Supplemental Materials for: "Anomalous linear elasticity of disordered networks"

Edan Lerner*
Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, the Netherlands

Eran Bouchbinder†
Chemical and Biological Physics Department, Weizmann Institute of Science, Rehovot 7610001, Israel

I. The response function $c(r)$ and its scaling properties

We construct the response function $c(r)$ as follows:

1. We define the dimensionless unit force dipole $\boldsymbol{d}^{(ij)}$ applied on the pair of nodes i,j as

$$\boldsymbol{d}^{(ij)}_k = \frac{\partial r_{ij}}{\partial \boldsymbol{x}_k} = (\delta_{jk} - \delta_{ik})\,\hat{\boldsymbol{n}}_{ij}\,, \tag{S1}$$

where the k^{th} node coordinates are denoted by $\boldsymbol{x}_k$, $r_{ij} \equiv |\boldsymbol{x}_{ij}|$ is the (scalar) pairwise distance between nodes i and j, $\boldsymbol{x}_{ij} \equiv \boldsymbol{x}_j - \boldsymbol{x}_i$ is the vector difference, and $\hat{\boldsymbol{n}}_{ij} \equiv \boldsymbol{x}_{ij}/r_{ij}$ is the unit vector pointing from node i to node j.

We note that the contraction of the dipole $\boldsymbol{d}$ — as defined in Eq. (S1) above — with a field corresponds to taking the difference of the field across a bond, i.e. to a discrete (network-level) gradient in the bond direction.

2. The response of the network to a unit dimensionless force dipole $\boldsymbol{d}^{(ij)}$ acting on the (ij) spring is calculated through

$$\boldsymbol{u}^{(ij)} = \boldsymbol{\mathcal{M}}^{-1} \cdot \boldsymbol{d}^{(ij)}\,, \tag{S2}$$

where $\boldsymbol{\mathcal{M}} \equiv \frac{\partial^2 U}{\partial \boldsymbol{x} \partial \boldsymbol{x}}$ is the Hessian matrix. An example of such a dipolar response is presented in Fig. 1a in the manuscript. We then define the normalized response as $\hat{\boldsymbol{u}}^{(ij)} \equiv \boldsymbol{u}^{(ij)}/\sqrt{\boldsymbol{u}^{(ij)} \cdot \boldsymbol{u}^{(ij)}}$ and subsequently use it.

3. We are next interested in the *extension/compression* of each spring connecting nodes m,n, associated with the normalized response $\hat{\boldsymbol{u}}^{(ij)}$ to a dimensionless unit force dipole $\boldsymbol{d}^{(ij)}$ applied on the pair of nodes i,j. This quantity is given by

$$A_{(ij),(nm)} = \hat{\boldsymbol{u}}^{(ij)} \cdot \boldsymbol{d}^{(nm)} = \frac{\boldsymbol{d}^{(ij)} \cdot \boldsymbol{\mathcal{M}}^{-1} \cdot \boldsymbol{d}^{(nm)}}{\sqrt{\boldsymbol{d}^{(ij)} \cdot \boldsymbol{\mathcal{M}}^{-2} \cdot \boldsymbol{d}^{(ij)}}}\,. \tag{S3}$$

Recall that a contraction of a vectorial field with $\boldsymbol{d}^{(nm)}$ corresponds to the difference of the vectorial field across the nodes m,n, projected on the direction of the (mn) bond.

4. We are then interested in the average dimensionless energy density associated with the bond extension/compression of Eq. (S3) at a distance r from the position where the dimensionless unit force dipole $\boldsymbol{d}^{(ij)}$ is applied. This quantity, denoted by $c_{ij}(r)$, is given (to quadratic order) by

$$c_{ij}(r) = \langle A^2_{(ij),(nm)} \rangle_{r_{ij,nm}=r}\,, \tag{S4}$$

which is an angular average of $A^2_{(ij),(nm)}$ over all (mn) bonds at a distance r from the (ij) bond.

5. Finally, the response function $c(r)$ reported in Fig. 1b-c in the manuscript is given by an average of the individual functions $c_{ij}(r)$ over a large set of random edges i,j.

In Fig. 1c in the main text, we presented a scaling collapse of the products $r^6 c(r)$ measured in 3D relaxed Hookean spring networks. This is achieved by rescaling the abscissa by $1/\sqrt{z-z_c}$ (since the correlation length $\xi \sim 1/\sqrt{z-z_c}$) and the ordinate by $\sqrt{z-z_c}$. The latter rescaling is motivated as follows.

