$. If the elements of W_θ and $V_\theta$are real numbers, the intervals are multiplied by positive real numbers, then the elements of $\phi(W_l)$,$\phi(W_r)$and $\phi(V_\theta)$are positive real numbers. Hence, the order of x and y coordinates are preserved after transformation M_θ. Also, $M_\theta$is an identity transformation if all elements of W_θ and $V_\theta$equal zero. The transformation $M_\theta$does not affect the origin of the coordinates (0,0). Hence, we define an additive component of the transformation A_θ, which performs a translation to the origin:

$$A_\theta(X^t) = A_x(E_\theta) = E \cdot I_n^T \text{and}$$

$$A_\theta(Y^t) = A_y(F_\theta) = I_m \cdot F^T \quad (8)$$

where E_θ and F_θ are column matrices of real numbers parametrized by θ with size $m \times 1$and $n \times 1$, and I_m andI_n are a column matrix of ones of size $m \times 1$and $n \times 1$.

Any number of transformations can be combined without violating the order-preserving properties if we define a new operator $\boxplus$as follows:

$$T_{\alpha\beta}(G) = T_\alpha(G) \boxplus T_\beta(G) = T_\beta(G) \boxplus T_\alpha(G) =$$

$$\left(M_x\left(W_\alpha \circ W_\beta\right) + A_x\left(G_\alpha + G_\beta\right), M_y\left(V_\alpha \circ V_\beta\right) + A_y\left(H_\alpha + H_\beta\right)\right) \qquad (9)$$