First, we write the sum-of-squares of the dipole displacement response function $\boldsymbol{u}$ as (omiting the (ij) superscript for the ease of notation)

$$\boldsymbol{u} \cdot \boldsymbol{u} = \boldsymbol{d} \cdot \boldsymbol{\mathcal{M}}^{-2} \cdot \boldsymbol{d} = \sum_\ell \frac{(\boldsymbol{\psi}_\ell \cdot \boldsymbol{d})^2}{\omega_\ell^4}\,, \tag{S5}$$

where $\boldsymbol{\psi}_\ell$ are the eigenfunctions of the Hessian $\boldsymbol{\mathcal{M}}$, ω_ℓ^2 are the eigenvalues associated with the eigenfunctions $\boldsymbol{\psi}_\ell$ and $\boldsymbol{d}$ is a dimensionless unit force dipole. It is known that in relaxed Hookean spring networks with $z \to z_c$, one has $\boldsymbol{\psi}_\ell \cdot \boldsymbol{d} \sim \omega_\ell$ [1]. The sum above can thus be approximated by an integral over the vibrational density of states as [2]

$$\sum_\ell \frac{(\boldsymbol{\psi}_\ell \cdot \boldsymbol{d})^2}{\omega_\ell^4} \sim \int_{\omega_\star} \frac{\omega^2\,\mathcal{D}(\omega)}{\omega^4} \sim \frac{1}{\omega}\bigg|_{\omega_\star} \sim \frac{1}{z-z_c}\,, \tag{S6}$$

where $\omega_\star$ is a characteristic frequency in the unjamming of relaxed Hookean spring networks, whose scaling $\omega_\star \sim z - z_c$ is well established [3].

* e.lerner@uva.nl
† eran.bouchbinder@weizmann.ac.il

Combining the results in Eqs. (S5)-(S6), i.e. $\boldsymbol{u}\cdot\boldsymbol{u}\sim(z-z_{\rm c})^{-1}$, with the spatial scaling of $\boldsymbol{u}(\boldsymbol{r})$ discussed in the manuscript, $\boldsymbol{u}(\boldsymbol{r})\sim r^{-(\text{đ}-2)/2}$, we obtain for the *normalized* response $\hat{\boldsymbol{u}}(\boldsymbol{r})$ the following prediction

$$|\hat{\boldsymbol{u}}(\boldsymbol{r})|\sim\sqrt{z-z_{\rm c}}\,|\boldsymbol{u}(\boldsymbol{r})|\sim\frac{\sqrt{z-z_{\rm c}}}{r^{(\text{đ}-2)/2}}\,. \tag{S7}$$

To proceed, we consider the energy e associated with a unit dimensionless dipole response $\boldsymbol{u}$ in a relaxed spring network, summed over all springs *apart* from the perturbed spring. It was shown that [4]

$$e\sim\int_{a_0}^{\xi}(\boldsymbol{u}\cdot\boldsymbol{d})^2\,r^{\text{đ}-1}dr\sim z-z_{\rm c}\,, \tag{S8}$$

where a_0 is the bond length. If we then invoke the *hypothesis* that $\boldsymbol{u}\cdot\boldsymbol{d}\sim1/r^{(\text{đ}-2)/2}$, spelled out in the manuscript, and account for coordination-dependence of $\boldsymbol{u}\cdot\boldsymbol{d}$ by setting $(\boldsymbol{u}\cdot\boldsymbol{d})^2\equiv f(z)/r^{\text{đ}-2}$, we obtain

$$e\sim f(z)\int_1^{\xi}r^{-(\text{đ}-2)}\,r^{\text{đ}-1}dr\sim f(z)\,\xi^2\sim z-z_{\rm c}\,. \tag{S9}$$

The last relation, together with $\xi\sim1/\sqrt{z-z_{\rm c}}$, implies that $f(z)\sim(z-z_{\rm c})^2$. Consequently, $A_{(ij),(nm)}$ defined in Eq. (S3), which scales as $\boldsymbol{u}\cdot\boldsymbol{d}$, satisfies

$$A_{(ij),(nm)}\sim\frac{\sqrt{z-z_{\rm c}}\sqrt{f(z)}}{r^{(\text{đ}-2)/2}}\sim\frac{(z-z_{\rm c})^{3/2}}{r^{(\text{đ}-2)/2}}\,. \tag{S10}$$

Finally, since $c(r)$ scales as $(A_{(ij),(nm)})^2$, we obtain

$$c(r)\sim(z-z_{\rm c})^3\,r^{-(\text{đ}-2)}\,, \tag{S11}$$

for $r<\xi$, which was used in the manuscript in 3D ($\text{đ}=3$).

II. The response function $C(r)$

For completeness, we present in Fig. S1 the response function $C(r)\sim\langle\boldsymbol{u}(\boldsymbol{r})\cdot\boldsymbol{u}(\boldsymbol{r})\rangle$ (for various level of connectivity z), where $\langle\cdot\rangle$ stands for an angular average — see also the main text —, as measured in our disordered networks of Hookean springs. $C(r)$ indeed reveals two different power laws at small and large r's, which are more cleanly quantified by analyzing $c(r)$, defined above and presented in Fig. 1b,c in the main text.

III. Computer disordered networks

We created disordered networks of 16 million nodes each, composed of relaxed Hookean springs connecting (unit) point masses, with both positional and topological (i.e. degree of connectivity) disorder. This is achieved by adopting the interaction networks of simple, three-dimensional (3D) soft-spheres glasses (see Ref. [5] for a description of the soft-spheres model), where we place a Hookean spring between every pair of interacting particles in the original glass.

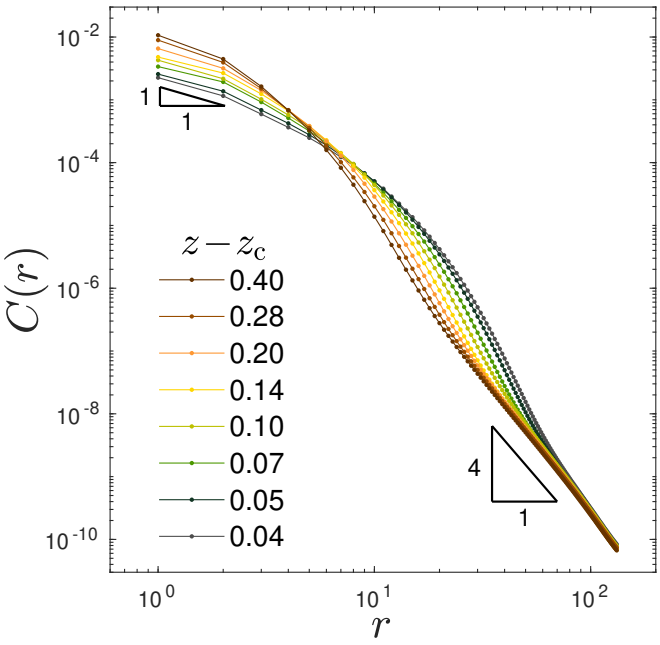

FIG. S1. Response functions $C(r)$ are shown to decay as r^{-1} for $r\ll\xi$ and as r^{-4} for $r\gg\xi$, as expected.

This procedure results in a disordered spring network of initial coordination $z\approx16$, which is much larger than the Maxwell threshold $z_{\rm c}=6$ in 3D. We then systematically remove bonds (springs) by considering in each iteration the bond i,j whose combined connectivity z_i+z_j is largest. Since there are many bonds that share the same combined connectivity z_i+z_j, we consider a secondary bond-removal criterion: amongst all bonds i,j whose z_i+z_j is maximal, we select to remove a bond whose difference $|z_i-z_j|$ is smallest. These two criteria ensure that the connectivity fluctuations of the resulting disordered spring network are small. This procedure is iteratively applied until a target connectivity z is reached. We present the bond dilution algorithm in great detail next.

S-1. Network dilution algorithm

We assume having in hand a soft-sphere-packing-derived, highly coordinated initial network of N nodes, and a set of edges $\mathcal{E}$. An edge $e^{ij}\in\mathcal{E}$ connects between a pair of neighboring nodes i,j, with ranks z^i and z^j respectively. For every edge e^{ij} we define the sum of ranks $s^{ij}=z^i+z^j$ and the absolute difference $d^{ij}=|z^i-z^j|$.

The algorithm first pre-processes the initial network as follows: each edge in the network is represented by a link which is stored in a data-structure — illustrated in Fig. S2 — that is tailored for efficient access to edges with

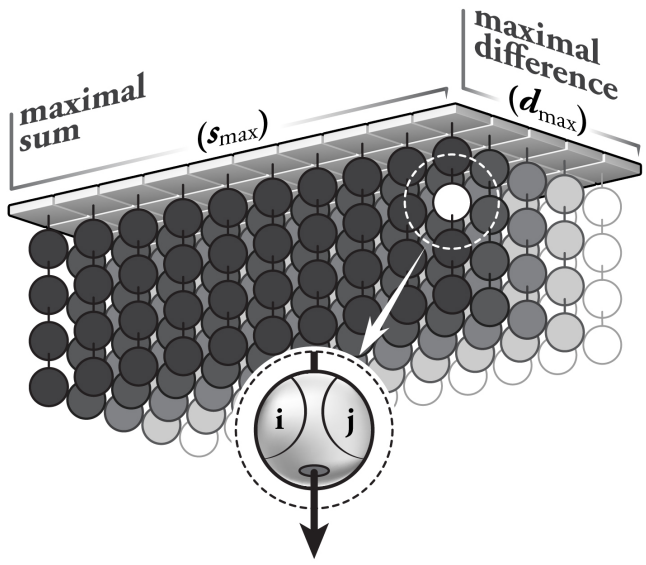

FIG. S2. Illustration of the data-structure used by the algorithm for creating homogeneous random networks. Each link holds the indices i, j of an edge e^{ij}, in addition to a pointer to the next link. See text for further details.

respect to the sum and absolute difference of their ranks during the dilution process. The data-structure consists of a two dimensional array-of-linked-lists $\mathcal{A}$ of dimensions $s_{\max} \times d_{\max}$, where $s_{\max}$ is the maximal sum of ranks, and $d_{\max}$ is the maximal absolute difference of ranks, amongst all of the network edges. Each list is a concatenation of links that represent edges; each link holds a pointer to the next link in a list, and the two indices i and j that define the edge e^{ij} represented by that link, as illustrated in Fig. S2. Links that share same sum of ranks s and same absolute difference d are concatenated into a single list which is anchored at $\mathcal{A}(s, d)$. The pre-processing is described by the following pseudo-code:

Algorithm 1 Pre-processing : $\mathcal{O}(N)$

```
Assign an s_max × d_max array-of-lists A
for each edge e^ij ∈ E do
    s^ij ← z^i + z^j
    d^ij ← |z^i − z^j|
    Add chain link L(i, j) at A(s^ij, d^ij)
end for
```

We can now proceed with deriving a network with some desired average connectivity $\bar{z}_f$ from the initial network using the production stage of the procedure. The main idea is to proceed as follows: we will start from the first non-empty list $\mathcal{A}(s, d)$ with the highest index s and lowest index d; while a list is found and is non-empty, we will remove the first link from that list, and check for its validity in terms of its attributes s^{ij} and d^{ij} (the two latter could have changed by previous edge removals). If valid, we remove the represented edge from the network, or, otherwise, we will re-insert the link into the appropriate list. In this way edges are consecutively removed from the network until reaching the target mean connectivity $\bar{z}_f$, following the min-max scheme discussed above.

The number of edges which need to be removed from the initial network in order to reach $\bar{z}_f < \bar{z}$ is $k \equiv \lfloor N(\bar{z} - \bar{z}_f) \rfloor$, where $\bar{z}$ is the mean connectivity of the initial network. The production stage is described by the following pseudo-code:

Algorithm 2 Production : $\mathcal{O}(N)$

```
s ← s_max
d ← 0
counter ← 0
while counter < k do
    while A(s, d) is empty do
        if d < d_max then
            Increase d
        else
            Decrease s
            d ← 0
        end if
    end while
    Read L(i, j) at A(s, d)
    if (z^i + z^j = s)  AND  (|z^i − z^j| = d) then
        Remove L(i, j) from A(s, d)
        Update z^i and z^j, and remove e^ij from the network
        Increase counter
    else
        Insert L(i, j) to A(z^i + z^j, |z^i − z^j|)
    end if
end while
```

To estimate the complexity of this algorithm, we consider the worst case scenario in which each link visits all of the $s_{\max} \times d_{\max}$ lists before removal, then the running time would be $\propto s_{\max} \times d_{\max} \times N$. However, since both parameters $s_{\max}$ and $d_{\max}$ are independent of N, the complexity remains $\mathcal{O}(N)$.

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